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ON MAXIMAL ESSENTIAL EXTENSIONS OF RINGS

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Abstract

The main purpose of this paper is to give a new, elementary proof of Flanigan's theorem, which says that a given ring A has a maximal essential extension $\mathrm{ME}(A)$ if and only if the two-sided annihilator of A is zero. Moreover, we discuss the problem of description of $\mathrm{ME}(A)$ for a given right ideal A of a ring with an identity.

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1. Introduction

All rings in this paper are associative but we do not assume that each ring has an identity element. To denote that I is an ideal (respectively a left ideal, a right ideal) of a ring R, we write $I \lhd R$ (respectively $I <_l R$, $I <_r R$). We say that a subring A of a ring R is *essential* in R or that R is an *essential extension* of A, if $A \cap I \neq 0$ for every nonzero ideal I of R.

The *idealizer* $\mathrm{Id}_R(A)$ of a subring A of a ring R is the largest subring of R in which A is an ideal, that is, $\mathrm{Id}_R(A) = \{r \in R : rA \subseteq A \text{ and } Ar \subseteq A\}$. For a nonempty subset X of a ring R the *left annihilator* of X is $l_R(X) = \{r \in R : rX = 0\}$ and the *right annihilator* of X is $r_R(X) = \{r \in R : Xr = 0\}$. The *two-sided annihilator* of X is defined to be the subring $a_R(X) = l_R(X) \cap r_R(X)$. To simplify the notation we write $a(R)$ instead of $a_R(R)$.

In [2] Beidar introduced the notion of a maximal essential extension of a ring.

DEFINITION 1.1 (Beidar). A ring R is said to be a **maximal essential extension** of a ring A ($R = \mathrm{ME}(A)$), if A is an essential ideal of R and, for any ring S which contains A as an ideal, there exists a ring homomorphism $h : S \to R$ such that $h(a) = a$ for all $a \in A$.

There are many important applications of maximal essential extension of rings (see [5, 6]), especially in the theory of radicals. We refer the reader to [2] for a thorough discussion of the various applications of this idea in solving significant problems in the theory of rings. In [2] Beidar asked when a given ring A has $\mathrm{ME}(A)$. It turned out that an answer this question was already given in [3] by Flanigan. However, the author used advanced methods and techniques of category theory. The Flanigan theorem can be presented in the following form.

THEOREM 1.2 (Flanigan). *A ring A has* $\mathrm{ME}(A)$ *if and only if* $a(A) = \{0\}$.

The main purpose of this paper is to give a new elementary proof of Flanigan's theorem (see Section 3). Moreover, we present an example which shows that in [2, Theorem 6], given without a proof, there are few assumptions missing. For the proof the author refers to [1], but that paper only gives a proof of a special case of the theorem under discussion. We give necessary assumptions for [2, Theorem 6] and show that they cannot be weakened.

2. Preliminaries

For any element a of a ring R we denote the ideal of R generated by a by $R^1 a R^1$. We denote the additive group of A by A^+ and we denote the ring of endomorphisms of the right (respectively left) A-module A by $\mathrm{End}(A_A)$ (respectively $\mathrm{End}({}_A A)$). For ${}_A A$ we write endomorphisms on the right of the arguments and consequently we will use the right-hand rule for composition of mappings. The ring $\mathrm{End}(A_A) \oplus \mathrm{End}({}_A A)$ will be denoted by $E(A)$. It is easy to check that

$$\Omega(A) = \{(\lambda, \rho) \in E(A) : a \cdot \lambda(b) = (a)\rho \cdot b \ \forall_{a,b \in A}\}$$

is a subring of $E(A)$. If $A \lhd S$ then for $s \in S$ we define $l_s : A \to A$ by $l_s(a) = s \cdot a$ and $r_s : A \to A$ by $(a)r_s = a \cdot s$. A trivial verification shows that $l_s \in \mathrm{End}(A_A)$, $r_s \in \mathrm{End}({}_A A)$ and $(l_s, r_s) \in \Omega(A)$ for any $s \in S$. Moreover, the map $f : S \to \Omega(A)$ given by $f(s) = (l_s, r_s)$ for $s \in S$ is a ring homomorphism such that $\mathrm{Ker}\, f = a_S(A)$. In particular, for $S = A$ we obtain that $\overline{A} = \{(l_a, r_a) : a \in A\}$ is a subring of $\Omega(A)$. It is seen at once that, for all $a \in A$ and $(\lambda, \rho) \in \Omega(A)$, we have $(\lambda, \rho) \cdot (l_a, r_a) = (l_{\lambda(a)}, r_{\lambda(a)})$ and $(l_a, r_a) \cdot (\lambda, \rho) = (l_{(a)\rho}, r_{(a)\rho})$. It follows that $\overline{A} \lhd \Omega(A)$. If, in addition, $a(A) = \{0\}$, then $\overline{A} \cong A$.

