SUMMABILITY METHODS DEFINED BY RIEMANN SUMS

J. D. HILL

1. Introduction. Let $f(x)$ be real valued, bounded and, integrable in the sense of Riemann on the interval $X \equiv (0 \leqslant x \leqslant 1)$, with the value of its integral over X equal to one. For brevity we call such a function *admissible*. The symbol X_k^n will always denote the interval $(k-1)/n \leqslant x \leqslant k/n$, x_k^n an arbitrarily chosen point of X_k^n, and δ any specified set of intermediate points

$$(x_k^n) \qquad\qquad (k = 1, 2, \dots, n;\ n = 1, 2, 3, \dots).$$

If $\{\alpha_k\}$ is a sequence of 0's and 1's such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = \alpha,$$

then it is known [**2**] that the "pattern integral," defined by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k^n)\, \alpha_k, \tag{1.1}$$

exists for all choices of δ and has the value α.

It is clear that (1.1) may also be regarded as defining a method of summability, which we denote by $(\mathfrak{R}, f, \delta)$, and in §2 we find the condition under which this method includes the method $(C, 1)$ of arithmetic means. In §3, by reinterpreting certain results of Agnew and Rado, we call attention to the existence of two classes of functions for which $(\mathfrak{R}, f, \delta)$ is equivalent to $(C, 1)$. We conclude with a pair of examples, the first of which shows that $(\mathfrak{R}, f, \delta)$, for certain f, may be definitely stronger than $(C, 1)$ for bounded sequences.

In terms of the pattern integral, the results exhibit conditions under which the existence of the pattern integral implies that the pattern $\{\alpha_k\}$ has a density in the sense of $(C, 1)$; and the first example shows that the pattern integral may exist without the pattern having a $(C, 1)$-density.

2. Inclusion of $(C, 1)$ by $(\mathfrak{R}, f, \delta)$. In addition to the definitions in §1 we need the following facts from the theory of summability. A transformation of the form

$$T_n = \sum_{k=1}^{n} a_{nk}\, s_k \qquad (n = 1, 2, 3, \dots) \tag{T}$$

defines a method of summability by means of which a sequence $\{s_k\}$ is said to be *summable-T to s* if $T_n \to s$ as $n \to \infty$. If every convergent sequence is summable-T

Received February 19, 1952; in revised form December 18, 1952.

https://doi.org/10.4153/CJM-1953-032-0 Published online by Cambridge University Press

to its ordinary limit, then T is said to be *regular*. In order that T be regular the following conditions are necessary and sufficient:

$$\lim_{n\to\infty} a_{nk} = 0 \qquad (k = 1, 2, 3, \ldots), \tag{2.1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1, \tag{2.2}$$

$$\sup_{n} \sum_{k=1}^{n} | a_{nk} | < \infty. \tag{2.3}$$

A method T_1 is said to *include* a method T_2 if every sequence summable-T_2 is summable-T_1 to the same value. If each of T_1 and T_2 includes the other, then they are *equivalent*. These definitions can be phrased to hold with respect to a specified class of sequences. For example, it will be necessary to employ the phrase, *equivalent for bounded sequences*, with its obvious meaning. A more restrictive concept than the latter is the following. The methods T_1 and T_2 are said to be *absolutely equivalent for bounded sequences* if for each bounded sequence $\{s_k\}$ the corresponding transforms are related by means of the condition

$$\lim_{n\to\infty} [T_n^{(1)} - T_n^{(2)}] = 0.$$

As indicated above, we use the notation $(\mathfrak{R}, f, \delta)$ for the method of summability defined by the transformation

$$T_n = \frac{1}{n} \sum_{k=1}^{n} f(x_k^n)\, s_k \qquad (n = 1, 2, 3, \ldots), \tag{2.4}$$

where f is admissible and $\delta = (x_k^n)$ is a given set of intermediate points. If $f(x) \equiv 1$ on X we note that (2.4) reduces to the Cesàro method $(C, 1)$.

THEOREM 1. *For arbitrary δ_1 and δ_2 the methods $(\mathfrak{R}, f, \delta_1)$ and $(\mathfrak{R}, f, \delta_2)$ are absolutely equivalent for bounded sequences.*

Proof. Let the set of intermediate points δ_1 be denoted by $(x_{k,1}^n)$ and the set δ_2 by $(x_{k,2}^n)$. By a theorem of Cooke [**3**, p. 105] we have only to show that

$$D_n \equiv \frac{1}{n} \sum_{k=1}^{n} | f(x_{k,1}^n) - f(x_{k,2}^n) | = o(1).$$

But this is immediate. For let $M_k^n = \sup f(x)$ on X_k^n, and $m_k^n = \inf f(x)$ on X_k^n $(k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots)$. Then

$$D_n \leqslant \frac{1}{n} \sum_{k=1}^{n} (M_k^n - m_k^n) = o(1).$$

THEOREM 2. *Every method $(\mathfrak{R}, f, \delta)$ includes $(C, 1)$ for bounded sequences.*

Proof. For sequences of 0's and 1's this theorem is merely a restatement of the "principal theorem" in [**2**]. For arbitrary bounded sequences the proof remains the same.

