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ON THE CHARACTERISTIC INITIAL-VALUE PROBLEM FOR LINEAR PARTIAL FUNCTIONAL-DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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Abstract Theorems on the Fredholm alternative and well-posedness of the characteristic initial-value problem

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = \ell(u)(t,x) + q(t,x),$$
$$u(t,c) = \varphi(t) \quad \text{for } t \in [a,b], \qquad u(a,x) = \psi(x) \quad \text{for } x \in [c,d],$$

are established, where $\ell : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator, $q \in L(\mathcal{D};\mathbb{R})$, $\varphi : [a,b] \to \mathbb{R}$, $\psi : [c,d] \to \mathbb{R}$ are absolutely continuous functions and $\mathcal{D} = [a,b] \times [c,d]$. Some solvability conditions of the problem considered are also given.

Keywords: functional-differential equation of hyperbolic type; characteristic initial-value problem; Fredholm alternative; well-posedness; existence of solutions

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1. Introduction

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$, we consider the linear partial functional-differential equation

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = \ell(u)(t,x) + q(t,x), \tag{1.1}$$

where $\ell : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ is a linear bounded operator and $q \in L(\mathcal{D};\mathbb{R})$. As usual, $C(\mathcal{D};\mathbb{R})$ and $L(\mathcal{D};\mathbb{R})$ denote the Banach spaces of continuous and Lebesgue integrable functions, respectively, equipped with the standard norms.

A function $u \in C^*(\mathcal{D};\mathbb{R})$ is said to be a solution to Equation (1.1) if it satisfies the equality (1.1) almost everywhere on the set $\mathcal{D}$.

Various initial- and boundary-value problems for hyperbolic differential equations and their systems have been studied in the literature (see, for example, [**3**, **6**, **7**, **9**–**12**, **16**, **19**–**21**] and the references therein). We shall consider the so-called characteristic initial-value problem (Darboux problem). In this case, the values of the solution u of (1.1) are prescribed on both characteristics $t = a$ and $x = c$, i.e. the initial conditions are

$$u(t,c) = \varphi(t) \quad \text{for } t \in [a,b], \qquad u(a,x) = \psi(x) \quad \text{for } x \in [c,d], \tag{1.2}$$

where $\varphi : [a,b] \to \mathbb{R}$, $\psi : [c,d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$.

The aim of the paper is to prove the Fredholm alternative and well-posedness of problem (1.1), (1.2) (see §§ 4 and 6). Moreover, in § 5 some conditions are given under which problem (1.1), (1.2) has a unique solution. The results obtained are applied for the equation with deviating arguments

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = p(t,x)u(\tau(t,x), \mu(t,x)) + q(t,x), \tag{1.1$'$}$$

where $p, q \in L(\mathcal{D}; \mathbb{R})$ and $\tau : \mathcal{D} \to [a,b]$, $\mu : \mathcal{D} \to [c,d]$ are measurable functions.

Let us note that analogous results for the 'ordinary' functional-differential equations and their systems are given in [**1**, **8**, **13**, **14**].

2. Notation and definitions

The following notation is used throughout the paper.

$\mathbb{N}$ is the set of all natural numbers; $\mathbb{R}$ is the set of all real numbers; $\mathrm{Ent}(x)$ denotes the entire part of the number $x \in \mathbb{R}$.

$\mathcal{D} = [a,b] \times [c,d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.

$C(\mathcal{D}; \mathbb{R})$ is the Banach space of continuous functions $v : \mathcal{D} \to \mathbb{R}$ equipped with the norm

$$\|v\|_C = \max\{|v(t,x)| : (t,x) \in \mathcal{D}\}.$$

$\tilde{C}([\alpha,\beta]; \mathbb{R})$, where $-\infty < \alpha < \beta < +\infty$, is the set of absolutely continuous functions $u : [\alpha,\beta] \to \mathbb{R}$.

$C^*(\mathcal{D}; \mathbb{R})$ is the set of functions $v : \mathcal{D} \to \mathbb{R}$ admitting the representation

$$v(t,x) = v_1(t) + v_2(x) + \int_a^t \int_c^x h(s,\eta) \, \mathrm{d}\eta \, \mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},$$

where $v_1 \in \tilde{C}([a,b], \mathbb{R})$, $v_2 \in \tilde{C}([c,d], \mathbb{R})$ and $h \in L(\mathcal{D}; \mathbb{R})$. Equivalent definitions of the class $C^*(\mathcal{D}; \mathbb{R})$ are given in Remark 2.2, below.

$L(\mathcal{D}; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : \mathcal{D} \to \mathbb{R}$ equipped with the norm

$$\|p\|_L = \iint_{\mathcal{D}} |p(t,x)| \, \mathrm{d}t \, \mathrm{d}x.$$

$\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell : C(\mathcal{D}; \mathbb{R}) \to L(\mathcal{D}; \mathbb{R})$.

The Lebesgue measure of the set $A \subset \mathbb{R}^2$ is denoted by $\operatorname{meas} A$.

If X, Y are Banach spaces and $T : X \to Y$ is a linear bounded operator, then $\|T\|$ denotes the norm of the operator T, i.e.

$$\|T\| = \sup\{\|T(z)\|_Y : z \in X, \ \|z\|_X \leqslant 1\}.$$

Definition 2.1. An operator $\ell \in \mathcal{L}(\mathcal{D})$ is said to be an (a, c)–Volterra operator if, for arbitrary rectangle $[a, t_0] \times [c, x_0] \subseteq \mathcal{D}$ and function $v \in C(\mathcal{D}; \mathbb{R})$ such that

$$v(t, x) = 0 \quad \text{for } (t, x) \in [a, t_0] \times [c, x_0],$$

the relation

$$\ell(v)(t, x) = 0 \quad \text{for a.e. } (t, x) \in [a, t_0] \times [c, x_0]$$

is fulfilled.

Remark 2.2. One can verify (see, for example, [**5**, **18**]) that $v \in C^*(\mathcal{D}; \mathbb{R})$ if and only if the function v satisfies the following conditions:

(i) $v(\cdot\,, x) \in \tilde{C}([a, b], \mathbb{R})$ for every $x \in [c, d]$, $v(a, \cdot) \in \tilde{C}([c, d], \mathbb{R})$;

(ii) $v_t(t, \cdot) \in \tilde{C}([c, d], \mathbb{R})$ for almost all $t \in [a, b]$;

(iii) $v_{tx} \in L(\mathcal{D}; \mathbb{R})$.

Using Fubini's theorem, it is clear that the order of the integration can be changed in the integral representation of the function $v \in C^*(\mathcal{D}; \mathbb{R})$ and thus the conditions stated above can be replaced by the following symmetric ones:

(i′) $v(\cdot\,, c) \in \tilde{C}([a, b], \mathbb{R})$, $v(t, \cdot) \in \tilde{C}([c, d], \mathbb{R})$ for every $t \in [a, b]$;

(ii′) $v_x(\cdot\,, x) \in \tilde{C}([a, b], \mathbb{R})$ for almost all $x \in [c, d]$;

(iii′) $v_{xt} \in L(\mathcal{D}; \mathbb{R})$.

Note also that the set $C^*(\mathcal{D}; \mathbb{R})$ coincides with the class of functions of two variables, which are absolutely continuous on $\mathcal{D}$ in Carathéodory's sense (see, for example, [**2**, **5**, **15**, **20**]).

3. Auxiliary statements

The following proposition plays a crucial role in the proofs of statements given in §§ 4–6.

Proposition 3.1. *Let $\ell \in \mathcal{L}(\mathcal{D})$. Then the operator $T : C(\mathcal{D}; \mathbb{R}) \to C(\mathcal{D}; \mathbb{R})$ defined by*

$$T(v)(t, x) \overset{\text{def}}{=} \int_a^t \int_c^x \ell(v)(s, \eta)\, \mathrm{d}\eta\, \mathrm{d}s \quad \text{for } (t, x) \in \mathcal{D},\ v \in C(\mathcal{D}; \mathbb{R}) \tag{3.1}$$

is completely continuous.