PROPOSITION 2.1. *If A is an ideal of a ring R and $a(A) = \{0\}$ then A is an essential ideal in R if and only if $a_R(A) = \{0\}$.*

PROOF. If $a_R(A) = \{0\}$ and J is a nonzero ideal of a ring R then $JA \neq \{0\}$ or $AJ \neq \{0\}$ and $JA \cup AJ \subseteq A \cap J$. Thus $A \cap J \neq \{0\}$. Hence A is an essential ideal of R. Conversely, assume that A is an essential ideal of R. Since $a_R(A) \lhd R$ and $a_R(A) \cap A = a(A)$, it follows that $a_R(A) = \{0\}$. □

PROPOSITION 2.2. *Let A be a ring such that $a(A) = \{0\}$. Assume that A is an essential ideal of a ring R and A is a subring of a ring S such that $a_S(A) = \{0\}$. Then:*

(i) *every ring homomorphism $f : R \to S$ such that $f(a) = a$ for every $a \in A$ is an embedding;*

(ii) *there exists at most one homomorphism $f : R \to S$ such that $f(a) = a$ for all $a \in A$.*

PROOF. (i) Assume that the homomorphism $f : R \to S$ is not an embedding. Then $\operatorname{Ker} f \neq \{0\}$, so $\operatorname{Ker} f \cap A \neq \{0\}$. Hence there exists $0 \neq a \in A$ such that $f(a) = 0$. But $a = f(a)$, which is a contradiction.

(ii) Suppose that f and g are homomorphisms of R into S such that $f|_A = \mathrm{id}_A$ and $g|_A = \mathrm{id}_A$. Let $x \in R$ and $a \in A$. Then

$$\begin{aligned}(f(x) - g(x))a &= f(x)a - g(x)a = f(x)f(a) - g(x)g(a) \\ &= f(xa) - g(xa) = xa - xa = 0,\end{aligned}$$

since $xa \in A$. So $(f(x) - g(x))A = \{0\}$. Similarly, one can prove that $A(f(x) - g(x)) = \{0\}$. From this we conclude that $f(x) - g(x) \in a_S(A)$. But $a_S(A) = \{0\}$, so $f(x) = g(x)$ for every $x \in R$. This shows that $f = g$. □

PROPOSITION 2.3. *Let A be a ring such that $a(A) = \{0\}$. Then $a_{E(A)}(\overline{A}) = \{0\}$ and $\Omega(A) = \mathrm{id}_{E(A)}(\overline{A})$. Moreover, if A is an essential ideal of a ring S then the function $s \mapsto (l_s, r_s)$ for $s \in S$ is the unique ring homomorphism of S into $\Omega(A)$ such that $a \mapsto (l_a, r_a)$ for $a \in A$.*

PROOF. Let $(f, g) \in a_{E(A)}(\overline{A})$. Then for any $a \in A$ we have $(f, g) \cdot (l_a, l_r) = (0, 0)$ and $(l_a, r_a) \cdot (f, g) = (0, 0)$, so that $fl_a = 0$, $l_a f = 0$, $gr_a = 0$ and $r_a g = 0$. Hence, for every $b \in A$ we have

$$\begin{gathered}0 = (fl_a)(b) = f(a \cdot b) = f(a) \cdot b, \quad 0 = (l_b f)(a) = b \cdot f(a), \\ 0 = (b)(gr_a) = (b)g \cdot a \quad \text{and} \quad 0 = (a)(r_b g) = (a \cdot b)g = a \cdot (b)g.\end{gathered}$$

This means that $f(a), (b)g \in a(A) = \{0\}$, hence $f = 0$, $g = 0$ and $a_{E(A)}(\overline{A}) = \{0\}$.