In order to discuss the inclusion of $(C, 1)$ by $(\mathfrak{R}, f, \delta)$ in the general case, we denote by t_n the $(C, 1)$ transform,

$$\frac{1}{n}\sum_{k=1}^{n} s_k,$$

of an arbitrary sequence $\{s_k\}$. Then

$$s_n = nt_n - (n-1)t_{n-1} \qquad (n = 1, 2, 3, \ldots; \quad t_0 \equiv 0)$$

and this expression for s_n in (2.4) yields

$$(2.5) \qquad T_n = \frac{1}{n}\sum_{k=1}^{n} k[f(x_k^n) - f(x_{k+1}^n)]t_k \qquad (n = 1, 2, 3, \ldots),$$

where $f(x_{n+1}^n)$ is understood to be zero.

THEOREM 3. *In order that* $(\mathfrak{R}, f, \delta)$ *include* $(C, 1)$ *for a given* δ *it is necessary and sufficient that*

$$(2.6) \qquad \sup_n \sum_{k=1}^{n} \frac{k}{n} | f(x_k^n) - f(x_{k+1}^n) | \equiv K(\delta) < \infty.$$

Proof. In the notation above it is clear that the statements "$\{s_k\}$ is an arbitrary $(C, 1)$-summable sequence" and "$\{t_n\}$ is an arbitrary convergent sequence" are equivalent. Consequently, convergence in (2.4) for every $(C, 1)$-summable $\{s_k\}$ is equivalent to convergence in (2.5) for every convergent $\{t_n\}$. In order that the latter be true it is necessary and sufficient that the matrix

$$\left(\frac{k}{n}[f(x_k^n) - f(x_{k+1}^n)]\right)$$

be regular, and the conditions (2.1), (2.2), (2.3) in this case reduce simply to (2.6).

It seems reasonable to expect that the satisfaction of (2.6) for all δ can be characterized by some simple property of the function $f(x)$. That this is in fact the case is shown by the next theorem, the proof of which is facilitated by the following lemma.

LEMMA 1. *If* (2.6) *holds for all* δ, *then* $\sup_\delta K(\delta) < \infty$.

Proof. Suppose to the contrary that $\sup_\delta K(\delta) = +\infty$. Then for each $i = 1, 2, 3, \ldots$ there exists a set of intermediate points

$$\delta_i = (x_{k,i}^n)$$

such that $K(\delta_i) > i$. Hence there exists a sequence of indices $\{n_i\}$ such that

$$\frac{1}{n_i}\sum_{k=1}^{n_i} k \, | f(x_{k,i}^{n_i}) - f(x_{k+1,i}^{n_i}) | > i,$$

and it is easily seen that $\{n_i\}$ must contain a strictly increasing subsequence $\{n_{i_j}\} \equiv \{m_j\}$. Let a set of intermediate points be defined as follows: x_k^n is arbitrary if $n \neq m_j$ $(k = 1, 2, \ldots, n)$; and

https://doi.org/10.4153/CJM-1953-032-0 Published online by Cambridge University Press

$$x_k^{m_j} = x_{k,i_j}^{m_j} \qquad (k = 1, 2, \ldots, m_j; \quad j = 1, 2, 3, \ldots).$$

Then (2.6) is evidently violated by this choice of δ.

THEOREM 4. *In order that* (2.6) *hold for all* δ *it is necessary and sufficient that the function* $x f(x)$ *be of bounded variation on* X.

Proof. We first observe that

$$(2.7) \qquad \sum_{k=1}^{n} \frac{k}{n} \,|f(x_k^n) - f(x_{k+1}^n)| \leqslant \sum_{k=1}^{n} |x_k^n f(x_k^n) - x_{k+1}^n f(x_{k+1}^n)| + O(1),$$

$$(2.8) \qquad \sum_{k=1}^{n} |x_k^n f(x_k^n) - x_{k+1}^n f(x_{k+1}^n)| \leqslant \sum_{k=1}^{n} \frac{k}{n} \,|f(x_k^n) - f(x_{k+1}^n)| + O(1),$$

where the quantities $O(1)$, entering here and below, are independent of δ. Then if $x f(x)$ is of bounded variation on X, we find from (2.7) that

$$\sum_{k=1}^{n} \frac{k}{n} \,|f(x_k^n) - f(x_{k+1}^n)| \leqslant V_0^1[x f(x)] + O(1) = O(1),$$

where V denotes total variation. This establishes the sufficiency.