The statement above can easily be proved in the case where the operator ℓ is strongly bounded, i.e. if there exists a function $\eta \in L(\mathcal{D}; \mathbb{R}_+)$ such that

$$|\ell(v)(t, x)| \leqslant \eta(t, x)\|v\|_C \quad \text{for a.e. } (t, x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D}; \mathbb{R}). \tag{3.2}$$

Schaefer proved, however, that there exists an operator $\ell \in \mathcal{L}(\mathcal{D})$ which is not strongly bounded (see [**17**]). To prove Proposition 3.1 without the additional requirement (3.2) we need a number of notions and statements from functional analysis. Note here that the proof is analogous to the proof of Proposition 2.9 in [**8**].

Definition 3.2. Let X be a Banach space and let X^* be its dual space.

We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is weakly convergent if there exists $x \in X$ such that $f(x) = \lim_{n\to+\infty} f(x_n)$ for every $f \in X^*$. The element x is said to be a weak limit of this sequence.

A set $M \subseteq X$ is called weakly relatively compact if every sequence of elements from M contains a subsequence which is weakly convergent in X.

A sequence $\{x_n\}_{n=1}^{+\infty}$ of elements from X is said to be weakly fundamental if the sequence $\{f(x_n)\}_{n=1}^{+\infty}$ is fundamental in $\mathbb{R}$ for every $f \in X^*$.

We say that the space X is weakly complete if every weakly fundamental sequence of elements from X possesses a weak limit in X.

Definition 3.3. Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear bounded operator. The operator T is said to be weakly completely continuous if it maps a unit ball of X into a weakly relatively compact subset of Y.

Definition 3.4. We say that a set $M \subseteq L(\mathcal{D};\mathbb{R})$ has an absolutely continuous integral property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\left| \iint_E p(t,x)\,\mathrm{d}t\,\mathrm{d}x \right| < \varepsilon \quad \text{for every } p \in M$$

is true whenever a measurable set $E \subseteq \mathcal{D}$ is such that $\operatorname{meas} E < \delta$.

The following three lemmas can be found in [**4**].

Lemma 3.5 (Theorem IV.8.6). *The space $L(\mathcal{D};\mathbb{R})$ is weakly complete.*

Lemma 3.6 (Theorem VI.7.6). *A linear bounded operator mapping the space $C(\mathcal{D};\mathbb{R})$ into a weakly complete Banach space is weakly completely continuous.*

Lemma 3.7 (Theorem IV.8.11). *If a set $M \subseteq L(\mathcal{D};\mathbb{R})$ is weakly relatively compact then it has a property of absolutely continuous integral.*

Proof of Proposition 3.1. Let $M \subseteq C(\mathcal{D};\mathbb{R})$ be a bounded set. We will show that the set $T(M) = \{T(v) : v \in M\}$ is relatively compact in $C(\mathcal{D};\mathbb{R})$. According to the Arzelà–Ascoli lemma, it is sufficient to show that the set $T(M)$ is bounded and equicontinuous.

(i) Boundedness. It is clear that

$$|T(v)(t,x)| \leqslant \int_a^t \int_c^x |\ell(v)(s,\eta)|\,\mathrm{d}\eta\,\mathrm{d}s \leqslant \|\ell(v)\|_L \leqslant \|\ell\|\,\|v\|_C$$

for $(t,x) \in \mathcal{D}$ and every $v \in M$. Therefore, the set $T(M)$ is bounded in $C(\mathcal{D};\mathbb{R})$.

(ii) Equicontinuity. Let $\varepsilon > 0$ be arbitrary but fixed. Lemmas 3.5 and 3.6 yield that the operator ℓ is weakly completely continuous, that is, the set $\ell(M) = \{\ell(v) : v \in M\}$ is weakly relatively compact subset of $L(\mathcal{D};\mathbb{R})$. Therefore, Lemma 3.7 guarantees that there exists $\delta > 0$ such that the relation

$$\left| \iint_E \ell(v)(t,x)\,\mathrm{d}t\,\mathrm{d}x \right| < \tfrac{1}{2}\varepsilon \quad \text{for } v \in M \tag{3.3}$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\operatorname{meas} E < \max\{b-a, d-c\}\delta$.

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On the other hand, for $(t_1, x_1), (t_2, x_2) \in \mathcal{D}$ and $v \in M$, we have

$$|T(v)(t_2, x_2) - T(v)(t_1, x_1)| = \left| \int_a^{t_2} \int_c^{x_2} \ell(v)(s, \eta) \, d\eta \, ds - \int_a^{t_1} \int_c^{x_1} \ell(v)(s, \eta) \, d\eta \, ds \right| \leqslant \left| \iint_{E_1} \ell(v)(s, \eta) \, ds \, d\eta \right| + \left| \iint_{E_2} \ell(v)(s, \eta) \, ds \, d\eta \right|,$$

where measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that meas $E_1 \leqslant (d-c)|t_2 - t_1|$ and meas $E_2 \leqslant (b-a)|x_2 - x_1|$. Hence, by virtue of (3.3), we get

$$|T(v)(t_2, x_2) - T(v)(t_1, x_1)| < \varepsilon \quad \text{for } (t_1, x_1), (t_2, x_2) \in \mathcal{D}, \ |t_2 - t_1| + |x_2 - x_1| < \delta \text{ and } v \in M,$$

i.e. the set $T(M)$ is equicontinuous in $C(\mathcal{D}; \mathbb{R})$. □

4. Fredholm property

The main result of this section is the following statement on the Fredholmity of problem (1.1), (1.2).

Theorem 4.1. *For the unique solvability of problem (1.1), (1.2) it is sufficient and necessary that the homogeneous problem*

$$\frac{\partial^2 u(t, x)}{\partial t \partial x} = \ell(u)(t, x), \tag{1.1$_0$}$$

$$u(t, c) = 0 \quad \text{for } t \in [a, b], \qquad u(a, x) = 0 \quad \text{for } x \in [c, d], \tag{1.2$_0$}$$

has only the trivial solution.

Proof. Let u be a solution to problem (1.1), (1.2). It is clear that u is a solution to the equation

$$v = T(v) + f \tag{4.1}$$

in the space $C(\mathcal{D}; \mathbb{R})$, where the operator T is given by (3.1) and

$$f(t, x) \stackrel{\text{def}}{=} -\varphi(a) + \varphi(t) + \psi(x) + \int_a^t \int_c^x q(s, \eta) \, d\eta \, ds \quad \text{for } (t, x) \in \mathcal{D}. \tag{4.2}$$

Conversely, if $v \in C(\mathcal{D}; \mathbb{R})$ is a solution to Equation (4.1) with f given by (4.2), then $v \in C^*(\mathcal{D}; \mathbb{R})$ and v is a solution to problem (1.1), (1.2). Hence, problem (1.1), (1.2) and Equation (4.1) are equivalent in this sense.

Note also that u is a solution to the homogeneous problem (1.1$_0$), (1.2$_0$) if and only if u is a solution to the homogeneous equation

$$v = T(v) \tag{4.3}$$

in the space $C(\mathcal{D}; \mathbb{R})$.

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According to Proposition 3.1, the operator T is completely continuous. It follows from the Riesz–Schauder theory that Equation (4.1) is uniquely solvable for every $f \in C(\mathcal{D};\mathbb{R})$ if and only if the homogeneous Equation (4.3) has only the trivial solution. Therefore, the assertion of the theorem is true. □

Definition 4.2. Let the problem (1.1_0), (1.2_0) have only the trivial solution. An operator $\Omega : L(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ which assigns to every $q \in L(\mathcal{D};\mathbb{R})$ the solution u of the problem (1.1), (1.2_0) is called the Darboux operator of the problem (1.1_0), (1.2_0).

Remark 4.3. It is clear that the Darboux operator Ω is linear.

If the homogeneous problem (1.1_0), (1.2_0) has a non-trivial solution then, by virtue of Theorem 4.1, there exist functions q, φ and ψ such that problem (1.1), (1.2) has either no solution or infinitely many solutions. However, as follows from the proof of Theorem 4.1, a stronger assertion can be shown in this case.