Recall that $\overline{A} \lhd \Omega(A)$ implies that $\Omega(A) \subseteq \mathrm{Id}_{E(A)}(\overline{A})$. Moreover, $\overline{A} \lhd \mathrm{Id}_{E(A)}(\overline{A})$ and $a_{E(A)}(\overline{A}) = \{0\}$ and $\overline{A} \cong A$, so Proposition 2.1 gives that $\overline{A}$ is an essential ideal of $\mathrm{Id}_{E(A)}(\overline{A})$ and there exists an embedding of rings $h : \mathrm{Id}_{E(A)}(\overline{A}) \to \Omega(A)$ such that $h|_{\overline{A}} = \mathrm{id}_{\overline{A}}$. By Propositions 2.1 and 2.2, we get that the identity mapping on $\mathrm{Id}_{E(A)}(\overline{A})$ is the unique ring homomorphism of $\mathrm{Id}_{E(A)}(\overline{A})$ into $E(A)$ which is an identity map on $\overline{A}$. Hence $h(\mathrm{Id}_{E(A)}(\overline{A})) = \mathrm{Id}_{E(A)}(\overline{A})$ implies that $\mathrm{Id}_{E(A)}(\overline{A}) \subseteq \Omega(A)$ and $\mathrm{Id}_{E(A)}(\overline{A}) = \Omega(A)$.

Finally, suppose that A is an essential ideal of S. Applying Proposition 2.2, we get that the function $g : S \to \Omega(A)$ given by $g(s) = (l_s, r_s)$ for $s \in S$ is the unique ring homomorphism such that $g(a) = (l_a, r_a)$ for $a \in A$. □

3. Proof of the Flanigan theorem

A few facts in this section are well known, but we will prove them for completeness.

LEMMA 3.1. *Let A, B be nonzero rings such that $a(B)$ is an essential ideal of B and $f : a(A) \to a(B)$ is an isomorphism of rings. Then*

$$\begin{gathered}I = \{(a, f(a)) : a \in a(A)\} \lhd A \times B, \\ [(A \times \{0\}) + I]/I \cong A, \quad [(\{0\} \times B) + I]/I \cong B \quad \text{and} \quad [(A \times \{0\}) + I]/I\end{gathered}$$

is an essential ideal of the ring $(A \times B)/I$.

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PROOF. Directly from the assumptions we get that I is a subgroup of $(A \times B)^+$ and $I \subseteq a(A \times B)$. Hence, in particular, $I \lhd A \times B$. Moreover,

$$(A \times \{0\}) \cap I = \{(a, f(a)) : a \in a(A), f(a) = 0\} = \{(0, 0)\},$$

so $[(A \times \{0\}) + I]/I \cong A$. Next,

$$(\{0\} \times B) \cap I = \{(a, f(a)) : a = 0\} = \{(0, 0)\}$$

implies that $[(\{0\} \times B) + I]/I \cong B$.

Suppose that $J \lhd A \times B$ and $I \subset J$. Recall that it is sufficient to prove that

$$[(A \times \{0\}) + I]/I \cap J/I \neq 0.$$

There exists $(a, b) \in A \times B$ such that $(a, b) \in J \setminus I$. If $a \notin a(A)$ then there exists $c \in A$ such that $ca \neq 0$ or $ac \neq 0$. If $ca \neq 0$ then

$$I \neq (ca, 0) + I = [(c, 0) + I] \cdot [(a, b) + I] \in [(A \times \{0\}) + I]/I \cap J/I.$$

Likewise, if $ac \neq 0$ then

$$I \neq (ac, 0) + I = [(a, b) + I] \cdot [(c, 0) + I] \in [(A \times \{0\}) + I]/I \cap J/I.$$

Assume that $a \in a(A)$. Then $I \neq (a, b) + I = (0, b - f(a)) + I$. Consequently $0 \neq b - f(a) \in B$. Essentiality of $a(B)$ implies that there exists

$$0 \neq y \in [B^1(b - f(a))B^1] \cap a(B).$$

Then $(0, y) + I \in J$. But f is onto, so $y = f(x)$ for some $x \in a(A)$. Notice that $x \neq 0$, because $y \neq 0$. Thus

$$\begin{aligned} I \neq (0, y) + I &= [(x, f(x)) + (-x, 0)] + I \\ &= (-x, 0) + I \in [(A \times \{0\}) + I]/I \cap J/I. \end{aligned}$$