To prove the necessity, let $0 = x_0 < x_1 < \ldots < x_m = 1$ be an arbitrary partition of the interval X. Fix an integer p so large that at most one of the points x_i lies in any sub-interval X_k^p, and let

$$\delta = (x_k^n)$$

be any set of intermediate points such that the set $(x_0, x_1, \ldots, x_m)$ is contained in the set $(x_1^p, x_2^p, \ldots, x_p^p)$. Then using (2.8) and Lemma 1, we have

$$\sum_{i=1}^{m} |x_{i-1} f(x_{i-1}) - x_i f(x_i)| \leqslant \sum_{k=1}^{p} |x_k^p f(x_k^p) - x_{k+1}^p f(x_{k+1}^p)|$$

$$\leqslant \sum_{k=1}^{p} \frac{k}{p} \,|f(x_k^p) - f(x_{k+1}^p)| + O(1) \leqslant \sup_\delta K(\delta) + O(1) = O(1)$$

This completes the proof.

Combining Theorems 3 and 4 we obtain

THEOREM 5. *In order that* $(\mathfrak{R}, f, \delta)$ *include* $(C, 1)$ *for all* δ *it is necessary and sufficient that* $x f(x)$ *be of bounded variation on* X.

3. Equivalence of $(C, 1)$ **and** $(\mathfrak{R}, f, \delta)$. For the sake of completeness we now wish to point out that results of Agnew and Rado yield two classes of monotone functions for which $(\mathfrak{R}, f, \delta)$ is equivalent to $(C, 1)$. It is convenient, however, to begin with the following obvious lemma.

LEMMA 2. *In order that* $(\mathfrak{R}, f, \delta)$ *be equivalent to* $(C, 1)$ *for all* δ *it is necessary and sufficient that* $x f(x)$ *be of bounded variation on* X, *and that the matrix*

https://doi.org/10.4153/CJM-1953-032-0 Published online by Cambridge University Press

$$\left(\frac{k}{n}\,[f(x_k^n) - f(x_{k+1}^n)]\right)$$

in (2.5) *define a method* $(\mathfrak{R}^*, f, \delta)$ *equivalent to convergence for all* δ.

The next lemma is a result of Rado [**5**, p. 274] adapted to the present situation. Essentially the same result was given earlier by Agnew [**1**, p. 245].

Lemma 3. *If* $(\mathfrak{R}^*, f, \delta)$ *is regular for a given* δ *and if there exists constants* θ_δ $(0 < \theta_\delta < 1)$ *and* $N_\delta > 0$, *such that*

$$\sum_{k=1}^{n-1} \frac{k}{n}\, |f(x_k^n) - f(x_{k+1}^n)| \leqslant \theta_\delta\, |f(x_n^n)| \qquad (\text{all } n \geqslant N_\delta),$$

then $(\mathfrak{R}^*, f, \delta)$ *is equivalent to convergence.*

Using these lemmas we easily deduce the following theorems which are essentially contained in results of Agnew [**1**, p. 251].

Theorem 6. *If* $f(x)$ *is non-decreasing then* $(\mathfrak{R}, f, \delta)$ *is equivalent to* $(C, 1)$ *for all* δ.

Proof. To show that the hypotheses of Lemma 2 are satisfied, we first observe that $x\,f(x)$ is of bounded variation if $f(x)$ is non-decreasing. This implies, by Theorem 4, that $(\mathfrak{R}^*, f, \delta)$ is regular for all δ. Turning now to Lemma 3, we have to show that there exist constants θ $(0 < \theta < 1)$ and $N > 0$, independent of δ, such that

$$f(x_n^n) - \frac{1}{n}\sum_{k=1}^{n} f(x_k^n) \leqslant \theta\, |f(x_n^n)| \qquad (\text{all } n \geqslant N; \text{all } \delta). \tag{3.1}$$

To accomplish this we recall the assumption

$$\int_0^1 f(x)\, dx = 1, \tag{3.2}$$

which, together with the fact that $f(x)$ is non-decreasing, implies that $f(x) > 0$ throughout an interval X_m^m. Condition (3.2) also implies the existence of an integer $N \geqslant m$ such that

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k^n) \geqslant \tfrac{1}{2}$$

for all $n \geqslant N$ and all δ. Now fix θ $(0 < \theta < 1)$ so that $(1 - \theta)f(1) \leqslant \frac{1}{2}$. Then we have

$$(1-\theta)\, f(x_n) \leqslant (1-\theta)\, f(1) \leqslant \tfrac{1}{2} \leqslant \frac{1}{n}\sum_{k=1}^{n} f(x_k^n),$$

for all $n \geqslant N$ and all δ, and (3.1) follows at once.