Proposition 4.4. *Let problem (1.1_0), (1.2_0) have a non-trivial solution. Then, for arbitrary $\varphi \in \tilde{C}([a,b],\mathbb{R})$ and $\psi \in \tilde{C}([c,d],\mathbb{R})$ satisfying $\varphi(a) = \psi(c)$, there exists a function $q \in L(\mathcal{D};\mathbb{R})$ such that problem (1.1), (1.2) has no solution.*

Proof. Let u_0 be a non-trivial solution to the problem (1.1_0), (1.2_0) and let $\varphi \in \tilde{C}([a,b],\mathbb{R})$ and $\psi \in \tilde{C}([c,d],\mathbb{R})$ be such that $\varphi(a) = \psi(c)$.

It follows from the proof of Theorem 4.1 that u_0 is also a non-trivial solution to the homogeneous Equation (4.3). Therefore, by the Riesz–Schauder theory, there exists $f \in C(\mathcal{D};\mathbb{R})$ such that Equation (4.1) has no solution.

Then problem (1.1), (1.2) has no solution for $q \equiv \ell(z)$, where

$$z(t,x) = f(t,x) + \varphi(a) - \varphi(t) - \psi(x) \quad \text{for } (t,x) \in \mathcal{D}.$$

Indeed, if the problem indicated has a solution u, then the function $u + z$ is a solution to Equation (4.1), which is a contradiction. □

5. Existence and uniqueness theorems

In this section, we shall establish some efficient condition guaranteeing the unique solvability of the problems (1.1), (1.2) and (1.1′), (1.2). We will prove, in particular, that problem (1.1), (1.2) has a unique solution provided that the operator ℓ is an (a,c)–Volterra one. We first formulate all the results; their proofs are given later.

We introduce the following notation.

Notation 5.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Define operators $\vartheta_k : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$, $k = 0,1,2,\dots$, by setting

$$\vartheta_0(v) \stackrel{\text{def}}{=} v, \quad \vartheta_k(v) \stackrel{\text{def}}{=} T(\vartheta_{k-1}(v)) \quad \text{for } v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}, \tag{5.1}$$

where the operator T is given by (3.1).

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Theorem 5.2. *Let there exist $m \in \mathbb{N}$ and $\alpha \in [0,1[$ such that the inequality*

$$\|\vartheta_m(u)\|_C \leqslant \alpha\|u\|_C \tag{5.2}$$

is satisfied for every solution u of the homogeneous problem (1.1$_0$), (1.2$_0$). Then problem (1.1), (1.2) is uniquely solvable.

Remark 5.3. The assumption that $\alpha \in [0,1[$ in the previous theorem cannot be replaced by the assumption that $\alpha \in [0,1]$ (see Example 7.1).

Corollary 5.4. *Let there exist $j \in \mathbb{N}$ such that*

$$\iint_{\mathcal{D}} p_j(t,x)\,\mathrm{d}t\,\mathrm{d}x < 1, \tag{5.3}$$

where $p_1 \equiv |p|$ and

$$p_{k+1}(t,x) \stackrel{\mathrm{def}}{=} |p(t,x)| \int_a^{\tau(t,x)} \int_c^{\mu(t,x)} p_k(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for a.e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \tag{5.4}$$

Then the problem (1.1′), (1.2) is uniquely solvable.

Remark 5.5. Example 7.1 shows that the strict inequality (5.3) in Corollary 5.4 cannot be replaced by the non-strict one.

Theorem 5.6. *Let ℓ be an (a,c)–Volterra operator. Then problem (1.1), (1.2) has a unique solution.*

Corollary 5.7. *Let*

$$|p(t,x)|(\tau(t,x) - t) \leqslant 0 \quad \text{for a.e. } (t,x) \in \mathcal{D} \tag{5.5}$$

and

$$|p(t,x)|(\mu(t,x) - x) \leqslant 0 \quad \text{for a.e. } (t,x) \in \mathcal{D}. \tag{5.6}$$

Then problem (1.1′), (1.2) has a unique solution.

5.1. Proofs

Proof of Theorem 5.2. According to Theorem 4.1, it is sufficient to show that the homogeneous problem (1.1$_0$), (1.2$_0$) has only the trivial solution.

Let u be a solution to the problem (1.1$_0$), (1.2$_0$). Then it is clear that

$$u(t,x) = \int_a^t \int_c^x \ell(u)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = T(u)(t,x) = \vartheta_1(u)(t,x) \quad \text{for } (t,x) \in \mathcal{D}.$$

Using the above relation, we get

$$u(t,x) = T(\vartheta_1(u))(t,x) = \vartheta_2(u)(t,x) \quad \text{for } (t,x) \in \mathcal{D},$$

and thus $u = \vartheta_k(u)$ for every $k \in \mathbb{N}$. Therefore, (5.2) implies that

$$\|u\|_C = \|\vartheta_m(u)\|_C \leqslant \alpha\|u\|_C,$$

which guarantees $u \equiv 0$. □

Proof of Corollary 5.4. Let $\ell \in \mathcal{L}(\mathcal{D})$ be defined by

$$\ell(v)(t,x) \overset{\text{def}}{=} p(t,x)v(\tau(t,x),\mu(t,x)) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}). \tag{5.7}$$

It is clear that

$$\begin{aligned}|\vartheta_k(v)(t,x)| &\leqslant \int_a^t \int_c^x |p(s,\eta)\vartheta_{k-1}(v)(\tau(s,\eta),\mu(s,\eta))| \,\mathrm{d}\eta \,\mathrm{d}s \\ &\leqslant \|v\|_C \int_a^t \int_c^x p_k(s,\eta) \,\mathrm{d}\eta \,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N},\ v \in C(\mathcal{D};\mathbb{R}).\end{aligned}$$

Therefore, the assumptions of Theorem 5.2 are satisfied with $m = j$ and

$$\alpha = \iint_{\mathcal{D}} p_j(t,x) \,\mathrm{d}t \,\mathrm{d}x.$$

□

To prove Theorem 5.6 we need the following lemma.

Lemma 5.8. *Let $\ell \in \mathcal{L}(\mathcal{D})$ be an (a,c)–Volterra operator. Then*

$$\lim_{k \to +\infty} \|\vartheta_k\| = 0, \tag{5.8}$$

where the operators ϑ_k are defined by (5.1).

Proof. Let $\varepsilon \in \,]0,1[$. According to Proposition 3.1, the operator ϑ_1 is completely continuous. Therefore, by virtue of the Arzelà–Ascoli lemma, there exists $\delta > 0$ such that

$$\left| \int_a^{t_2} \int_c^{x_2} \ell(w)(s,\eta) \,\mathrm{d}\eta \,\mathrm{d}s - \int_a^{t_1} \int_c^{x_1} \ell(w)(s,\eta) \,\mathrm{d}\eta \,\mathrm{d}s \right| \leqslant \varepsilon \|w\|_C$$
$$\text{for } (t_1,x_1),(t_2,x_2) \in \mathcal{D},\ |t_2 - t_1| + |x_2 - x_1| < \delta,\ w \in C(\mathcal{D};\mathbb{R}). \tag{5.9}$$

Put

$$n = \max\left\{ \operatorname{Ent}\left(\frac{2(b-a)}{\delta}\right), \operatorname{Ent}\left(\frac{2(d-c)}{\delta}\right) \right\},$$

$$t_i = a + i\frac{b-a}{n+1}, \quad x_i = c + i\frac{d-c}{n+1} \quad \text{for } i = 0,1,\dots,n+1,$$

$$\mathcal{D}_i = [a,t_i] \times [c,x_i] \quad \text{for } i = 1,2,\dots,n+1.$$

It is clear that, for any $j,r = 0,1,\dots,n$, we have

$$|\tilde{t}_2 - \tilde{t}_1| + |\tilde{x}_2 - \tilde{x}_1| < \delta \quad \text{for } (\tilde{t}_1,\tilde{x}_1),(\tilde{t}_2,\tilde{x}_2) \in [t_j,t_{j+1}] \times [x_r,x_{r+1}]. \tag{5.10}$$