This concludes the proof. □

LEMMA 3.2. *Let M^+ be a nonzero abelian group and T be a nonempty set. Denote $M_t = M$ for $t \in T$ and $N = M \oplus \bigoplus_{t \in T} M_t$, $S = \{f \in \mathrm{End}(N) : f(M) = 0\}$. Then $S <_l \mathrm{End}(N)$ and $\left[\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right] = a(B)$ is an essential ideal of the ring $B = \left[\begin{smallmatrix} S & N \\ 0 & 0 \end{smallmatrix}\right]$.*

PROOF. An easy computation shows that $S <_l \mathrm{End}(N)$ and

$$\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \subseteq a(B).$$

Let

$$\begin{bmatrix} f_0 & x_0 \\ 0 & 0 \end{bmatrix} \in a(B).$$

Suppose that $f_0 \neq 0$. Then there exists $y \in N$ such that $f_0(y) \neq 0$. Hence

$$\begin{bmatrix} f_0 & x_0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & f_0(y) \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is a contradiction. Thus $f_0 = 0$. Assume that $x_0 \notin M$. For $t \in T$ let π_t denote a projection of N onto M_t. It is clear that $\pi_t \in S$ for $t \in T$. But $x_0 \notin M$, so there

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exists $t_0 \in T$ such that $\pi_{t_0}(x_0) \neq 0$. Moreover,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} \pi_{t_0} & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x_0 \\ 0 & 0 \end{bmatrix}$$

which is impossible. Hence $x_0 \in M$ and, finally,

$$a(B) = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}.$$

Let J be a nonzero ideal of B. Then there exists

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix} \in J.$$

If $f \neq 0$ then we can take $y \in N$ such that $f(y) \neq 0$. Next,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & f(y) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \in J.$$

So $\left[\begin{smallmatrix} 0 & n \\ 0 & 0 \end{smallmatrix}\right] \in J$ for some $0 \neq n \in N$. If $n \in M$ then

$$\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \in J \cap a(B).$$

If $n \notin M$ then there exists $t_0 \in T$ such that $\pi_{t_0}(n) \neq 0$. Let $g_0 : M_{t_0} \to M$ be a natural isomorphism. Then $g_0\pi_{t_0} \in S$, $0 \neq (g_0\pi_{t_0})(n) \in M$ and

$$\begin{bmatrix} 0 & (g_0\pi_{t_0})(n) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} g_0\pi_{t_0} & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \in J \cap a(B).$$

This proves that $a(B)$ is an essential ideal of B. □

THEOREM 3.3. *Let A be a ring such that $a(A) \neq \{0\}$. Then for any cardinal number α there exists a ring R such that* card $R \geq \alpha$ *and A is an essential ideal of R.*

PROOF. Let T be a set of cardinality $\alpha \geq 1$. Let $M_t = a(A)$ for $t \in T$. Then $N = a(A) \oplus \bigoplus_{t \in T} M_t$ has cardinality greater than or equal to α. Put

$$S = \{f \in \mathrm{End}(N) : f(a(A)) = 0\}.$$

According to Lemma 3.2

$$a(A) \cong \begin{bmatrix} 0 & a(A) \\ 0 & 0 \end{bmatrix} = a(B) \quad \text{for } B = \begin{bmatrix} S & N \\ 0 & 0 \end{bmatrix}.$$

Moreover, $a(B)$ is an essential ideal of B and card $B \geq \alpha$. The function $f : a(A) \to a(B)$ defined by

$$f(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \quad \text{for } x \in a(A)$$

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is an isomorphism of rings. Denote $I = \{(a, f(a)) : a \in a(A)\}$. By Lemma 3.1 we get that

$$A \cong [(A \times \{0\}) + I]/I \lhd (A \times B)/I, \quad \operatorname{card}((A \times B)/I) \geq \operatorname{card} B \geq \alpha$$

and $[(A \times \{0\}) + I]/I$ is an essential ideal of $(A \times B)/I$. □

THEOREM 3.4 (Flanigan). *A ring A has* $\mathrm{ME}(A)$ *if and only if* $a(A) = \{0\}$. *Moreover, if* $a(A) = \{0\}$ *then* $\Omega(A) = \mathrm{ME}(A)$.

PROOF. Suppose that $R = \mathrm{ME}(A)$. If $a(A) \neq \{0\}$ then by Theorem 3.3 there exists a ring S of cardinality less than card R which is an essential extension of A. Hence there exists a ring homomorphism $f : S \to R$ such that $F|_A = \mathrm{id}_A$. Thus $\operatorname{Ker} f \cap A = \{0\}$ and consequently $\operatorname{Ker} f = \{0\}$ which means that f is an embedding. But card $R <$ card S, a contradiction. Finally, $a(A) = \{0\}$.