Theorem 7. *If* $f(x)$ *is non-increasing with* $f(1) > \frac{1}{2}$, *then* $(\mathfrak{R}, f, \delta)$ *is equivalent to* $(C, 1)$ *for all* δ.

Proof. The proof parallels the preceding one except that (3.1) is now replaced by

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k^n) - f(x_n^n) \leqslant \theta f(x_n^n). \tag{3.3}$$

In this case, since $f(1) > \frac{1}{2}$, we can fix θ $(0 < \theta < 1)$ so that $q \equiv (1+\theta)f(1) > 1$. We can then choose N so large that

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k^n) \leqslant q$$

for all $n \geqslant N$ and all δ. Then

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k^n) \leqslant (1+\theta)f(1) \leqslant (1+\theta)f(x_n^n),$$

and (3.3) follows.

In terms of the pattern integral, Theorems 6 and 7 provide instances in which the existence of the pattern integral implies that the pattern $\{\alpha_k\}$ has a $(C, 1)$-density. Such examples were lacking in [**2**].

It is of interest to ask if the restriction $f(1) > \frac{1}{2}$ in Theorem 7 is essential. In this connection we have the following example in which $f(1) = \frac{1}{2}$, and the theorem fails to hold.

Example 1. Let $f^*(x)$ be defined as $\frac{3}{2}$ for $0 \leqslant x < \frac{1}{2}$, and as $\frac{1}{2}$ for $\frac{1}{2} \leqslant x \leqslant 1$. Then $f^*(x)$ is admissible and non-increasing but $(\mathfrak{N}, f^*, \delta)$, which includes $(C, 1)$ for all δ by Theorem 5, is definitely stronger than $(C, 1)$. To prove this we consider the sequence

$$\{\alpha_k^*\} \equiv (0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, \ldots),$$

composed of groups of 0's and 1's, where each group beyond the second contains twice as many elements as the preceding group. Let $\{t_n^*\}$ be the $(C, 1)$-transform of $\{\alpha_k^*\}$ and use the notation $t^*(n)$ as alternative to t_n^*. Then it is easy to see that $t^*(2^{2i}) \to \frac{1}{3}$, while $t^*(2^{2i+1}) \to \frac{2}{3}$, so that $\{\alpha_k^*\}$ is not summable-$(C, 1)$.

On the other hand, we can show that $\{\alpha_k^*\}$ is summable-$(\mathfrak{N}, f^*, \delta)$ to the value $\frac{1}{2}$. For let n be given and determine the unique integer $i = i(n)$ such that either (a) $2^{2i-1} < n \leqslant 2^{2i}$, or (b) $2^{2i} < n \leqslant 2^{2i+1}$. Then in case (a) we find that

$$\begin{aligned}
T_n^* &= \frac{1}{n}\sum_{k=1}^{n} f^*(x_k^n)\,\alpha_k^* \\
&= \frac{3}{2n}\sum_{k=1}^{2^{2i-2}} \alpha_k^* + \frac{3}{2n}\sum_{k=2^{2i-2}+1}^{[\frac{1}{2}n]} \alpha_k^* + \frac{1}{2n}\sum_{k=[\frac{1}{2}n]+1}^{2^{2i-1}} \alpha_k^* \\
&= \frac{3}{2n}\sum_{j=1}^{i-2} 2^{2j} + \frac{3}{2n}([\tfrac{1}{2}n] - 2^{2i-2}) + \frac{1}{2n}(2^{2i-1} - [\tfrac{1}{2}n]) \\
&= \frac{[\frac{1}{2}n]}{n} - \frac{2}{n}.
\end{aligned}$$

https://doi.org/10.4153/CJM-1953-032-0 Published online by Cambridge University Press

The choise of δ is obviously immaterial here except in one interval, and this interval yields a term $o(1)$ for either functional value. A similar calculation in case (b) shows that $T_n^* = \frac{1}{2} - (2/n)$. Consequently, the sequence $\{\alpha_k^*\}$ is summable-$(\mathfrak{R}, f^*, \delta)$ to the value $\frac{1}{2}$.