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If $w \in C(\mathcal{D};\mathbb{R})$, then we define

$$\|w\|_i = \|w\|_{C(\mathcal{D}_i;\mathbb{R})} \quad \text{for } i = 1, 2, \dots, n+1.$$

Let $v \in C(\mathcal{D};\mathbb{R})$ be arbitrary but fixed. We shall show that the relation

$$\|\vartheta_k(v)\|_i \leqslant \alpha_i(k)\varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N} \tag{5.11}$$

holds for every $i = 1, 2, \dots, n+1$, where

$$\alpha_i(k) = \alpha_i k^{i-1} \quad \text{for } k \in \mathbb{N},\ i = 1, 2, \dots, n+1, \tag{5.12}$$

$$\alpha_1 = 1, \qquad \alpha_{i+1} = i + 1 + i\alpha_i \quad \text{for } i = 1, 2, \dots, n. \tag{5.13}$$

By virtue of (5.9) and (5.10), it is easy to verify that, for any $w \in C(\mathcal{D};\mathbb{R})$, we have

$$\left| \int_a^{t_j} \int_c^{x_r} \ell(w)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \leqslant \min\{j,r\}\varepsilon \|w\|_C \quad \text{for } j, r = 0, 1, \dots, n+1. \tag{5.14}$$

Firstly, note that

$$\|\vartheta_1(v)\|_i \leqslant i\varepsilon\|v\|_C \quad \text{for } i = 1, 2, \dots, n+1. \tag{5.15}$$

Indeed, according to (5.9), (5.10) and (5.14), it is obvious that

$$\begin{aligned}
\|\vartheta_1(v)\|_i &= \max\left\{ \left| \int_a^t \int_c^x \ell(v)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| : (t,x) \in \mathcal{D}_i \right\} \\
&= \left| \int_a^{t_i^*} \int_c^{x_i^*} \ell(v)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \\
&\leqslant \left| \int_a^{t_i^*} \int_c^{x_i^*} \ell(v)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s - \int_a^{t_{j_0(i)}} \int_c^{x_{r_0(i)}} \ell(v)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \\
&\qquad\qquad + \left| \int_a^{t_{j_0(i)}} \int_c^{x_{r_0(i)}} \ell(v)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \\
&\leqslant \varepsilon\|v\|_C + (i-1)\varepsilon\|v\|_C \\
&= i\varepsilon\|v\|_C \quad \text{for } i = 1, 2, \dots, n+1,
\end{aligned}$$

where $(t_i^*, x_i^*) \in \mathcal{D}_i$ and

$$\left.\begin{aligned}
j_0(i) &= \begin{cases} \dfrac{t_i^* - t_0}{t_1 - t_0} - 1 & \text{if } \dfrac{t_i^* - t_0}{t_1 - t_0} \in \mathbb{N}, \\[2ex] \operatorname{Ent}\left(\dfrac{t_i^* - t_0}{t_1 - t_0}\right) & \text{otherwise,} \end{cases} \\[2ex]
r_0(i) &= \begin{cases} \dfrac{x_i^* - x_0}{x_1 - x_0} - 1 & \text{if } \dfrac{x_i^* - x_0}{x_1 - x_0} \in \mathbb{N}, \\[2ex] \operatorname{Ent}\left(\dfrac{x_i^* - x_0}{x_1 - x_0}\right) & \text{otherwise.} \end{cases}
\end{aligned}\right\} \tag{5.16}$$

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Furthermore, on account of (5.9) and the fact that ℓ is an (a,c)–Volterra operator, we have

$$|\vartheta_{k+1}(v)(t,x)| = \left| \int_a^t \int_c^x \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \leqslant \varepsilon \|\vartheta_k(v)\|_1$$

for $(t,x) \in \mathcal{D}_1$ and $k \in \mathbb{N}$. Hence, by virtue of (5.15), we get

$$\|\vartheta_k(v)\|_1 \leqslant \varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N},$$

i.e. (5.11) is true for $i = 1$.

Now suppose that the relation (5.11) holds for some $i \in \{1, 2, \dots, n\}$. We will show that the relation indicated is true also for $i+1$. With respect to (5.9), (5.10), (5.14) and the fact that ℓ is an (a,c)–Volterra operator, we obtain

$$\begin{aligned}
\|\vartheta_{k+1}(v)\|_{i+1} &= \max\left\{ \left| \int_a^t \int_c^x \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| : (t,x) \in \mathcal{D}_{i+1} \right\} \\
&= \left| \int_a^{t_k^*} \int_c^{x_k^*} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \\
&\leqslant \left| \int_a^{t_k^*} \int_c^{x_k^*} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s - \int_a^{t_{j_0(k)}} \int_c^{x_{r_0(k)}} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \\
&\qquad + \left| \int_a^{t_{j_0(k)}} \int_c^{x_{r_0(k)}} \ell(\vartheta_k(v))(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right| \\
&\leqslant \varepsilon \|\vartheta_k(v)\|_{i+1} + i\varepsilon \|\vartheta_k(v)\|_i \\
&\leqslant \varepsilon \|\vartheta_k(v)\|_{i+1} + i\alpha_i(k)\varepsilon^{k+1} \|v\|_C \quad \text{for } k \in \mathbb{N},
\end{aligned}$$

where $(t_k^*, x_k^*) \in \mathcal{D}_{i+1}$ and $j_0(k)$, $r_0(k)$ are given by (5.16). Whence, we get

$$\|\vartheta_{k+1}(v)\|_{i+1} \leqslant \varepsilon(\varepsilon \|\vartheta_{k-1}(v)\|_{i+1} + i\alpha_i(k-1)\varepsilon^k \|v\|_C) + i\alpha_i(k)\varepsilon^{k+1}\|v\|_C \quad \text{for } k \in \mathbb{N}.$$

To continue this procedure, on account of (5.15), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} \leqslant (i + 1 + i(\alpha_i(1) + \cdots + \alpha_i(k)))\varepsilon^{k+1} \|v\|_C \quad \text{for } k \in \mathbb{N}. \tag{5.17}$$

With respect to (5.12) and (5.13), it is easy to verify that

$$\begin{aligned}
i + 1 + i(\alpha_i(1) + \cdots + \alpha_i(k)) &= i + 1 + i\alpha_i(1^{i-1} + \cdots + k^{i-1}) \\
&\leqslant i + 1 + i\alpha_i k k^{i-1} \\
&= i + 1 + i\alpha_i k^i \\
&\leqslant (i + 1 + i\alpha_i) k^i \\
&= \alpha_{i+1} k^i \\
&\leqslant \alpha_{i+1}(k+1).
\end{aligned}$$

Therefore, (5.15) and (5.17) imply that

$$\|\vartheta_k(v)\|_{i+1} \leqslant \alpha_{i+1}(k)\varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N}.$$

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Thus, by induction, we have proved that the relation (5.11) is true for every $i = 1, 2, \dots, n+1$.

Now it is already clear that, for any $k \in \mathbb{N}$, the estimate

$$\|\vartheta_k(v)\|_C = \|\vartheta_k(v)\|_{n+1} \leqslant \alpha_{n+1} k^n \varepsilon^k \|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R})$$

holds. Therefore,

$$\|\vartheta_k\| \leqslant \alpha_{n+1} k^n \varepsilon^k \quad \text{for } k \in \mathbb{N}.$$

Since we suppose that $\varepsilon \in \,]0,1[$, the last relation yields (5.8). □

Proof of Theorem 5.6. According to Lemma 5.8, there exists $m_0 \in \mathbb{N}$ such that $\|\vartheta_{m_0}\| < 1$. Moreover, it is clear that

$$\|\vartheta_{m_0}(v)\|_C \leqslant \|\vartheta_{m_0}\| \, \|v\|_C \quad \text{for } v \in C(\mathcal{D}; \mathbb{R}),$$

because the operator ϑ_{m_0} is bounded. Therefore, the assumptions of Theorem 5.2 are satisfied with $m = m_0$ and $\alpha = \|\vartheta_{m_0}\|$. □

Proof of Corollary 5.7. The assumptions (5.5) and (5.6) guarantee that the operator ℓ given by (5.7) is an (a,c)–Volterra one. Therefore, the validity of the corollary follows immediately from Theorem 5.6. □

6. Well-posedness

In this part, the well-posedness of the problems (1.1), (1.2) and (1.1′), (1.2) is investigated. We first formulate all the results; their proofs are given later.