Conversely, assume that $a(A) = \{0\}$. The mapping $a \mapsto (l_a, r_a)$ for $a \in A$ is an isomorphism of A onto a ring $\overline{A}$. By Propositions 2.1 and 2.3, $\overline{A}$ is an essential ideal of the ring $\Omega(A)$. Let A be an ideal of S. Then the function $f : S \to \Omega(A)$ given by $f(s) = (l_s, r_s)$ for $s \in S$ is a ring homomorphism and $f(a) = (l_a, r_a)$ for $a \in A$. Thus $\Omega(A) = \mathrm{ME}(A)$. □

PROPOSITION 3.5. *If* $R = \mathrm{ME}(A)$ *and* $S = \mathrm{ME}(A)$ *then there exists a unique ring homomorphism* $f : R \to S$ *such that* $f|_A = \mathrm{id}_A$ *and this homomorphism is an isomorphism.*

PROOF. Theorem 3.4 implies that $a(A) = \{0\}$. By definition of $\mathrm{ME}(A)$ we get that there exist ring homomorphisms $f : R \to S$, $g : S \to R$ such that $f|_A = \mathrm{id}_A$ and $g|_A = \mathrm{id}_A$. Thus $g \circ f : R \to R$ and $f \circ g : S \to S$ are homomorphisms of rings such that $g \circ f|_A = \mathrm{id}_A$ and $f \circ g|_A = \mathrm{id}_A$. From this and Propositions 2.1 and 2.2 we conclude that $g \circ f = \mathrm{id}_R$ and $f \circ g = \mathrm{id}_S$. Hence f and g are isomorphisms. Moreover, applying Proposition 2.2, we see that f is the unique homomorphism of R into S. □

4. Examples and applications

In many problems concerning the structure of rings, properties of the ring $\mathrm{ME}(A)$ play an important role, especially in the case where $\mathrm{ME}(A)$ is precisely described. In [5, Theorem 2] it was proved that if a is a regular element of a ring R with identity then $\mathrm{Id}_R(aR) = \mathrm{ME}(aR)$.

DEFINITION 4.1 (Beidar). Let R be a ring with identity and let M be a right unital R-module. We say that M is a *generator* if

$$\sum \{f(M) : f \in \operatorname{Hom}(M_R, R_R)\} = R,$$

that is, there exist $f_1, \ldots, f_s \in \operatorname{Hom}(M_R, R_R)$ and $m_1, \ldots, m_s \in M$ such that

$$1 = f_1(m_1) + \cdots + f_s(m_s).$$

EXAMPLE 4.2.

(a) If $a \in R$ is a right regular element of a ring R with an identity then the function $h : R \to aR$ given by $h(x) = a \cdot x$ is an isomorphism of a right regular R-module.

Thus $h^{-1} : aR \to R$ is also an isomorphism of a right regular R-module and $M = aR$ is a generator.

(b) Let J be a right ideal of a ring R with an identity such that $RJ = R$. Then there exist $j_1, \ldots, j_s \in J$ and $a_1, \ldots, a_s \in R$ such that $1 = a_1 j_1 + \cdots + a_s j_s$, so $1 = l_{a_1}(j_1) + \cdots + l_{a_s}(j_s)$. Hence, $M = J$ is a generator.

In [2, Theorem 6] Beidar stated without proof that if I is a right ideal of a ring R with an identity and if I_R is a generator, then $\mathrm{Id}_R(I) = \mathrm{ME}(I)$. Unfortunately, this theorem does not hold in general. However, the theorem is true if we add following assumption:

$$\text{for any } h \in \mathrm{Hom}({}_I I, l_I(I)) \text{ there exists } c \in R \text{ such that } h = r_c. \tag{4.1}$$

There are many examples of rings R with an identity and $I <_r R$ such that I_R is a generator, but for which condition (4.1) does not hold.