In so far as the pattern integral is concerned, this example shows that the latter may exist without the pattern $\{\alpha_k\}$ having a $(C, 1)$-density. This question was left open in [**2**].

In connection with Theorem 7 and the fact that the condition $f(1) > \frac{1}{2}$ cannot be weakened, the following example is of interest.

Example 2. For any $\alpha > 1$ the function $f_\alpha(x) \equiv \alpha(1 - x)^{\alpha-1}$ is admissible and strictly decreasing, with $f_\alpha(1) = 0$. Moreover, it can be shown that $(\mathfrak{R}, f_\alpha, \delta)$ is equivalent to $(C, 1)$ for bounded sequences. In view of Theorem 1 we can make any convenient choice of δ, and we select δ^- defined by

$$x_k^n = (k - 1)/n \qquad (k = 1, 2, \ldots, n; \quad n = 1, 2, 3, \ldots)$$

Then the matrix of $(\mathfrak{R}, f_\alpha, \delta^-)$ reduces to $(\alpha(n - k + 1)^{\alpha-1}/n^\alpha)$, which is equivalent to the Norlund matrix corresponding to the defining sequence $\{k^{\alpha-1}\}$. Therefore, any bounded sequence summable-$(\mathfrak{R}, f_\alpha, \delta^-)$, say to s, is summable to s by the classical Abel method [**7**, p. 426], and hence summable-$(C, 1)$ to s [**4**, p. 37].

4. Some further remarks. One observes that $(\mathfrak{R}, f_2, \delta^-)$ of the preceding Example 2 is equivalent to $(C, 2)$, and this raises the question of the relationship between $(\mathfrak{R}, f, \delta)$ and (C, α) in general. In this regard we state without proof the following facts.

(4.1) *If there exists a Riemann integrable function $f_\alpha^*(x)$ and a set of subdivision points δ_α such that $(\mathfrak{R}, f_\alpha^*, \delta_\alpha)$ coincides with (C, α) for $\alpha \geqslant 1$, then $f_\alpha^*(x)$ is equal to $f_\alpha(x) = \alpha(1 - x)^{\alpha-1}$ almost everywhere.*

(4.2) *In order that there exist a set of subdivision points δ_α such that $(\mathfrak{R}, f_\alpha, \delta_\alpha)$ coincides with (C, α), it is necessary and sufficient that $1 \leqslant \alpha \leqslant 2$.*

(4.3) *The sequence $\{(-1)^{k-1} k^3\}$, which is not summable-$(C, 3)$, is summable-$(\mathfrak{R}, f_3, \delta^-)$ to zero.*

A connection between general triangular methods (a_{nk}) and the methods $(\mathfrak{R}, f, \delta)$ may be established as follows.

(4.4) *Let (a_{nk}) be triangular and regular and let $\phi_n(x) \equiv na_{nk}$ for $(k - 1)/n \leqslant x < k/n$ $(k = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots)$. Suppose that $|\phi_n(x)| \leqslant \phi(x)$ a.e. for all $n \geqslant N$, where $\phi(x)$ is positive and Lebesgue integrable; and that there exists a Riemann integrable function $f(x)$ such that $\phi_n(x) \to f(x)$ a.e. Then, for all δ, $(\mathfrak{R}, f, \delta)$ is absolutely equivalent to (a_{nk}) for bounded sequences.*

The conclusion in (4.4) cannot in general be strengthened to *equivalence*. To see this we choose for (a_{nk}) the matrix of $(C, 3)$, so that $f(x)$ in (4.4) can be taken as $f_3(x)$ in (4.1). The assertion then follows from (4.3).

References

1. R. P. Agnew, *On equivalence of methods of evaluation of sequences*, Tôhoku Math. J., *35* (1932), 244–252.
2. R. E. Carr and J. D. Hill, *Pattern integration*, Proc. Amer. Math. Soc., *2* (1951), 242–245.
3. R. G. Cooke, *Infinite matrices and sequence spaces* (London, 1950).
4. E. Kogbetliantz, *Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques* (Paris, 1931).
5. R. Rado, *Some elementary Tauberian theorems* I, Quarterly J. Math., *9* (1938), 274–282.
6. M. Riesz, *Sur l'équivalence de certaines méthodes de sommation*, Proc. London Math. Soc. (2), *22* (1923), 412–419.
7. G. F. Woronoi and J. D. Tamarkin, *Extensions of the notion of the limit of the sum of the terms of an infinite series*, Ann. Math. (2), *33* (1932), 422–428.

Michigan State College