For any $k \in \mathbb{N}$, along with problem (1.1), (1.2) we consider the perturbed problem

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = \ell_k(u)(t,x) + q_k(t,x), \tag{1.1$_k$}$$

$$u(t,c) = \varphi_k(t) \quad \text{for } t \in [a,b], \qquad u(a,x) = \psi_k(x) \quad \text{for } x \in [c,d], \tag{1.2$_k$}$$

where $\ell_k \in \mathcal{L}(\mathcal{D})$, $q_k \in L(\mathcal{D}; \mathbb{R})$ and $\varphi_k \in \tilde{C}([a,b]; \mathbb{R})$, $\psi_k \in \tilde{C}([c,d]; \mathbb{R})$ are such that $\varphi_k(a) = \psi_k(c)$.

We introduce the following notation.

Notation 6.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Denote by $M(\ell)$ the set of all functions $y \in C^*(\mathcal{D}; \mathbb{R})$ admitting the representation

$$y(t,x) = -z(a,c) + \int_a^t \int_c^x \ell(z)(s,\eta) \, d\eta \, ds \quad \text{for } (t,x) \in \mathcal{D},$$

where $z \in C(\mathcal{D}; \mathbb{R})$ and $\|z\|_C = 1$.

Theorem 6.2. *Let problem (1.1), (1.2) have a unique solution u,*

$$\lim_{k \to +\infty} \lambda_k = 0, \tag{6.1}$$

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where

$$\lambda_k = \sup\left\{\left|\int_a^t\int_c^x (\ell_k(y)(s,\eta) - \ell(y)(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s\right| : (t,x)\in\mathcal{D},\ y\in M(\ell_k)\right\} \quad \textit{for } k\in\mathbb{N},$$

and let

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\int_a^t\int_c^x (\ell_k(y)(s,\eta) - \ell(y)(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s = 0$$
$$\textit{uniformly on } \mathcal{D} \textit{ for every } y\in C^*(\mathcal{D};\mathbb{R}). \tag{6.2}$$

Moreover, let

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\int_a^t\int_c^x (q_k(s,\eta) - q(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s = 0 \quad \textit{uniformly on } \mathcal{D} \tag{6.3}$$

and

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\|\varphi_k-\varphi\|_C = 0, \qquad \lim_{k\to+\infty}(1+\|\ell_k\|)\|\psi_k-\psi\|_C = 0. \tag{6.4}$$

Then there exists $k_0\in\mathbb{N}$ such that, for every $k>k_0$, the problem (1.1$_k$), (1.2$_k$) has a unique solution u_k and

$$\lim_{k\to+\infty}\|u_k-u\|_C = 0. \tag{6.5}$$

If we suppose that the operators ℓ_k are uniformly bounded in the sense of (6.6), then we obtain the following statement.

Corollary 6.3. *Let problem (1.1), (1.2) have a unique solution u, let there exist a function $\omega\in L(\mathcal{D};\mathbb{R}_+)$ such that*

$$|\ell_k(y)(t,x)| \leqslant \omega(t,x)\|y\|_C \quad \textit{for a.e. } (t,x)\in\mathcal{D} \textit{ and all } y\in C(\mathcal{D};\mathbb{R}),\ k\in\mathbb{N}, \tag{6.6}$$

and let

$$\lim_{k\to+\infty}\int_a^t\int_c^x (\ell_k(y)(s,\eta) - \ell(y)(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s = 0 \quad \textit{uniformly on } \mathcal{D} \tag{6.7}$$

for every $y\in C^*(\mathcal{D};\mathbb{R})$. Moreover, let

$$\lim_{k\to+\infty}\int_a^t\int_c^x (q_k(s,\eta) - q(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s = 0 \quad \textit{uniformly on } \mathcal{D}, \tag{6.8}$$

and

$$\lim_{k\to+\infty}\|\varphi_k-\varphi\|_C = 0, \qquad \lim_{k\to+\infty}\|\psi_k-\psi\|_C = 0. \tag{6.9}$$

Then the conclusion of Theorem 6.2 holds.

Remark 6.4. Assumption (6.6) in the previous corollary is essential and cannot be omitted (see Example 7.2).

Corollary 6.3 yields the following.

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Corollary 6.5. *Let the homogeneous problem* (1.1_0), (1.2_0) *have only the trivial solution. Then the Darboux operator* of the problem* (1.1_0), (1.2_0) *is continuous.*

Now we shall give a statement on the well-posedness of the problem $(1.1')$, (1.2). For any $k \in \mathbb{N}$, along with Equation $(1.1')$ we consider the perturbed equation

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = p_k(t,x) u(\tau_k(t,x), \mu_k(t,x)) + q_k(t,x), \tag{$1.1'_k$}$$

where $p_k, q_k \in L(\mathcal{D};\mathbb{R})$ and $\tau_k : \mathcal{D} \to [a,b]$, $\mu_k : \mathcal{D} \to [c,d]$ are measurable functions.

Theorem 6.6. *Let the problem* $(1.1')$, (1.2) *have a unique solution* u, *let there exist a function* $\omega \in L(\mathcal{D};\mathbb{R})$ *such that*

$$|p_k(t,x)| \leqslant \omega(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \tag{6.10}$$

and let

$$\lim_{k\to+\infty} \int_a^t \int_c^x (p_k(s,\eta) - p(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s = 0 \quad \text{uniformly on } \mathcal{D}. \tag{6.11}$$

Moreover, let conditions (6.8) and (6.9) be satisfied and let

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\{|\tau_k(t,x) - \tau(t,x)| : (t,x) \in \mathcal{D}\} = 0, \tag{6.12}$$

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\{|\mu_k(t,x) - \mu(t,x)| : (t,x) \in \mathcal{D}\} = 0. \tag{6.13}$$

Then there exists $k_0 \in \mathbb{N}$ *such that, for every* $k > k_0$, *problem* $(1.1'_k)$, (1.2_k) *has a unique solution* u_k *and (6.5) holds.*

Remark 6.7. Assumption (6.10) in the previous theorem is essential and cannot be omitted (see Example 7.2).

6.1. Proofs

To prove Theorem 6.2 we need the following lemma.

Lemma 6.8. *Let problem* (1.1_0), (1.2_0) *have only the trivial solution and let condition (6.1) be satisfied. Then there exist* $k_0 \in \mathbb{N}$ *and* $r_0 > 0$ *such that*

$$\|z\|_C \leqslant r_0 \rho_k(z) \quad \text{for } k > k_0,\ z \in C^*(\mathcal{D};\mathbb{R}), \tag{6.14}$$

where

$$\rho_k(v) \overset{\text{def}}{=} |v(a,c)| + (1 + \|\ell_k\|)\|\varGamma_k(v)\|_C \quad \text{for } v \in C^*(\mathcal{D};\mathbb{R}) \tag{6.15}$$

and

$$\varGamma_k(v)(t,x) \overset{\text{def}}{=} v(t,c) + v(a,x) + \int_a^t \int_c^x \left(\frac{\partial^2 v(s,\eta)}{\partial s \partial \eta} - \ell_k(v)(s,\eta) \right) \mathrm{d}\eta\,\mathrm{d}s \tag{6.16}$$

for $(t,x) \in \mathcal{D}$ *and* $v \in C^*(\mathcal{D};\mathbb{R})$.

* The notion of Darboux operator is given in Definition 4.2.

