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EXTREME VALUES FOR DIVISOR FUNCTIONS

R.A. MACLEOD

Maximum and minimum values are obtained for some sums invovling the divisor functions of number theory.

1. INTRODUCTION

Let t be an arbitrary real number, and f any numerical function. In number theory we often study the summatory functions

$$\sum_{n \leqslant x} n^t f(n);$$

the cases $t = 0$ and $t = -1$ tend to be of particular interest. In many cases, there are known elementary functions $g_t(x)$ such that

$$\sum_{n \leqslant x} n^t f(n) = g_t(x) + E_t(x), \tag{1}$$

where $E_t(x)$, the "error term", is usually known in terms of rather awkward sums or to an order of magnitude. For example, if $\sigma(n)$ is the sum of the divisors of n, and $t = 0$, then we have

$$\sum_{n \leqslant x} \sigma(n) = \frac{\pi^2}{12} x^2 + 0(x \log x), \tag{2}$$

or, more precisely,

$$\sum_{n \leqslant x} \sigma(n) = \frac{\pi^2}{12} x^2 - x \sum_{d \leqslant \sqrt{x}} \frac{1}{d} \hat{B}_1\left(\frac{x}{d}\right) - \frac{1}{2} x - \sum_{d \leqslant \sqrt{x}} d \hat{B}_1\left(\frac{x}{d}\right) + 0\left(x^{\frac{1}{2}}\right), \tag{3}$$

where $\hat{B}_1(x)$ is the quantity $\{x\} - \frac{1}{2}$ and $\{x\}$ denotes the fractional part of x; here $g_0(x)$ is $\frac{\pi^2}{12} x^2$, while $E_t(x)$ depends (primarily) on the sum $\sum_{d \leqslant \sqrt{x}} \frac{1}{d} \hat{B}_1\left(\frac{x}{d}\right)$. Suprisingly

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often, the values of $E_t(x)$, over the integers, show at least initially a preponderance of either positive values or negative values. For the example of (2), $E_t(x)$ is positive for the first ten integers and for 88 of the first 100. Since we presumably have

$$\lim_{x\to\infty}\frac{E_t(x)}{g_t(x)}=0,$$

a fairly natural question seems to be to establish the minimum or maximum values of $(E_t(x)/g_t(x))$ over the integers $x \geqslant 1$. We obtain several results along this line for functions related to divisor functions. If we write $d(n)$ for the number of divisors of n, and $\sigma_a(n)$ for the sum of the a-th powers of the divisors of n, then some of the more interesting results are:

$$\frac{1}{x^2}\left(\sum_{n\leqslant x}\sigma(n)-\frac{\pi^2}{12}x^2\right)$$

has its minimum at $x = 23$ (maximum at $x = 1, 2$);

$$\frac{1}{x\log x+(2\gamma-1)x}\left(\sum_{n\leqslant x}d(n)-x\log x-(2\gamma-1)x\right)$$

has its minimum at $x = 179$ (maximum at $x = 1$);

$$\frac{1}{x}\left(\sum_{n\leqslant x}\left\{\frac{x}{n}\right\}-(1-\gamma)x\right)$$

has its maximum at $x = 179$ (minimum at $x = 1, 2$);

$$\frac{1}{x^{a+1}}\left(\sum_{n\leqslant x}\sigma_a(n)-\frac{\zeta(a+1)}{a+1}\right)$$

is always positive for $a = 2, 3, 4$, has limit 0, and is negative for at most a finite number of positive integers x for any integer $a \geqslant 5$;

$$\frac{1}{\frac{1}{2}\log^2 x+2\gamma\log x}\left(\sum_{n\leqslant x}\frac{d(n)}{n}-\frac{1}{2}\log^2 x-2\gamma\log x\right)$$

is always positive and has limit 0;

$$\frac{1}{\frac{\pi^2}{6}x}\left(\sum_{n\leqslant x}\frac{\sigma(n)}{n}-\frac{\pi^2}{6}x\right)$$

is always positive and has limit 0;

$$\left|\sum_{n\leqslant x} d(n) - x\log x - (2\gamma - 1)x\right| \leqslant \sqrt{x} + \frac{1}{2};$$

$$\left|\sum_{n\leqslant x} \sigma(n) - \frac{\pi^2}{12}x^2\right| \leqslant \frac{1}{4}x\log x + \left(\frac{1}{2}\gamma + \frac{3}{4}\right) + 2\sqrt{x}.$$

Our objects, therefore, are twofold: to obtain good usable upper bounds for the functions $E_t(x)$, as in the last two mentioned above, and to use these to study the relavent extreme values.

Our methods are completely elementary.

Among results known for other arithmetic functions, we mention the following:

$$\frac{1}{x^2}\left(\sum_{n\leqslant x} \phi(n) - \frac{3}{\pi^2}x^2\right)$$

has its minimum at $x = 1276$ (where 1276 gives only the second negative value);

$$\sum_{n\leqslant x} \frac{\mu(n)}{n}$$

has its minimum at $x = 13$;

$$\frac{1}{x}\left(\sum_{n\leqslant x} |\mu(n)| - \frac{6}{\pi^2}x\right)$$

has its minimum at $x = 176$. For proofs, see respectively [**2**], [**3**] and [**5**].

2. Some preliminaries

In what follows, let $[x]$ and $\{x\}$ denote respectively the integer and fractional parts of x, and let $\hat{B}_t(x) = B_t(\{x\})$ where $B_t(x)$ is the Bernoulli polynomial. Let $k = k(x)$ be defined by $k = [\sqrt{x}]$, and let $B_t = B_t(0)$ be the Bernoulli number. The following are well-known:

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \tag{4}$$

$$|\hat{B}_1(\mu)| \leqslant \frac{1}{2}, \quad |\hat{B}_2(\mu)| \leqslant \frac{1}{6}, \quad |\hat{B}_3(\mu)| \leqslant \frac{\sqrt{3}}{36} < \frac{1}{20}, \qquad \text{real } \mu. \tag{5}$$

Lemma 1.

