A New Proof of the Formulae for Right-Angled Spherical Triangles.

By Professor JOHN JACK.

It is assumed that the sines of the sides are proportional to the sines of the opposite angles.

ACB (Fig. 9) is a spherical △, with C a right angle.

Produce AC, AB to D and E so that $AD = AE = \frac{\pi}{2}$.

Draw the great ⊙ DEF and produce CB to meet it in F.

Then F is the pole of AD and A the pole of DF.

Then sides and angles of ABC are $a \quad b \quad c \quad A \quad B$

BEF are $\frac{\pi}{2} - A \quad \frac{\pi}{2} - c \quad \frac{\pi}{2} - a \quad B \quad \frac{\pi}{2} - b.$

$$\therefore \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{1} \quad - \quad - \quad - \quad \text{I.}$$

and
$$\frac{\sin(\frac{\pi}{2} - A)}{\sin B} = \frac{\sin(\frac{\pi}{2} - c)}{\sin(\frac{\pi}{2} - b)} = \frac{\sin(\frac{\pi}{2} - a)}{1}$$

that is
$$\frac{\cos A}{\sin B} = \frac{\cos c}{\cos b} = \frac{\cos a}{1} \quad - \quad - \quad - \quad \text{II.}$$

and ∴
$$\frac{\cos B}{\sin A} = \frac{\cos c}{\cos a} = \frac{\cos b}{1} \quad - \quad - \quad - \quad \text{III.}$$

by interchange of a, A and b, B.

$$\therefore \quad \left.\begin{array}{l} \sin a = \sin c \sin A \\ \sin b = \sin c \sin B \end{array}\right\} \text{ from I.} \quad - \quad - \quad - \quad 1.$$

$$\text{and} \quad \left.\begin{array}{l} \cos A = \cos a \sin B \\ \cos B = \cos b \sin A \end{array}\right\} \text{ from II., III.} \quad - \quad - \quad 2.$$

$$\text{and} \quad \cos c = \cos a \,.\, \cos b \quad \text{from II. or III.} \quad - \quad 3.$$

From 2
$$\cos A \cos B = \cos a \cos b \sin A \sin B$$

$$\therefore \quad \cot A \cot B = \cos a \cos b = \cos c \text{ by } 3 \quad - \quad - \quad - \quad 4.$$

Again
$$\sin c = \frac{\sin b}{\sin B} \quad \text{by I.}$$

$$\cos c = \frac{\cos A \cos b}{\sin B} \quad \text{by II.} \qquad \text{Divide}$$

$$\therefore \quad \tan c = \frac{\tan b}{\cos A} \qquad \therefore \quad \left.\begin{array}{l} \tan b = \tan c \cos A \\ \text{and } \tan a = \tan c \cos B \end{array}\right\} \quad - \quad - \quad 5$$

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Again $\sin a = \dfrac{\sin b \sin A}{\sin B}$ by I.

and $\cos a = \dfrac{\cos A}{\sin B}$ by II. Divide

$$\left.\begin{array}{ll}\therefore & \tan a = \tan A \sin b \\ \text{so} & \tan b = \tan B \sin a\end{array}\right\} \quad . \quad . \quad . \quad . \quad 6.$$

Note on Napier's Rules.

By Professor John Jack.

Denote the parts b A c B a of $\triangle$ ABC (Fig. 9)
by 1 2 3 4 5

then the parts corresponding of the $\triangle$ BEF, namely,

$$\frac{\pi}{2} - c, \quad B, \quad \frac{\pi}{2} - a, \quad \frac{\pi}{2} - b, \quad \frac{\pi}{2} - A$$

will be denoted by $\frac{\pi}{2} - 3, \quad 4, \quad \frac{\pi}{2} - 5, \quad \frac{\pi}{2} - 1, \quad \frac{\pi}{2} - 2.$

Now a third $\triangle$ can similarly be derived from this second, a fourth from the third, and a fifth from the fourth. But when the process is applied to the fifth, the first $\triangle$ is obtained. Hence only 5 $\triangle$s can be obtained, which are the following :—

1	2	3	4	5
$\frac{\pi}{2} - 3$	4	$\frac{\pi}{2} - 5$	$\frac{\pi}{2} - 1$	$\frac{\pi}{2} - 2$
5	$\frac{\pi}{2} - 1$	2	3	$\frac{\pi}{2} - 4$
$\frac{\pi}{2} - 2$	3	4	$\frac{\pi}{2} - 5$	1
$\frac{\pi}{2} - 4$	$\frac{\pi}{2} - 5$	$\frac{\pi}{2} - 1$	2	$\frac{\pi}{2} - 3$
1	2	3	4	5

where the mid-column contains the hypotenuse, the two next to it contain the angles, and the extreme columns the sides of the several right-angled triangles.