EXAMPLE 4.3. Let S be a polynomial ring in two noncommuting variables X, Y over the ring $\mathbb{Z}$ of integers. Let J be an ideal of S generated by YX and Y^2. Let $x = X + J$, $y = Y + J$ and $R = S/J$. Then

$$R = \mathbb{Z}[x] + \mathbb{Z}[x]y.$$

Moreover, x is a right regular element of R. Let $I = xR$. It is obvious that $I = x\mathbb{Z}[x] + x\mathbb{Z}[x]y$, I_R is a generator and $\mathrm{Id}_R(I) = R$. We see at once that $l_I(I) = x\mathbb{Z}[x]y$. Let $h : I \to l_I(I)$ be defined by

$$h(a + by) = (a + b)y \quad \text{for } a, b \in x\mathbb{Z}[x].$$

Then $h \in \mathrm{Hom}({}_I I, l_I(I))$. If $h = r_c$ for some $c \in R$ then $h(x) = (x)r_c$ and consequently $xy = xc$ and $c = y$. But this and $y^2 = 0$ imply that $xy = h(xy) = (xy)r_y = xy^2 = 0$, which is a contradiction.

THEOREM 4.4. *Let I be a right ideal of a ring R with an identity and I_R be a generator. Then $\mathrm{Id}_R(I) = \mathrm{ME}(I)$ if and only if for every $h \in \mathrm{Hom}({}_I I, l_I(I))$ there exists $c \in R$ such that $h = r_c$.*

PROOF. By assumption, there exist $f_1, \ldots, f_s \in \mathrm{Hom}(I_R, R_R)$ and $i_1, \ldots, i_s \in I$ such that

$$1 = f_1(i_1) + \cdots + f_s(i_s). \tag{4.2}$$

From (4.2) we conclude that for any $r \in R$ we have $r = f_1(i_1 r) + \cdots + f_s(i_s r)$. Therefore $r_R(I) = \{0\}$. Hence $a_R(I) = \{0\}$, so by Theorem 3.4, $\Omega(A) = \mathrm{ME}(A)$, I is an essential in R and, moreover, I is an essential ideal of $T = \mathrm{Id}_R(I)$.

Suppose that $T = \mathrm{ME}(I)$. Applying Theorem 3.4 and Proposition 3.5, we see that the function $f : T \to \Omega(I)$ given by $f(t) = (l_t, r_t)$ for $t \in T$ is an isomorphism of rings. Choose any $h \in \mathrm{Hom}({}_I I, l_I(I))$. Then $(0, h) \in \Omega(I)$, so there exists $c \in T$ such that $(0, h) = (l_c, r_c)$ and $h = r_c$.

Conversely, assume that for every $h \in \mathrm{Hom}({}_I I, l_I(I))$ there exists $c \in R$ such that $h = r_c$. Then $Ic \subseteq l_I(I)$, which implies that $IcI = \{0\}$. Hence $cI \subseteq r_I(I) = \{0\}$

and $c \in l_I(I)$. Therefore $c \in T$ and $(0, h) = (l_c, r_c) \in \Omega(I)$. By Propositions 2.2, 2.3 and Theorem 3.4 it suffices to prove that the function $f : T \to \Omega(I)$ given by $f(t) = (l_t, r_t)$ for $t \in T$ is onto. Fix any $(\lambda, \rho) \in \Omega(I)$. Then for all $i, j \in I$, $i \cdot \lambda(j) = (i)\rho \cdot j$. Let

$$b = f_1((i_1)\rho) + \cdots + f_s((i_s)\rho).$$

Notice that $b \in R$ and, for any $j \in I$,

$$\begin{aligned} b \cdot j &= f_1((i_1)\rho) \cdot j + \cdots + f_s((i_s)\rho) \cdot j = f_1((i_1)\rho \cdot j) + \cdots + f_s((i_s)\rho \cdot j) \\ &= f_1(i_1\lambda(j)) + \cdots + f_k(i_s\lambda(j)) = f_1(i_1) \cdot \lambda(j) + \cdots + f_s(i_s) \cdot \lambda(j) = \lambda(j) \end{aligned}$$

by (4.1). Hence, $\lambda = l_b$. From this, $b \in T$. Next, for

$$i, j \in I : (j \cdot b - (j)\rho) \cdot i = j \cdot (b \cdot i) - (j)\rho \cdot i = j \cdot \lambda(i) - j \cdot \lambda(i) = 0,$$

so $j \cdot b - (j)\rho \in l_I(I)$. By the above, the function h defined by $h(j) = j \cdot b - (j)\rho$ for $j \in I$ is a homomorphism of the left I-module I into the left I-module $l_I(I)$. Thus there exists $c \in l_I(I)$ such that $h = r_c$ and $(l_c, r_c) = (0, h) \in \Omega(I)$. But $h = r_b - \rho$ so