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(a) $$\sum_{n\leqslant x}\frac{1}{n}=\log x+\gamma-\left(\hat{B}(x)/x\right)+R_1(x),\ \textit{where } |R_1(x)|\leqslant\frac{1}{6}x^{-2}.$$

(b) $$\sum_{n\leqslant x}\frac{1}{n}=\log x+\gamma-\left(\hat{B}_1(x)/x\right)-\frac{1}{2}\left(\hat{B}_2(x)/x^2\right)+R_1'(x),$$

$$\textit{where } |R'(x)|\leqslant\frac{1}{30}x^{-3}.$$

(c) $$\sum_{n\leqslant x}\frac{1}{n^t}=\zeta(t)-\frac{1}{t-1}x^{-(t-1)}-\left(\hat{B}_1(x)/x^t\right)+R_t(x),$$

$$\textit{where } |R_t(x)|\leqslant\frac{1}{6}tx^{-(t+1)},\, t\geqslant 2.$$

(d) $$\sum_{n\leqslant x}\frac{1}{n^t}=\zeta(t)-\frac{1}{t-1}x^{-(t-1)}-\left(\hat{B}_1(x)/x^t\right)-\frac{1}{2}t\left(\hat{B}_2(x)/x^{t+1}\right)+R_t'(x),$$

$$\textit{where } |R_t'(x)|\leqslant\frac{1}{60}t(t+1)x^{-(t+2)},\, t\geqslant 2.$$

(e) $$\sum_{n\leqslant k}\log n=\left(k+\frac{1}{2}\right)\log k-k+\frac{1}{2}\log 2\pi+\frac{1}{12}k^{-1}+Q(k),$$

$$\textit{where } |Q(k)|\leqslant\frac{1}{180}k^{-3}.$$

(f) $$\sum_{n\leqslant x}\frac{\log n}{n}=\frac{1}{2}\log^2 x+A_{-1}-\hat{B}_1(x)\frac{\log x}{x}+W_{-1}(x),$$

$$\textit{where } |W_{-1}(x)|\leqslant\frac{1}{6}\frac{\log x-1}{x^2}.$$

(g) *Let* $L_\mu(x)=\sum_{d\leqslant x}d^\mu$. *Then for* $\mu\geqslant 0$

$$L_\mu(x)=\frac{1}{\mu+1}x^{\mu+1}-\sum_{r=0}^{\mu}\frac{(-1)^r}{r+1}\binom{\mu}{r}\hat{B}_{r+1}(x)x^{\mu-r}$$
$$-\frac{1}{\mu+1}B_{\mu+1}-\left[\frac{1}{\mu+1}\right].$$

PROOF: These follow immediately from the Euler-Maclaurin summation formulae.

Note that, for an integer x,

$$\hat{B}_1(x)=-\frac{1}{2},\qquad \hat{B}_2(x)=\frac{1}{6},\qquad \hat{B}_3(x)=0.$$

From Lemma 1(b) and (5), Lemma 1(a) could be strengthened to

(6) $$|R_1(x)|\leqslant\frac{1}{12}x^{-2}+\frac{1}{30}x^{-3},$$

while from Lemma 1(d) and (5), Lemma 1(c) could be strengthened to

$$|R_t(x)| \leqslant \frac{t}{12}x^{-(t+1)} + \frac{t(t+1)}{60}x^{-(t+2)}, \qquad t \geqslant 2. \tag{7}$$

LEMMA 2.

$$\zeta(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + h(t), \text{ with } h(t) \leqslant \frac{1}{2^{t-1}(2^{t-1}-1)}.$$

COROLLARY 2.1.

$$\sum_{d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right) \leqslant \begin{cases} -\frac{1}{2} + \frac{1}{2}\frac{1}{3^t} + \dfrac{1}{2^t(2^{t-1}-1)}, & \text{for } t \geqslant 2 \text{ (integer } x). \\ -0.3, & \text{for } t = 1. \end{cases}$$

PROOF OF LEMMA:

$$\begin{aligned} h(t) &= \frac{1}{4^t} + \frac{1}{5^t} + \dots \\ &\leqslant \frac{4}{4^t} + \frac{8}{8^t} + \frac{16}{16^t} + \dots \\ &= \frac{1}{2^{t-1}(2^{t-1}-1)}. \end{aligned}$$

∎

PROOF OF COROLLARY: For x even, we have

$$\sum_{d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right) = -\frac{1}{2} - \frac{1}{2^{t+1}} + \sum_{3 \leqslant d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right),$$

while for x odd, we have

$$\sum_{d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right) = -\frac{1}{2} + \sum_{3 \leqslant d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right);$$

$$\left|\sum_{3 \leqslant d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right)\right| \leqslant \frac{1}{2}\sum_{3 \leqslant d \leqslant x} \frac{1}{d^t} \leqslant \frac{1}{2}\left(\zeta(t) - 1 - \frac{1}{2^t}\right).$$

the result now follows from Lemma 2; $\zeta(2) - 1 - \frac{1}{4} < 0.4$. ∎

We note that Corollary 2.1 can be sharpened slightly; for integer x, $\hat{B}_1\left(\frac{x}{d}\right) \leqslant \frac{1}{2} - \frac{1}{d}$, so that

$$\sum_{d \leqslant x} \frac{1}{d^t}\hat{B}_1\left(\frac{x}{d}\right) \leqslant \frac{1}{2}\zeta(t) - \zeta(t+1) < 0. \tag{8}$$

LEMMA 3. *Let* $L_\mu(x) = \sum_{d \leqslant x} d^\mu$, *let* c *be any numerical function, let* $f(n) = \sum_{d|n} c(d)$, *and let* t *be any real number. Then*

$$\sum_{n \leqslant x} n^t f(n) = \sum_{n \leqslant x} n^t \sum_{m \leqslant \frac{x}{n}} m^t c(m).$$

In particular, for $c(m) = m^a$, *we have*

$$\sum_{n \leqslant x} \frac{\sigma_a(n)}{n^t} = \sum_{n \leqslant x} \frac{1}{n^t} L_{a-t}\Big(\frac{x}{n}\Big).$$

PROOF:

$$\sum_{n \leqslant x} n^t f(n) = \sum_{n \leqslant x} n^t \sum_{m|n} c(m) = \sum_{md \leqslant x} m^t d^t c(m) = \sum_{n \leqslant x} n^t \sum_{m \leqslant \frac{x}{n}} m^t c(m).$$

∎

COROLLARY 3.1. *Let* a *and* t *be integers, with* $0 \leqslant t \leqslant a$. *Then*

$$\sum_{n \leqslant x} \frac{\sigma_a(n)}{n^t} = \frac{1}{a-t+1} x^{a-t+1} \sum_{d \leqslant x} \frac{1}{d^{a+1}} + \frac{1}{a-t+1} \sum_{r=1}^{a-t+1} (-1)^r$$
$$\times \binom{a-t+1}{r} x^{a-t+1-r} \sum_{d \leqslant x} \frac{1}{d^{a+1-r}} \hat{B}_r\Big(\frac{x}{d}\Big)$$
$$- \Big(\frac{1}{a-t+1} B_{a-t+1} + \Big[\frac{1}{a-t+1}\Big]\Big) \sum_{d \leqslant x} \frac{1}{d^t}$$

PROOF: Put Lemma 1(g) into Lemma 3. ∎

LEMMA 4. *Let* c *be any numerical function and* t *a positive integer, and define* $C_r(x)$ *by*

$$C_r(x) = \sum_{n \leqslant x} c(n) n^r.$$

Then for any integer k *such that* $1 \leqslant k \leqslant [x]$,

$$\sum_{n \leqslant x} c(n)\Big\{\frac{x}{n}\Big\}^t = \sum_{n \leqslant \frac{x}{k}} c(n)\Big\{\frac{x}{n}\Big\}^t + x^t\Big(C_{-t}(x) - C_{-t}\Big(\frac{x}{k}\Big)\Big)$$
$$+ \sum_{r=1}^{t} \binom{t}{r} (-1)^r x^{t-r} \Big(\sum_{s=1}^{r} \binom{r}{s} (-1)^{s+1} \sum_{d \leqslant k} d^{r-s} C_{-(t-r)}\Big(\frac{x}{d}\Big)$$
$$- k^r C_{-(t-r)}\Big(\frac{x}{k}\Big)\Big).$$

PROOF: This is Lemma 6 in [4]. ∎

In our applications, we will take k to be $[\sqrt{x}]$ and take $c(n)$ to be 1, in which case C_r is L_r.