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Proof. Let $T, T_k : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ be operators defined by (3.1) and

$$T_k(v)(t,x) \overset{\text{def}}{=} \int_a^t \int_c^x \ell_k(v)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},\ v \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}. \tag{6.17}$$

Obviously,

$$\|T_k(y)\|_C \leqslant \|\ell_k\|\,\|y\|_C \quad \text{for } y \in C(\mathcal{D};\mathbb{R}),\ k \in \mathbb{N}.$$

Therefore, the operators T_k, $k \in \mathbb{N}$, are linear and bounded and the relation

$$\|T_k\| \leqslant \|\ell_k\| \quad \text{for } k \in \mathbb{N} \tag{6.18}$$

holds. Moreover, the condition (6.1) can be rewritten in the form

$$\sup\{\|T_k(y) - T(y)\|_C : y \in M(\ell_k)\} \to 0 \quad \text{as } k \to +\infty. \tag{6.19}$$

Assume that, on the contrary, the assertion of the lemma is not true. Then there exist an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers and a sequence $\{z_m\}_{m=1}^{+\infty}$ of functions from $C^*(\mathcal{D};\mathbb{R})$ such that

$$\|z_m\|_C > m\rho_{k_m}(z_m) \quad \text{for } m \in \mathbb{N}. \tag{6.20}$$

For any $m \in \mathbb{N}$ and $(t,x) \in \mathcal{D}$, we set

$$y_m(t,x) = \frac{z_m(t,x)}{\|z_m\|_C}, \tag{6.21}$$

$$v_m(t,x) = y_m(t,c) + y_m(a,x) + \int_a^t \int_c^x \left(\frac{\partial^2 y_m(s,\eta)}{\partial s \partial \eta} - \ell_{k_m}(y_m)(s,\eta) \right) \mathrm{d}\eta\,\mathrm{d}s, \tag{6.22}$$

$$y_{0m}(t,x) = y_m(t,x) - v_m(t,x), \tag{6.23}$$

$$w_m(t,x) = T_{k_m}(y_{0m})(t,x) - T(y_{0m})(t,x) + T_{k_m}(v_m)(t,x). \tag{6.24}$$

Obviously,

$$\|y_m\|_C = 1 \quad \text{for } m \in \mathbb{N}, \tag{6.25}$$

$$y_{0m}(t,x) = -y_m(a,c) + T_{k_m}(y_m)(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}, \tag{6.26}$$

and

$$y_{0m}(t,x) = -y_m(a,c) + T(y_{0m})(t,x) + w_m(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}. \tag{6.27}$$

On the other hand, from (6.15), (6.16), (6.18), (6.21) and (6.22), by virtue of (6.20), we get

$$\|v_m\|_C \leqslant \frac{\rho_{k_m}(z_m)}{\|z_m\|_C(1 + \|\ell_{k_m}\|)} < \frac{1}{m(1 + \|\ell_{k_m}\|)} \quad \text{for } m \in \mathbb{N}, \tag{6.28}$$

$$\|T_{k_m}(v_m)\|_C \leqslant \|T_{k_m}\|\,\|v_m\|_C < \frac{\|\ell_{k_m}\|}{m(1 + \|\ell_{k_m}\|)} < \frac{1}{m} \quad \text{for } m \in \mathbb{N}, \tag{6.29}$$

and

$$|y_m(a,c)| \leqslant \frac{\rho_{k_m}(z_m)}{\|z_m\|_C} < \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \tag{6.30}$$

The relations (6.25) and (6.26) guarantee that $y_{0m} \in M(\ell_{k_m})$ for $m \in \mathbb{N}$ and, therefore, in view of (6.19), we obtain

$$\lim_{m\to+\infty} \|T_{k_m}(y_{0m}) - T(y_{0m})\|_C = 0. \tag{6.31}$$

According to (6.29) and (6.31), it follows from (6.24) that

$$\lim_{m\to+\infty} \|w_m\|_C = 0, \tag{6.32}$$

and, by virtue of (6.25) and (6.28), the equality (6.23) implies

$$\|y_{0m}\|_C \leqslant \|y_m\|_C + \|v_m\|_C < 2 \quad \text{for } m \in \mathbb{N}.$$

Since the sequence $\{\|y_{0m}\|_C\}_{m=1}^{+\infty}$ is bounded and the operator T is completely continuous (see Proposition 3.1), there exists a subsequence of $\{T(y_{0m})\}_{m=1}^{+\infty}$ which is convergent. Without loss of generality we can assume that the sequence $\{T(y_{0m})\}_{m=1}^{+\infty}$ is convergent, i.e. there exists $y_0 \in C(\mathcal{D};\mathbb{R})$ such that

$$\lim_{m\to+\infty} \|T(y_{0m}) - y_0\|_C = 0.$$

Then it is clear that

$$\lim_{m\to+\infty} \|y_{0m} - y_0\|_C = 0 \tag{6.33}$$

because the functions y_{0m} admit the representation (6.27) and (6.30) and (6.32) are satisfied.

However, the estimate (6.28) holds for v_m and, thus, the equality (6.23) yields

$$\lim_{m\to+\infty} \|y_m - y_0\|_C = 0,$$

which, together with (6.25), guarantees that

$$\|y_0\|_C = 1.$$

Since the operator T is continuous and the conditions (6.30), (6.32) and (6.33) are fulfilled, the representation (6.27) of y_{0m} results in

$$y_0(t,x) = T(y_0)(t,x) \quad \text{for } (t,x) \in \mathcal{D}.$$

Consequently, $y_0 \in C^*(\mathcal{D};\mathbb{R})$ and y_0 is a non-trivial solution to the problem (1.1_0), (1.2_0). However, this is a contradiction because, according to our assumption, the problem indicated has no non-trivial solution. □

Proof of Theorem 6.2. Let $r_0 > 0$ and $k_0 \in \mathbb{N}$ be numbers appearing in Lemma 6.8. If, for some $k \in \mathbb{N}$, u_0 is a solution to the equation

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = \ell_k(u)(t,x) \tag{6.33_k}$$

satisfying (1.2_0), then $\rho_k(u_0) = 0$, where ρ_k is given by (6.15) and (6.16). Therefore, Lemma 6.8 guarantees that, for every $k > k_0$, the homogeneous problem (6.33_k), (1.2_0) has only the trivial solution. Hence, for every $k > k_0$, the problem (1.1_k), (1.2_k) has a unique solution u_k. We shall show that (6.5) is satisfied, where u is a solution to problem (1.1), (1.2).

For any $k > k_0$, we set

$$v_k(t,x) = u_k(t,x) - u(t,x) \quad \text{for } (t,x) \in \mathcal{D}.$$

Then it is clear that $v_k \in C^*(\mathcal{D};\mathbb{R})$ for $k > k_0$,

$$\frac{\partial^2 v_k(t,x)}{\partial t \partial x} = \ell_k(v_k)(t,x) + \tilde{q}_k(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D},\ k > k_0, \tag{6.34}$$

and

$$\left.\begin{aligned} v_k(t,c) &= \tilde{\varphi}_k(t) && \text{for } t \in [a,b],\ k > k_0, \\ v_k(a,x) &= \tilde{\psi}_k(x) && \text{for } x \in [c,d],\ k > k_0, \end{aligned}\right\} \tag{6.35}$$

where

$$\begin{aligned} \tilde{q}_k(t,x) &= \ell_k(u)(t,x) - \ell(u)(t,x) + q_k(t,x) - q(t,x) && \text{for a.e. } (t,x) \in \mathcal{D},\ k > k_0, \\ \tilde{\varphi}_k(t) &= \varphi_k(t) - \varphi(t) && \text{for } t \in [a,b], \qquad k > k_0, \\ \tilde{\psi}_k(x) &= \psi_k(x) - \psi(x) && \text{for } x \in [c,d], \qquad k > k_0. \end{aligned}$$

For any $k > k_0$, we set

$$\delta_k = (1 + \|\ell_k\|) \max\left\{ \left| \tilde{\varphi}_k(t) + \tilde{\psi}_k(x) + \int_a^t \int_c^x \tilde{q}_k(s,\eta)\,d\eta\,ds \right| : (t,x) \in \mathcal{D} \right\}.$$

Assumptions (6.2), (6.3) and (6.4) yield

$$\lim_{k \to +\infty} \delta_k = 0 \quad \text{and} \quad \lim_{k \to +\infty} |v_k(a,c)| = 0. \tag{6.36}$$

On the other hand, using Lemma 6.8, we get

$$\|v_k\|_C \leqslant r_0 \rho_k(v_k) = r_0(|v_k(a,c)| + \delta_k) \quad \text{for } k > k_0. \tag{6.37}$$

Therefore, (6.36) and (6.37) result in

$$\lim_{k \to +\infty} \|v_k\|_C = 0,$$

i.e. the relation (6.5) is satisfied. □

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Proof of Corollary 6.3. We will show that the assumptions of Theorem 6.2 are satisfied. Indeed, the relation (6.6) yields

$$\|\ell_k\| \leqslant \|\omega\|_L \quad \text{for } k \in \mathbb{N}.$$

Therefore, it is clear that, by virtue of (6.7)–(6.9), the assumptions (6.2)–(6.4) of Theorem 6.2 are fulfilled. It remains to show that the condition (6.1) is true.