$$\rho = r_b - h = r_b - r_c = r_{b-c}, \quad \lambda = l_b = l_b - l_c = l_{b-c}$$

and, finally, $(\lambda, \rho) = (\lambda_{b-c}, \rho_{b-c}) \in \Omega(I)$. □

COROLLARY 4.5. *Let R be a ring with an identity and assume that $I <_r R$ is a generator. If $l_I(I) = \{0\}$ then $\mathrm{Id}_R(I) = \mathrm{ME}(I)$. In particular, if a is a two-sided regular element of R then $\mathrm{Id}_R(aR) = \mathrm{ME}(aR)$.*

COROLLARY 4.6 (Beidar). *Let R be a ring with an identity and let $I <_r R$ and $RI = R$. Then $\mathrm{Id}_R(I) = \mathrm{ME}(I)$. In particular, $\mathrm{Id}_R(J) = \mathrm{ME}(J)$ for every nonzero right ideal J of a simple ring R with an identity.*

PROOF. By assumption, there exist $r_1, \ldots, r_s \in R$ and $i_1, \ldots, i_s \in I$ such that $1 = r_1 i_1 + \cdots + r_s i_s$. Hence $r_R(I) = \{0\}$ and $1 = l_{r_1}(i_1) + \cdots + l_{r_s}(i_s)$. Thus I_R is a generator. Choose $h \in \mathrm{Hom}({}_I I, l_I(I))$. Let $b = r_1 h(i_1) + \cdots + r_s h(i_s)$. Then $b \in R$ and, for every $i \in I$,

$$i = (ir_1)i_1 + \cdots + (ir_s)i_s \quad \text{and} \quad h(i) = (ir_1)h(i_1) + \cdots + (ir_s)h(i_s) = i \cdot b.$$

Thus $h = r_b$. From Theorem 4.4 it follows that $\mathrm{Id}_R(I) = \mathrm{ME}(I)$. □

DEFINITION 4.7. A simple ring $A = A^2$ satisfies a *Gardner condition* if, for any ring R, $A \lhd R$ and $R/A \cong A$ imply that there exists $I \lhd R$ such that $R = A \oplus I$.

PROPOSITION 4.8. *A simple ring $A = A^2$ satisfies a Gardner condition if and only if A is not embeddable in a ring $\mathrm{ME}(A)/A$.*

PROOF. Let $A = A^2$ be a simple ring. Then $a(A) = \{0\}$, so by Theorem 3.4 there exists $\mathrm{ME}(A)$. First, assume that A does not satisfy a Gardner condition. Then there exists a ring R in which $A \lhd R$, $R/A \cong A$ and there does not exist $I \lhd R$ such that $R = A \oplus I$. Suppose that A is not essential in R. Then there exists a nonzero ideal J

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of R such that $A \cap J = \{0\}$ and $(A + J)/A$ is a nonzero ideal of a simple ring R/A. Thus $A + J = R$. This means that $A \oplus J = R$, which is a contradiction. So A is essential in R. According to Proposition 2.2 and Theorem 3.4 there exists an embedding of rings $h : R \to \mathrm{ME}(A)$ such that $h|A = \mathrm{id}_A$. From this it follows that $h(R)/A \cong R/A$ and $A \cong h(R)/A$. This shows that A is embeddable in $\mathrm{ME}(A)/A$.

Conversely, suppose that A is embeddable in $\mathrm{ME}(A)/A$. Then there exists a subring S of $\mathrm{ME}(A)$ such that $A \lhd S$ and $S/A \cong A$. But $a_{\mathrm{ME}(A)}(A) = \{0\}$ by Proposition 2.1. Thus $a_S(A) = \{0\}$ and, again by Proposition 2.1, A is essential in S. Hence, there does not exist $I \lhd S$ such that $A \oplus I = S$. It follows that A does not satisfy a Gardner condition. □

In [4] it was proved that if a simple ring $A = A^2$ satisfies a Gardner condition then the lower radical determined by the class of all rings isomorphic to A is an atom in the lattice of all radicals. By the above remark and Proposition 4.8 we obtain the following theorem.

THEOREM 4.9. *If a simple ring $A = A^2$ is not embeddable in a ring* $\mathrm{ME}(A)/A$ *then the lower radical determined by the class of all rings isomorphic to A is an atom in the lattice of all radicals.*

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