3. The Main Results.

Theorem 5. *Let $d(n)$ be the number of divisors of n. Let $R_1(x)$ be the error term in the Euler-MacLaurin expansion of $\sum_{n\leqslant x}\frac{1}{n}$; that is*

$$\sum_{n\leqslant x}\frac{1}{n}=\log x+\gamma-\frac{\hat{B}_1(x)}{x}+R_1(x).$$

Then, writing $k=[\sqrt{x}]$,

$$\sum_{n\leqslant x}d(n)=x\log x+(2\gamma-1)x-\sum_{d\leqslant k}\hat{B}_1\Big(\frac{x}{d}\Big)-\sum_{d\leqslant\frac{x}{k}}\hat{B}_1\Big(\frac{x}{d}\Big)+T_0(x),$$

where we have for $x\geqslant 1$:

(i) $T_0(x)=xR_1\big(\frac{x}{k}\big)+xR_1(k)+\frac{1}{2}\{\frac{x}{k}\}$;

(ii) $|T_0(x)|\leqslant 1$;

(iii) $|\sum_{n\leqslant x}d(n)-x\log x-(2\gamma-1)x|\leqslant\sqrt{x}+\frac{1}{2}$.

Proof: We have, putting $t=0$ and $c(n)=1$ in Lemma 3,

$$\sum_{n\leqslant x}d(n)=\sum_{d\leqslant x}\Big[\frac{x}{d}\Big]=xL_{-1}(x)-\sum_{d\leqslant x}\Big\{\frac{x}{d}\Big\}. \tag{9}$$

Now, by Lemma 4,

$$\sum_{d\leqslant x}\Big\{\frac{x}{d}\Big\}=\sum_{d\leqslant\frac{x}{k}}\Big\{\frac{x}{d}\Big\}+xL_{-1}(x)-xL_{-1}\Big(\frac{x}{k}\Big)-\sum_{d\leqslant k}L_0\Big(\frac{x}{d}\Big)+kL_0\Big(\frac{x}{k}\Big). \tag{10}$$

Hence we have

$$\sum_{n\leqslant x}d(n)=xL_{-1}\Big(\frac{x}{k}\Big)+\sum_{d\leqslant k}L_0\Big(\frac{x}{d}\Big)-kL_0\Big(\frac{x}{k}\Big)-\sum_{d\leqslant\frac{x}{k}}\Big\{\frac{x}{d}\Big\}. \tag{11}$$

Here, we have

$$\begin{aligned}xL_{-1}\Big(\frac{x}{k}\Big)&=x\left(\log x-\log k+\gamma-\frac{\{\frac{x}{k}\}-\frac{1}{2}}{\frac{x}{k}}+R_1\Big(\frac{x}{k}\Big)\right)\ \text{(from Lemma 1(a))}\\&=x\log x-x\log k+x\gamma-k\Big\{\frac{x}{k}\Big\}+\frac{1}{2}k+xR_1\Big(\frac{x}{k}\Big),\end{aligned} \tag{12}$$

$$\begin{aligned}\sum_{d\leqslant k}L_0\Big(\frac{x}{d}\Big)-kL_0\Big(\frac{x}{k}\Big)&=\sum_{d\leqslant k}\Big[\frac{x}{d}\Big]-k\Big[\frac{x}{k}\Big]\\&=xL_{-1}(k)-\sum_{d\leqslant k}\Big\{\frac{x}{d}\Big\}-x+k\Big\{\frac{x}{k}\Big\}\\&=x\log k+x\gamma+\frac{1}{2}\frac{x}{k}+xR_1(k)-\sum_{d\leqslant k}\Big\{\frac{x}{d}\Big\}-x+k\Big\{\frac{x}{k}\Big\}.\end{aligned} \tag{13}$$

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Hence, putting (12) and (13) into (11), we get

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma-1)x + \frac{1}{2}k + \frac{1}{2}\frac{x}{k} \tag{14}$$
$$- \sum_{d\leqslant k}\left\{\frac{x}{d}\right\} - \sum_{d\leqslant \frac{x}{k}}\left\{\frac{x}{d}\right\} + xR_1\left(\frac{x}{k}\right) + xR_1(k)$$
$$= x\log x + (2\gamma-1)x - \sum_{d\leqslant k}\hat{B}_1\left(\frac{x}{d}\right) \tag{15}$$
$$- \sum_{d\leqslant \frac{x}{k}}\hat{B}_1\left(\frac{x}{d}\right) + xR_1\left(\frac{x}{k}\right) + xR_1(k) + \frac{1}{2}\left\{\frac{x}{k}\right\}.$$

As regards results (ii) and (iii), we have, since

$$|R_1(x)| \leqslant \frac{1}{6}\frac{1}{x^2}, \left|xR_1\left(\frac{x}{k}\right) + xR_1(k)\right| \leqslant \frac{1}{6}\frac{k^2}{x} + \frac{1}{6}\frac{x}{k^2}. \tag{16}$$

Also,

$$\left|-\sum_{d\leqslant k}\hat{B}_1\left(\frac{x}{d}\right) - \sum_{d\leqslant \frac{x}{k}}\hat{B}_1\left(\frac{x}{d}\right) + \frac{1}{2}\left\{\frac{x}{k}\right\}\right| \leqslant \sum_{d\leqslant k}\frac{1}{2} + \sum_{d\leqslant \frac{x}{k}}\frac{1}{2} + \frac{1}{2}\left\{\frac{x}{k}\right\} \tag{17}$$
$$= \frac{1}{2}k + \frac{1}{2}\left[\frac{x}{k}\right] + \frac{1}{2}\left\{\frac{x}{k}\right\}$$
$$= \frac{1}{2}k + \frac{1}{2}\frac{x}{k}.$$

Now,

$$\sqrt{x} = k + \{\sqrt{x}\} \implies x = k^2 + 2k\{\sqrt{x}\} + \{\sqrt{x}\}^2 \tag{18}$$
$$\implies \frac{x}{k} = k + 2\{\sqrt{x}\} + \frac{\{\sqrt{x}\}^2}{k}$$
$$\implies k + \frac{x}{k} = 2(k + \{\sqrt{x}\}) + \frac{\{\sqrt{x}\}^2}{k} \tag{19}$$
$$= 2\sqrt{x} + \frac{\{\sqrt{x}\}^2}{k}.$$