Assume that, on the contrary, the condition (6.1) does not hold. Then there exist $\varepsilon_0 > 0$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers and a sequence $\{y_m\}_{m=1}^{+\infty}$ such that

$$y_m \in M(\ell_{k_m}) \quad \text{for } m \in \mathbb{N} \tag{6.38}$$

and

$$\max\left\{\left|\int_a^t \int_c^x (\ell_{k_m}(y_m)(s,\eta) - \ell(y_m)(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s\right| : (t,x) \in \mathcal{D}\right\} \geqslant \varepsilon_0 \quad \text{for } m \in \mathbb{N}. \tag{6.39}$$

From (6.38) and Notation 6.1 we get

$$y_m(t,x) = -z_m(a,c) + \int_a^t \int_c^x \ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N},$$

where $z_m \in C(\mathcal{D};\mathbb{R})$ and $\|z_m\|_C = 1$ for $m \in \mathbb{N}$. Since we suppose that the operators ℓ_k are uniformly bounded in the sense of condition (6.6), we obtain

$$\|y_m\|_C \leqslant 1 + \|\omega\|_L \quad \text{for } m \in \mathbb{N}.$$

Furthermore, for any $(t_1,x_1),(t_2,x_2) \in \mathcal{D}$ and $m \in \mathbb{N}$, we get

$$\begin{aligned}|y_m(t_2,x_2) - y_m(t_1,x_1)| &= \left|\int_a^{t_2}\int_c^{x_2} \ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s - \int_a^{t_1}\int_c^{x_1} \ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right| \\ &\leqslant \iint_{E_1} \omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta + \iint_{E_2} \omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta,\end{aligned}$$

where the measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\operatorname{meas} E_1 \leqslant (d-c)|t_2 - t_1|$ and $\operatorname{meas} E_2 \leqslant (b-a)|x_2 - x_1|$.

Consequently, the sequence $\{y_m\}_{m=1}^{+\infty}$ is bounded and equicontinuous in $C(\mathcal{D};\mathbb{R})$. Thus, according to the Arzelà–Ascoli lemma, we can assume without loss of generality that the sequence indicated is convergent. Therefore, there exists $p_0 \in \mathbb{N}$ such that

$$\|y_m - y_{p_0}\|_C < \frac{\varepsilon_0}{2(\|\omega\|_L + \|\ell\| + 1)} \quad \text{for } m \geqslant p_0. \tag{6.40}$$

Since $y_{p_0} \in C^*(\mathcal{D};\mathbb{R})$ and the relation (6.7) holds, there exists $p_1 \in \mathbb{N}$ such that

$$\max\left\{\left|\int_a^t \int_c^x (\ell_k(y_{p_0})(s,\eta) - \ell(y_{p_0})(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s\right| : (t,x) \in \mathcal{D}\right\} < \tfrac{1}{2}\varepsilon_0 \quad \text{for } k \geqslant p_1. \tag{6.41}$$

Now choose a number $M \in \mathbb{N}$ satisfying $M \geqslant p_0$ and $k_M \geqslant p_1$. Then

$$\begin{aligned}\max\biggl\{\biggl|\int_a^t\int_c^x(\ell_{k_M}(y_M)(s,\eta)-\ell(y_M)(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s\biggr| : (t,x)\in\mathcal{D}\biggr\}\\ &\leqslant (\|\omega\|_L+\|\ell\|)\|y_M-y_{p_0}\|_C\\ &\quad+\max\biggl\{\biggl|\int_a^t\int_c^x(\ell_{k_M}(y_{p_0})(s,\eta)-\ell(y_{p_0})(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s\biggr| : (t,x)\in\mathcal{D}\biggr\}\\ &< \frac{\varepsilon_0}{2}\frac{\|\omega\|_L+\|\ell\|}{\|\omega\|_L+\|\ell\|+1}+\frac{\varepsilon_0}{2}\\ &< \varepsilon_0,\end{aligned}$$

which contradicts (6.39). □

To prove Theorem 6.6 we need the following statement, which is a two-dimensional analogy of the well-known Krasnoselskii–Krein lemma.

Lemma 6.9. *Let $p, p_k \in L(\mathcal{D};\mathbb{R})$ and let $\alpha, \alpha_k : \mathcal{D} \to \mathbb{R}$ be measurable and essentially bounded functions for $k \in \mathbb{N}$. Assume that the relations (6.10) and (6.11) are satisfied and that*

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\{|\alpha_k(t,x)-\alpha(t,x)| : (t,x)\in\mathcal{D}\} = 0. \tag{6.42}$$

Then

$$\lim_{k\to+\infty}\int_a^t\int_c^x(p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s = 0 \quad \textit{uniformly on } \mathcal{D}. \tag{6.43}$$

Proof. Without loss of generality we can assume that

$$|p(t,x)| \leqslant \omega(t,x) \quad \text{for a.e. } (t,x)\in\mathcal{D}. \tag{6.44}$$

Let $\varepsilon > 0$ be arbitrary but fixed. According to (6.42), there exists $k_0 \in \mathbb{N}$ such that

$$\iint_{\mathcal{D}} \omega(t,x)|\alpha_k(t,x)-\alpha(t,x)|\,\mathrm{d}t\,\mathrm{d}x < \tfrac{1}{4}\varepsilon \quad \text{for } k \geqslant k_0. \tag{6.45}$$

Since the function α is measurable and essentially bounded, there exists a function $w \in C(\mathcal{D};\mathbb{R})$, which has continuous derivatives up to second order and such that

$$\iint_{\mathcal{D}} \omega(t,x)|\alpha(t,x)-w(t,x)|\,\mathrm{d}t\,\mathrm{d}x < \tfrac{1}{4}\varepsilon. \tag{6.46}$$

For any $k \in \mathbb{N}$, we set

$$f_k(t,x) = \int_a^t\int_c^x(p_k(s,\eta)-p(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D}.$$

Clearly, (6.11) can be rewritten in the form

$$\lim_{k\to+\infty}\|f_k\|_C = 0. \tag{6.47}$$

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It can be verified by direct computation that

$$\int_a^t\int_c^x (p_k(s,\eta)-p(s,\eta))w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$
$$= f_k(t,x)w(t,x) - \int_a^t f_k(s,x)\frac{\partial w(s,x)}{\partial s}\,\mathrm{d}s - \int_c^x f_k(t,\eta)\frac{\partial w(t,\eta)}{\partial \eta}\,\mathrm{d}\eta$$
$$+ \int_a^t\int_c^x f_k(s,\eta)\frac{\partial^2 w(s,\eta)}{\partial s\partial \eta}\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

Consequently, using (6.47), we get

$$\lim_{k\to+\infty}\int_a^t\int_c^x (p_k(s,\eta)-p(s,\eta))w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = 0 \quad \text{uniformly on } \mathcal{D}.$$