Also,

$$\frac{x}{k^2} = 1 + \frac{2}{k}\{\sqrt{x}\} + \frac{\{\sqrt{x}\}^2}{k^2}, \frac{k^2}{x} = 1 - \frac{2k}{x}\{\sqrt{x}\} - \frac{\{\sqrt{x}\}^2}{x} \tag{20}$$
$$\implies \frac{x}{k^2} + \frac{k^2}{x} = 2 + \frac{4\{\sqrt{x}\}^2}{x} + \frac{4\{x\}^3}{kx} + \frac{\{\sqrt{x}\}^4}{k^2x}. \tag{21}$$

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Hence, putting (16)–(21) into (15), we have

$$\left|\sum_{n\leqslant x} d(n) - x\log x - (2\gamma-1)x\right| \leqslant \sqrt{x} + \frac{1}{2k} + \frac{1}{3} + \frac{2}{3}\frac{1}{x} + \frac{2}{3}\frac{1}{kx} + \frac{1}{6}\frac{1}{k^2x} \tag{22}$$
$$\leqslant \sqrt{x} + \frac{1}{2} \qquad \text{for } x \leqslant 21.$$

It is easy to check that (22) is still valid for all real x in the range $1 \leqslant x \leqslant 21$. ∎

COROLLARY 5.1.

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma-1)x - 2\sum_{d\leqslant k} \hat{B}_1\left(\frac{x}{d}\right) + 0(1).$$

Note. Solving the "Dirichlet divisor problem" is equivalent to showing

$$\sum_{d\leqslant k} \hat{B}_1\left(\frac{x}{d}\right) = 0\left(x^{\frac{1}{4}+\epsilon}\right).$$

COROLLARY 5.2. $\sum_{n\leqslant x} d(n) \geqslant x\log x$, *for all integers* $x \geqslant 1$ *and all real numbers* $x \geqslant 1$ *except for* x *in the following intervals (seven decimal accuracy):*

$[1.7632228, 2)$, $[2.8573908, 3)$, $[3.7686795, 4)$, $[4.9819032, 5)$,
$[5.7289256, 6)$, $[7.7927415, 8)$, $[9.9921716, 10)$, $[11.7643527, 12)$,
$[17.9931387, 18)$, *and* $[23.9345673, 24)$.

PROOF: $(2\gamma-1)x > \sqrt{x} + \frac{1}{2}$ for $x \geqslant 49$. Thus the result follows from Theorem 5 (iii) for $x \geqslant 49$, and it is not difficult to check for $1 \leqslant x < 49$. ∎

The first part of this Corollary appears in [**1**].

COROLLARY 5.3.

$$\left(\sum_{n\leqslant x} d(n) - x\log x - (2\gamma-1)x\right) \Big/ (x\log x + (2\gamma-1)x)$$

has its minimum over integers x *at* $x = 179$*; its value is* $-0.003331224\ldots$. *(Its maximum over integers is at* $x = 1$*.)*

PROOF:

$$\frac{\sqrt{x} + \frac{1}{2}}{x\log x + (2\gamma-1)x} \leqslant 0.0033 \quad \text{for } x \geqslant 1650;$$

thus it suffices to check $1 \leqslant x \leqslant 1650$. ∎

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COROLLARY 5.4. *Let* $k = [\sqrt{x}]$. *Then*

$$\sum_{n \leqslant x} \left\{ \frac{x}{n} \right\} = (1-\gamma)x + U(x),$$

where we have:

(i) $U(x) = \sum_{n \leqslant k} \hat{B}_1\left(\frac{x}{n}\right) + \sum_{n \leqslant \frac{x}{k}} \hat{B}_1\left(\frac{x}{n}\right) + xR_1(x) - xR_1\left(\frac{x}{k}\right) - xR_1(k) - \hat{B}_1(x) - \frac{1}{2}\left\{\frac{x}{k}\right\}$;

(ii) $|U(x)| \leqslant \sqrt{x} + 1$ *for* $x \geqslant 1$;

(iii) $\left(\sum_{n \leqslant x} \left\{\frac{x}{n}\right\} - (1-\gamma)x \right) / (1-\gamma)x$ *has its maximum over the integers* x *at* $x = 179$; *its value is* $0.048689\ldots$ *(minimum at* $x = 1$ *and* 2*).*

PROOF: Result (i) follows from (9), (14) and Lemma 1(a).
Result (ii) is proven the same way as part (iii) of Theorem 5.
For result (iii), we note

$$\frac{\sqrt{x} + 1}{(1-\gamma)x} < 0.048689\ldots \quad \text{for } x \geqslant 2460,$$

so that it suffices to check $1 \leqslant x \leqslant 2460$. ∎

We note that $\sum_{n \leqslant x} \left\{\frac{x}{n}\right\} - (1-\gamma)x$ is negative for the first 22 integers and for 92 of the first 100.

THEOREM 6. *Let* $R_2(x)$ *be the error term in the Euler-MacLaurin expansion of* $\sum_{n \leqslant x} \frac{1}{n^2}$; *that is,*

$$\sum_{n \leqslant x} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{1}{x} - \frac{1}{x^2}\hat{B}_1(x) + R_2(x).$$

Then, writing $k = [\sqrt{x}]$, *we have*

$$\sum_{n \leqslant x} \sigma(n) = \frac{\pi^2}{12}x^2 - x \sum_{d \leqslant \frac{x}{k}} \frac{1}{d}\hat{B}_1\left(\frac{x}{d}\right) - \sum_{d \leqslant k} d\hat{B}_1\left(\frac{x}{d}\right) - \frac{1}{2}x + T(x),$$

where we have, for $x \geqslant 1$:

(i) $T(x) = \frac{1}{2}x^2 R_2\left(\frac{x}{k}\right) + \frac{1}{2}kP_1\left(\frac{x}{k}\right) + \frac{1}{2}\sum_{n \leqslant \frac{x}{k}} \left\{\frac{x}{n}\right\}\left(\left\{\frac{x}{n}\right\} - 1\right)$;

(ii) $|T(x)| \leqslant \frac{11}{24}\sqrt{x} + \frac{1}{20}$;

(iii) $\left| \sum_{n \leqslant x} \sigma(n) - \frac{\pi^2}{12}x^2 \right| \leqslant \frac{1}{4}x \log x + \left(\frac{1}{2}\gamma + \frac{3}{4}\right)x + 2\sqrt{x}$.

PROOF: We have, putting $t = 0$ and $c(n) = n$ in Lemma 3,

$$\begin{aligned}\sum_{n \leqslant x} \sigma(n) &= \sum_{n \leqslant x} \frac{1}{2}\left[\frac{x}{n}\right]\left(\left[\frac{x}{n}\right] + 1\right) \\ &= \frac{1}{2}\sum_{n \leqslant x}\left[\frac{x}{n}\right]^2 + \frac{1}{2}\sum_{n \leqslant x}\left[\frac{x}{n}\right] \\ &= \frac{1}{2}x^2 L_{-2}(x) - x\sum_{n \leqslant x}\frac{1}{n}\left\{\frac{x}{n}\right\} + \frac{1}{2}\sum_{n \leqslant x}\left\{\frac{x}{n}\right\}^2 + \frac{1}{2}xL_{-1}(x) - \frac{1}{2}\sum_{n \leqslant x}\left\{\frac{x}{n}\right\}.\end{aligned} \tag{23}$$

Now, by Lemma 4, with $c(n) = \frac{1}{n}$ and $t = 1$, and $c(n) = 1$ and $t = 2$, 1,

$$\begin{aligned}\sum_{n \leqslant x}\frac{1}{n}\left\{\frac{x}{n}\right\} = \sum_{n \leqslant \frac{x}{k}}\frac{1}{n}\left\{\frac{x}{n}\right\} &+ xL_{-2}(x) - xL_{-2}\left(\frac{x}{k}\right) \\ &- \sum_{d \leqslant k} L_{-1}\left(\frac{x}{d}\right) + kL_{-1}\left(\frac{x}{k}\right),\end{aligned} \tag{24}$$

$$\begin{aligned}\sum_{n \leqslant x}\left\{\frac{x}{n}\right\}^2 = \sum_{n \leqslant \frac{x}{k}}\left\{\frac{x}{n}\right\}^2 &+ x^2 L_{-2}(x) \\ &- x^2 L_{-2}\left(\frac{x}{k}\right) - 2x\sum_{d \leqslant k} L_{-1}\left(\frac{x}{d}\right) + 2xkL_{-1}\left(\frac{x}{k}\right) \\ &+ 2\sum_{d \leqslant k} dL_0\left(\frac{x}{d}\right) - \sum_{d \leqslant k} L_0\left(\frac{x}{d}\right) - k^2 L_0\left(\frac{x}{k}\right),\end{aligned} \tag{25}$$

$$\begin{aligned}\sum_{n \leqslant x}\left\{\frac{x}{n}\right\} = \sum_{n \leqslant \frac{x}{k}}\left\{\frac{x}{n}\right\} &+ xL_{-1}(x) - xL_{-1}\left(\frac{x}{k}\right) \\ &- \sum_{d \leqslant k} L_0\left(\frac{x}{d}\right) + kL_0\left(\frac{x}{k}\right).\end{aligned} \tag{26}$$

Hence we have

$$\begin{aligned}&\sum_{n \leqslant x}\sigma(n) \\ &= \frac{1}{2}x^2 L_{-2}(x) - x\sum_{n \leqslant \frac{x}{k}}\frac{1}{n}\left\{\frac{x}{n}\right\} - x^2 L_{-2}(x) + x^2 L_{-2}\left(\frac{x}{k}\right) + x\sum_{d \leqslant k} L_1\left(\frac{x}{d}\right) \\ &- xkL_{-1}\left(\frac{x}{k}\right) + \frac{1}{2}\sum_{n \leqslant \frac{x}{k}}\left\{\frac{x}{n}\right\}^2 + \frac{1}{2}x^2 L_{-2}(x) - \frac{1}{2}x^2 L_{-2}\left(\frac{x}{k}\right) - x\sum_{d \leqslant k} L_1\left(\frac{x}{d}\right) \\ &+ xkL_{-1}\left(\frac{x}{k}\right) + \sum_{d \leqslant k} dL_0\left(\frac{x}{d}\right) - \frac{1}{2}\sum_{d \leqslant k} L_0\left(\frac{x}{d}\right) - \frac{1}{2}k^2 L_0\left(\frac{x}{k}\right) + \frac{1}{2}xL_{-1}(x)\end{aligned} \tag{27}$$

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$$-\frac{1}{2}\sum_{n\leqslant\frac{x}{k}}\left\{\frac{x}{n}\right\}-\frac{1}{2}xL_{-1}(x)+\frac{1}{2}xL_{-1}\left(\frac{x}{k}\right)+\frac{1}{2}\sum_{d\leqslant k}L_0\left(\frac{x}{d}\right)-\frac{1}{2}kL_0\left(\frac{x}{k}\right)$$
$$=\frac{1}{2}x^2L_{-2}\left(\frac{x}{k}\right)+\sum_{d\leqslant k}dL_0\left(\frac{x}{d}\right)+\frac{1}{2}xL_{-1}\left(\frac{x}{k}\right)-\frac{1}{2}k^2L_0\left(\frac{x}{k}\right)-\frac{1}{2}kL_0\left(\frac{x}{k}\right)$$
$$-x\sum_{n\leqslant\frac{x}{k}}\frac{1}{n}\left\{\frac{x}{n}\right\}+\frac{1}{2}\sum_{n\leqslant\frac{x}{k}}\left\{\frac{x}{n}\right\}\left(\left\{\frac{x}{n}\right\}-1\right).$$

In this last expression, we note that

(28)
$$\begin{aligned}\sum_{d\leqslant k}dL_0\left(\frac{x}{d}\right)=\sum_{d\leqslant k}d\left[\frac{x}{d}\right]&=xk-\sum_{d\leqslant k}d\left\{\frac{x}{d}\right\}\\&=xk-\sum_{d\leqslant k}d\hat{B}_1\left(\frac{x}{d}\right)-\frac{1}{2}\sum_{d\leqslant k}d\\&=xk-\frac{1}{4}k^2-\frac{1}{4}k-\sum_{d\leqslant k}d\hat{B}_1\left(\frac{x}{d}\right);\end{aligned}$$

and

(29)
$$\begin{aligned}&\frac{1}{2}x^2L_{-2}\left(\frac{x}{k}\right)-\frac{1}{2}k^2L_0\left(\frac{x}{k}\right)-\frac{1}{2}kL_0\left(\frac{x}{k}\right)\\&\quad=\frac{\pi^2}{12}x^2-\frac{1}{2}kx-\frac{1}{2}k^2\left\{\frac{x}{k}\right\}+\frac{1}{4}k^2+\frac{1}{2}x^2R_2\left(\frac{x}{k}\right)\\&\quad-\frac{1}{2}kx+\frac{1}{2}k^2\left\{\frac{x}{k}\right\}-\frac{1}{2}x+\frac{1}{2}k\left\{\frac{x}{k}\right\}\\&\quad=\frac{\pi^2}{12}x^2-kx+\frac{1}{4}k^2+\frac{1}{2}x^2R_2\left(\frac{x}{k}\right)-\frac{1}{2}x+\frac{1}{2}k\left\{\frac{x}{k}\right\};\end{aligned}$$

(30)
$$-x\sum_{n\leqslant\frac{x}{k}}\frac{1}{n}\left\{\frac{x}{n}\right\}+\frac{1}{2}xL_{-1}\left(\frac{x}{k}\right)=-x\sum_{n\leqslant\frac{x}{k}}\frac{1}{n}\hat{B}_1\left(\frac{x}{n}\right).$$