Hence, there exists $k_1 \geqslant k_0$ such that

$$\left|\int_a^t\int_c^x (p_k(s,\eta)-p(s,\eta))w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right| < \tfrac{1}{4}\varepsilon \quad \text{for } (t,x)\in\mathcal{D},\ k\geqslant k_1. \tag{6.48}$$

On the other hand, it is clear that, for any $(t,x)\in\mathcal{D}$ and $k\in\mathbb{N}$,

$$\int_a^t\int_c^x (p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s$$
$$= \int_a^t\int_c^x p_k(s,\eta)(\alpha_k(s,\eta)-\alpha(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s$$
$$+ \int_a^t\int_c^x (p_k(s,\eta)-p(s,\eta))w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$
$$+ \int_a^t\int_c^x (p_k(s,\eta)-p(s,\eta))(\alpha(s,\eta)-w(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s.$$

Therefore, in view of (6.10), (6.44)–(6.46) and (6.48), we get

$$\left|\int_a^t\int_c^x (p_k(s,\eta)\alpha_k(s,\eta)-p(s,\eta)\alpha(s,\eta))\,\mathrm{d}\eta\,\mathrm{d}s\right|$$
$$\leqslant \iint_{\mathcal{D}} \omega(s,\eta)|\alpha_k(s,\eta)-\alpha(s,\eta)|\,\mathrm{d}\eta\,\mathrm{d}s + \left|\int_a^t\int_c^x (p_k(s,\eta)-p(s,\eta))w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right|$$
$$+ 2\iint_{\mathcal{D}} \omega(s,\eta)|\alpha(s,\eta)-w(s,\eta)|\,\mathrm{d}\eta\,\mathrm{d}s$$
$$< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+2\frac{\varepsilon}{4}$$
$$= \varepsilon \quad \text{for } (t,x)\in\mathcal{D},\ k\geqslant k_1,$$

that is, the relation (6.43) holds. □

Proof of Theorem 6.6. Let $\ell\in\mathcal{L}(\mathcal{D})$ be defined by (5.7). For any $k\in\mathbb{N}$, we set

$$\ell_k(v)(t,x) \overset{\text{def}}{=} p_k(t,x)v(\tau_k(t,x),\mu_k(t,x)) \quad \text{for a.e. } (t,x)\in\mathcal{D} \text{ and all } v\in C(\mathcal{D};\mathbb{R}). \tag{6.49}$$

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We will show that (6.7) is satisfied for every $y \in C^*(\mathcal{D};\mathbb{R})$. Indeed, let $y \in C^*(\mathcal{D};\mathbb{R})$ be arbitrary but fixed. For any $k \in \mathbb{N}$, we set

$$\alpha_k(t,x) = y(\tau_k(t,x),\mu_k(t,x)), \quad \alpha(t,x) = y(\tau(t,x),\mu(t,x)) \quad \text{for } (t,x) \in \mathcal{D}.$$

Then it is clear that (6.12) and (6.13) guarantee the condition (6.42). Therefore, it follows from Lemma 6.9 that the condition (6.43) holds, i.e. the condition (6.7) is true.

Consequently, the assumptions of Corollary 6.3 are satisfied. □

7. Counter-examples

Example 7.1. Let $p \in L(\mathcal{D};\mathbb{R}_+)$ be such that

$$\iint_{\mathcal{D}} p(t,x)\,\mathrm{d}t\,\mathrm{d}x = 1$$

and let $\ell \in \mathcal{L}(\mathcal{D})$ be defined by

$$\ell(v)(t,x) \overset{\text{def}}{=} p(t,x)v(b,d) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}).$$

Then the condition (5.2) with $\alpha = 1$ is satisfied for every $m \in \mathbb{N}$ and $v \in C(\mathcal{D};\mathbb{R})$. Moreover,

$$\iint_{\mathcal{D}} p_j(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = 1 \quad \text{for every } j \in \mathbb{N},$$

where p_j is given by (5.4).

On the other hand, the problem (1.1_0), (1.2_0) has a non-trivial solution

$$u(t,x) = \int_a^t \int_c^x p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}.$$

This example shows that the assumption $\alpha \in [0,1[$ in Theorem 5.2 cannot be replaced by the assumption $\alpha \in [0,1]$, and the strict inequality (5.3) in Corollary 5.4 cannot be replaced by the non-strict one.

Example 7.2. Let

$$g_k(t) = k\cos(k^2 t), \quad h_k(t) = k\sin(k^2 t) \quad \text{for } t \geqslant 0,\ k \in \mathbb{N}, \tag{7.1}$$

and

$$y_k(t) = -k\int_0^t \exp\left(\frac{\sin(k^2 t)}{k} - \frac{\sin(k^2 s)}{k}\right)\sin(k^2 s)\,\mathrm{d}s \quad \text{for } t \geqslant 0,\ k \in \mathbb{N}. \tag{7.2}$$

It is not difficult to verify that, for every $k \in \mathbb{N}$,

$$y_k'(t) = g_k(t)y_k(t) + h_k(t) \quad \text{for } t \geqslant 0 \tag{7.3}$$

and

$$|y_k(t)| \leqslant 1 + \mathrm{e} + t\mathrm{e}^2 \quad \text{for } t \geqslant 0, \tag{7.4}$$

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because

$$y_k(t) = \frac{1}{k}\cos(k^2t) - \frac{1}{k}\exp\left(\frac{\sin(k^2t)}{k}\right) + \frac{1}{2}\int_0^t \exp\left(\frac{\sin(k^2t)}{k} - \frac{\sin(k^2s)}{k}\right)\mathrm{d}s$$
$$+ \frac{1}{2}\int_0^t \exp\left(\frac{\sin(k^2t)}{k} - \frac{\sin(k^2s)}{k}\right)\cos(2k^2s)\,\mathrm{d}s \quad \text{for } t \geqslant 0. \tag{7.5}$$

Moreover,

$$\lim_{k\to+\infty} y_k(t) = \tfrac{1}{2}t \quad \text{for } t \geqslant 0. \tag{7.6}$$

Now, let $p \equiv 0$, $q \equiv 0$, $\varphi \equiv 0$, $\psi \equiv 0$ and

$$\tau(t,x) = t, \quad \mu(t,x) = x \quad \text{for } (t,x) \in \mathcal{D}.$$

For any $k \in \mathbb{N}$, we set $\varphi_k \equiv 0$, $\psi_k \equiv 0$,

$$p_k(t,x) = g_k(t-a)g_k(x-c) \quad \text{for } (t,x) \in \mathcal{D},$$
$$q_k(t,x) = h_k(t-a)y_k'(x-c) + y_k'(t-a)h_k(x-c) - h_k(t-a)h_k(x-c) \quad \text{for } (t,x) \in \mathcal{D},$$

and

$$\tau_k(t,x) = t, \quad \mu_k(t,x) = x \quad \text{for } (t,x) \in \mathcal{D}.$$

According to (7.1), (7.3) and (7.4), it is clear that the assumptions of Theorem 6.6 are satisfied except for (6.10). Let $\ell, \ell_k \in \mathcal{L}(\mathcal{D})$ be operators defined by (5.7) and (6.49), respectively. Then it is not difficult to verify that the assumptions of Corollary 6.3 are satisfied except for (6.6).

On the other hand,

$$u(t,x) = 0 \quad \text{for } (t,x) \in \mathcal{D}$$

and

$$u_k(t,x) = y_k(t-a)y_k(x-c) \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N},$$

are solutions to problems (1.1′), (1.2) and ($1.1_k'$), (1.2_k), respectively, as well as problems (1.1), (1.2) and (1.1_k) and (1.2_k), respectively. However, in view of (7.6), we get

$$\lim_{k\to+\infty}(u_k(t,x) - u(t,x)) = \lim_{k\to+\infty} y_k(t-a)y_k(x-c)$$
$$= \frac{(t-a)(x-c)}{4} \quad \text{for } (t,x) \in \mathcal{D},$$

that is, the relation (6.5) is not true.

This example shows that the assumptions (6.6) in Corollary 6.3 and (6.10) in Theorem 6.6 are essential and they cannot be omitted.

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