Thus we have

(31)
$$\sum_{n\leqslant x}\sigma(n)=\frac{\pi^2}{12}x^2-\sum_{n\leqslant\frac{x}{k}}\frac{1}{n}\hat{B}_1\left(\frac{x}{n}\right)-\sum_{d\leqslant k}d\hat{B}_1\left(\frac{x}{d}\right)-\frac{1}{2}x+T(x),$$

where we have, as required,

(32)
$$T(x)=\frac{1}{2}x^2R_2\left(\frac{x}{k}\right)+\frac{1}{2}k\left\{\frac{x}{k}\right\}-\frac{1}{4}k+\frac{1}{2}\sum_{n\leqslant\frac{x}{k}}\left\{\frac{x}{n}\right\}\left(\left\{\frac{x}{n}\right\}-1\right).$$

Now from (7), with $t = 2$, we have

$$\left|\frac{1}{2}x^2 R_2\left(\frac{x}{k}\right)\right| \leqslant \frac{1}{12}\frac{k^3}{x} + \frac{1}{20}\frac{k^4}{x^2} \leqslant \frac{1}{12}k + \frac{1}{20} \leqslant \frac{1}{12}\sqrt{x} + \frac{1}{20}, \tag{33}$$

$$\left|\frac{1}{2}k\hat{B}_1\left(\frac{x}{k}\right)\right| \leqslant \frac{1}{2}k \cdot \frac{1}{2} = \frac{1}{4}\sqrt{x} - \frac{1}{4}\{\sqrt{x}\}. \tag{34}$$

Because $|x^2 - x|$ has a maximum on $[0,1)$ at $x = \frac{1}{2}$, we have

$$\left|\frac{1}{2}\sum_{n\leqslant\frac{x}{k}}\{\tfrac{x}{n}\}\left(\{\tfrac{x}{n}\}-1\right)\right| \leqslant \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\left[\frac{x}{k}\right] \leqslant \frac{1}{8}\frac{x}{k} = \frac{1}{8}\sqrt{x} + \frac{1}{8}\{\sqrt{x}\} + \frac{1}{8}\frac{\{\sqrt{x}\}}{k}, \tag{35}$$

and the second result now follows. Finally, we note that $\frac{x}{k} < \sqrt{x} + 2$, so that

$$\begin{aligned}\left|\sum_{n\leqslant\frac{x}{k}}\frac{1}{n}\hat{B}_1\left(\frac{x}{n}\right)\right| &\leqslant \frac{1}{2}\sum_{n\leqslant\sqrt{x}+2}\frac{1}{n} \leqslant \frac{1}{2}\sum_{n\leqslant\sqrt{x}}\frac{1}{n} + \frac{1}{2}\frac{2}{\sqrt{x}} \\ &\leqslant \frac{1}{2}\left(\frac{1}{2}\log x + \gamma + \frac{1}{2\sqrt{x}} + \frac{1}{6}\frac{1}{x}\right) + \frac{1}{\sqrt{x}} \\ &= \frac{1}{4}\log x + \frac{1}{2}\gamma + \frac{5}{4}\frac{1}{\sqrt{x}} + \frac{1}{12}\frac{1}{x},\end{aligned} \tag{36}$$

$$\left|\sum_{d\leqslant k} d\hat{B}_1\left(\frac{x}{d}\right)\right| \leqslant \frac{1}{2}\sum_{d\leqslant k} d = \frac{1}{4}k^2 + \frac{1}{4}k \leqslant \frac{1}{4}x + \frac{1}{4}\sqrt{x}, \tag{37}$$

and thus

$$\begin{aligned}\left|\sum_{n\leqslant x}\sigma(n) - \frac{\pi^2}{12}x^2\right| &\leqslant \frac{1}{4}x\log x + \left(\frac{1}{2}\gamma + \frac{3}{4}\right)x + \frac{47}{24}\sqrt{x} + \left(\frac{1}{12} + \frac{1}{20}\right) \\ &\leqslant \frac{1}{4}x\log x + \left(\frac{1}{2}\gamma + \frac{3}{4}\right)x + 2\sqrt{x} \qquad \text{for all } x > 11,\end{aligned} \tag{38}$$

and it is easy to check that (38) holds for $1 \leqslant x \leqslant 11$. ∎

Note. It is known that (see Walfisz [**6**, p. 99])

$$\sum_{n\leqslant x}\sigma(n) = \frac{\pi^2}{12}x^2 + 0\left(x\log^{2/3}x\right),$$

or, equivalently,

$$\sum_{n\leqslant k}\frac{1}{n}\hat{B}_1\left(\frac{x}{n}\right) = 0\left(\log^{2/3}x\right).$$

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COROLLARY 6.1. *Let* $f(x)=\frac{1}{x^2}\sum_{n\leqslant x}\sigma(n)-\frac{\pi^2}{12}$. *Then* $f(x)$ *has its minimum value over the positive integers at* $x=23$. *(Its maximum is at* $x=1$ *and* $x=2$.*)*

PROOF: The second negative value of $f(x)$ is at 23; indeed,

$$f(23)=-0.007722\ldots.$$

From Theorem 2(iii), we see

$$|f(x)|\leqslant 0.007722, \qquad \text{for } x\geqslant 338.$$

It is a simple matter to check the result for $1\leqslant x\leqslant 337$. ∎

THEOREM 7. *Let* $a\geqslant 2$. *Then*

$$\left(\sum_{n\leqslant x}\sigma_a(n)-\frac{\zeta(a+1)}{a+1}x^{a+1}\right)\Big/x^{a+1}$$

has limit 0 (as $x\to\infty$*), is always positive for* $a=2,3$ *and 4, and is negative for at most a finite number of positive integers* x *for any fixed integer* $a\geqslant 5$.

PROOF: By Corollary 3.1, the limit is 0, and by Corollary 3.1 and Lemma 1(c),

$$\begin{aligned}\sum_{n\leqslant x}\sigma_a(n)&=\frac{1}{a+1}\left(\zeta(a+1)-\frac{1}{a}\cdot\frac{1}{x^a}+\frac{1}{2x^{a+1}}+R_{a+1}(x)\right)x^{a+1}\\&\quad+\frac{1}{a+1}\sum_{r=1}^{a+1}(-1)^r\binom{a+1}{r}x^{a+1-r}\sum_{d\leqslant x}\frac{1}{d^{a+1-r}}\hat{B}_r\left(\frac{x}{d}\right)\\&\quad-\frac{1}{a+1}B_{a+1}x.\end{aligned}\tag{39}$$

For $a=2$, this gives

$$\begin{aligned}\sum_{n\leqslant x}\sigma_2(n)-\frac{1}{3}\zeta(3)x^3&=-\frac{1}{6}x+\frac{1}{6}+\frac{1}{3}x^3R_3(x)-x^2\sum_{d\leqslant x}\frac{1}{d^2}\hat{B}_1\left(\frac{x}{d}\right)\\&\quad+x\sum_{d\leqslant x}\frac{1}{d^2}\hat{B}_2\left(\frac{x}{d}\right)-\frac{1}{3}\sum_{d\leqslant x}\hat{B}_3\left(\frac{x}{d}\right).\end{aligned}\tag{40}$$

Now, from Corollary 2.1,

$$-x^2\sum_{d\leqslant x}\frac{1}{d^2}\hat{B}_1\left(\frac{x}{d}\right)>0.3x^2,\tag{41}$$

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while, by Lemma 1 and (5),

$$
\begin{aligned}
(42)\qquad & \Big| -\frac{1}{6}x+\frac{1}{6}+\frac{1}{3}x^3R_3(x)+x\sum_{d\leqslant x}\frac{1}{d}\hat{B}_2\Big(\frac{x}{d}\Big)-\frac{1}{3}\sum_{d\leqslant x}\hat{B}_3\Big(\frac{x}{d}\Big)\Big| \\
& \leqslant \frac{1}{6}x+\frac{1}{6}+\frac{1}{6}\frac{1}{x}+\frac{1}{6}x\log x+\frac{1}{6}x\gamma+\frac{1}{12}+\frac{1}{36}\frac{1}{x}+\frac{1}{3}\frac{\sqrt{3}}{36}\frac{1}{x} \\
& \leqslant 0.3x^2 \qquad \text{for } x\geqslant 2.
\end{aligned}
$$

For $a\geqslant 2$, we have

$$
(43)\qquad \sum_{n\leqslant x}\sigma_a(n)-\frac{1}{a+1}\zeta(a+1)x^{a+1}=-x^a\sum_{d\leqslant x}\frac{1}{d^a}\hat{B}_1\Big(\frac{x}{d}\Big)+Q_a(x),
$$

where we have

$$
\begin{aligned}
(44)\qquad Q_a(x)=&\frac{1}{a+1}\sum_{r=2}^{a+1}(-1)^r\binom{a+1}{r}x^{a+1-r}\sum_{d\leqslant x}\frac{1}{d^{a+1-r}}\hat{B}_r\Big(\frac{x}{d}\Big) \\
&-\frac{1}{a(a+1)}+\frac{B_{a+1}}{a+1}x+\frac{1}{2(a+1)}+\frac{1}{a+1}R_{a+1}(x)x^{a+1}.
\end{aligned}
$$

By Corollary 2.1,

$$
(45)\qquad -x^a\sum_{d\leqslant x}\frac{1}{d^a}\hat{B}_1\Big(\frac{x}{d}\Big)\geqslant\Big(\frac{1}{2}-\frac{1}{2}\frac{1}{3^a}-\frac{1}{2^a(2^{a-1}-1)}\Big)x^a.
$$

Since the terms of $Q_a(x)$ are of lower order (of order x^{a-1}), except possibly for small values of x we have that the right side of (43) is positive. It is easy enough to check small values of a. ∎

Note. These results do not hold for real x. Here, $\sum_{n\leqslant x}\sigma_2(n)-\frac{1}{3}\zeta(3)x^3$ is a decreasing function of x between integers; thus, in effect, one must examine $\sum_{n\leqslant x}\sigma_2(n)-\frac{1}{3}\zeta(3)(x+1)^3$;

$$
\sum_{n\leqslant x}\sigma_2(n)-\frac{1}{3}\zeta(3)(x+1)^3=\Big(\sum_{n\leqslant x}\sigma_2(n)-\frac{1}{3}\zeta(3)x^3\Big)-\zeta(3)x^2-\zeta(3)x-\frac{1}{3}\zeta(3),
$$

and the term $-\zeta(3)x^2$ now dominates $-x^2\sum_{d\leqslant x}\frac{1}{d^2}\hat{B}_1\big(\frac{x}{d}\big)$.

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Theorem 8. *Let A_{-1}, R_1 and W_{-1} be as defined in Lemma 1 and U as defined in Corollary 5.4; let $k = [\sqrt{x}]$. Then we have*

$$\sum_{n \leqslant x} \frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma \log x + W(x),$$

where we have, for $x \geqslant 1$:

$$\text{(i)} \qquad W(x) = \gamma^2 + \gamma - \frac{1}{2} - A_{-1} + \sum_{n \leqslant x} \frac{1}{n} R_1\Big(\frac{x}{n}\Big) - \gamma \frac{\hat{B}_1(x)}{x} + (\log x + \gamma) R_1(x) - W_{-1}(x) - \frac{U(x)}{x} - \frac{1}{2}\frac{\{x\}}{x};$$

$$\text{(ii)} \qquad 0.35 < W(x) < 0.60, \quad \textit{for } x \geqslant 2000;$$

$$\text{(iii)} \qquad \Big(\sum_{n \leqslant x} \frac{d(n)}{n} - \frac{1}{2}\log^2 x - 2\gamma \log x\Big) \Big/ \Big(\frac{1}{2}\log^2 x + 2\gamma\log x\Big)$$

is always positive, and goes to 0.

Proof:

$$\begin{aligned} \sum_{n \leqslant x} \frac{d(n)}{n} &= \sum_{n \leqslant x} \frac{1}{n} \sum_{m \mid n} 1 = \sum_{mm' \leqslant x} \frac{1}{mm'} = \sum_{n \leqslant x} \frac{1}{n} \sum_{m \leqslant \frac{x}{n}} \frac{1}{m} \\ &= \sum_{n \leqslant x} \frac{1}{n} L_{-1}\Big(\frac{x}{n}\Big) \qquad (46) \\ &= \sum_{n \leqslant x} \frac{1}{n} \Big(\log x + \gamma - \log n - \frac{n}{x} \hat{B}_1\Big(\frac{x}{n}\Big) + R_1\Big(\frac{x}{n}\Big)\Big). \end{aligned}$$

Result (i) now follows from Lemma 1 and Corollary 5.4. The constant A_{-1} can be evaluated by the method of ([4], p. 202); its value to six decimals is 0.0728158. Thus, to six decimals, $\gamma^2 + \gamma - \frac{1}{2} - A_{-1}$ has the value 0.483210. Now,

$$\Big| \sum_{n \leqslant x} \frac{1}{n} R_1\Big(\frac{x}{n}\Big) \Big| \leqslant \frac{1}{6x^2} \sum_{n \leqslant x} n \leqslant \frac{1}{12} + \frac{1}{12}\frac{1}{x};$$

$$\Big| -\gamma \frac{\hat{B}_1(x)}{x} + (\log x + \gamma) R_1(x) - W_{-1}(x) - \frac{U(x)}{x} - \frac{1}{2}\frac{\{x\}}{x} \Big| \leqslant \frac{1}{\sqrt{x}} + \Big(\frac{1}{2}\gamma + \frac{3}{2}\Big)\frac{1}{x} + \frac{1}{3}\frac{\log x}{x^2};$$

so that, for $x \geqslant 2000$, $W(x)$ differs from 0.483210 by less than 0.1067. To prove result (iii) it suffices to check $1 \leqslant x \leqslant 2000$. ∎

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Note. It is known (see, for example, ([4], p. 223)) that there exists a constant c_{-1} such that

$$\sum_{n\leqslant x}\frac{d(n)}{n}=\frac{1}{2}\log^2 x+2\gamma\log x+c_{-1}-\frac{2}{x}\sum_{d\leqslant k}d\hat{B}_2\Big(\frac{x}{d}\Big)+0\Big(\frac{1}{x}\Big).$$

Thus, there is the deeper question of the sign changes, or extrema, of

$$\Big(\sum_{n\leqslant x}\frac{d(n)}{n}-\frac{1}{2}\log^2 x-2\gamma\log x-c_{-1}\Big)\Big/\Big(\frac{1}{2}2\log^2 x+2\gamma\log x+c_{-1}\Big).$$

THEOREM 9. *Let R_1, R_2 and Q be defined as in Lemma 1; let $k=[\sqrt{x}]$. Then we have*

$$\sum_{n\leqslant x}\frac{\sigma(n)}{n}=\frac{\pi^2}{6}x-\frac{1}{2}\log x-\sum_{n\leqslant \frac{x}{k}}\frac{1}{n}\hat{B}_1\Big(\frac{x}{n}\Big)-\frac{1}{2}(\log 2\pi+\gamma)$$
$$-\frac{1}{x}\sum_{d\leqslant k}d\hat{B}_1\Big(\frac{x}{d}\Big)+V(x),$$

where we have for $x\geqslant 1$:
(i) $V(x)=xR_2\big(\frac{x}{k}\big)+\frac{1}{2}\frac{k}{x}\hat{B}_1\big(\frac{x}{k}\big)-\big(k+\frac{1}{2}\big)R_1\big(\frac{x}{k}\big)-\frac{1}{12}\frac{1}{k}-Q(k)+\sum_{d\leqslant k}R_1\big(\frac{x}{d}\big)$;
(ii) $|V(x)|\leqslant\frac{4}{3}\frac{1}{k}$;
(iii) $\Big(\sum_{n\leqslant x}\frac{\sigma(n)}{n}-\frac{\pi^2}{6}x\Big)/\frac{\pi^2}{6}x$ *is always negative, and goes to 0.*

PROOF: Putting $a=t=1$ in Lemma 3, we have

$$\sum_{n\leqslant x}\frac{\sigma(n)}{n}=\sum_{n\leqslant x}\frac{1}{n}\Big[\frac{x}{n}\Big]=x\sum_{n\leqslant x}\frac{1}{n^2}-\sum_{n\leqslant x}\frac{1}{n}\Big\{\frac{x}{n}\Big\}. \tag{47}$$

To prove result (iii), we re-write the right side of (47):

$$\sum_{n\leqslant x}\frac{\sigma(n)}{n}=\frac{\pi^2}{6}x-x\sum_{n>x}\frac{1}{n^2}-\sum_{n\leqslant x}\frac{1}{n}\Big\{\frac{x}{n}\Big\},$$

and the result is now obvious. On the other hand, from (47) and (24) we have

$$\sum_{n\leqslant x}\frac{\sigma(n)}{n}=xL_{-2}\Big(\frac{x}{k}\Big)-\sum_{n\leqslant\frac{x}{k}}\frac{1}{n}\hat{B}_1\Big(\frac{x}{n}\Big)-\frac{1}{2}L_{-1}\Big(\frac{x}{k}\Big)+\sum_{d\leqslant k}L_{-1}\Big(\frac{x}{d}\Big)-kL_{-1}\Big(\frac{x}{k}\Big).$$

Result (ii) now follows from Lemma 1; result (iii) follows from the estimates in Lemma 1. ∎

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Note. It is known (Walfisz ([6], p. 99)) that

$$\sum_{n\leqslant x}\frac{\sigma(n)}{n}=\frac{\pi^2}{6}x-\frac{1}{2}\log x+0\left(\log^{2/3}x\right).$$

Thus, there is the deeper question of the sign changes, or extrema, of

$$\left(\sum_{n\leqslant x}\frac{\sigma(n)}{n}-\left(\frac{\pi^2}{6}x-\frac{1}{2}\log x\right)\right)\Big/\left(\frac{\pi^2}{6}x-\frac{1}{2}\log x\right).$$

THEOREM 10. *Let $a\geqslant 2$. Then $\left(\sum_{n\leqslant x}\frac{\sigma_a(n)}{n}-\frac{\zeta(a+1)}{a}x^a\right)/x^a$ has limit 0 (as $x\to\infty$), is always positive for $a=2$, 3 and 4, and is negative for at most a finite number of positive integers x for any fixed integer $a\geqslant 5$.*

PROOF: By Corollary 3.1, with $t=1$, we have for $a\geqslant 2$

$$\sum_{n\leqslant x}\frac{\sigma_a(n)}{n}=\frac{1}{a}x^a\sum_{d\leqslant x}\frac{1}{d^{a+1}}-x^{a-1}\sum_{d\leqslant x}\frac{1}{d^a}\hat{B}_1\left(\frac{x}{d}\right)$$
$$+\frac{1}{a}\sum_{r=2}^{a}(-1)^r x^{a-r}\sum_{d\leqslant x}\frac{1}{d^{a+1-r}}\hat{B}_r\left(\frac{x}{d}\right)\qquad-\frac{B_a}{a}\sum_{d\leqslant x}\frac{1}{d}.$$

The argument now proceeds as in Theorem 7. ∎

Note. Had our object been solely to obtain our extreme values, then weaker and more easily proven estimates would have sufficed. For example, one easily sees that

$$\left|\sum_{n\leqslant x}\sigma(n)-\frac{\pi^2x^2}{12}\right|<\frac{1}{2}x\log x+x+\frac{1}{2}.$$

For we have

$$\sum_{n\leqslant x}\sigma(n)=\sum_{n\leqslant x}\frac{1}{2}\left[\frac{x}{n}\right]\left(\left[\frac{x}{n}\right]+1\right)\leqslant\frac{1}{2}\left(x^2\sum_{n\leqslant x}\frac{1}{n^2}+x\sum_{n\leqslant x}\frac{1}{n}\right)$$
$$\leqslant\frac{\pi^2x^2}{12}+\frac{x\log x}{2}+\frac{x}{2},$$

and also

$$\sum_{n\leqslant x}\sigma(n)\geqslant\sum_{n\leqslant x}\frac{1}{2}\frac{x}{n}\left(\frac{x}{n}-1\right)\geqslant\frac{1}{2}\left(x^2\sum_{n\leqslant x}\frac{1}{n^2}-x\sum_{n\leqslant x}\frac{1}{n}\right)$$
$$\geqslant\frac{\pi^2x^2}{12}-\frac{x^2}{2}\left(\sum_{n>x}\frac{1}{n^2}\right)-\frac{x\log x}{2}-\frac{x}{2}$$
$$\geqslant\frac{\pi^2x^2}{12}-\frac{x\log x}{2}-x-\frac{1}{2}.$$

We now obtain Corollary 6.1 by observing that $|f(x)|<\cdot 007722$ for $x>600$.

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Department of Mathematics
University of Victoria
P.O. Box 1700
Victoria, B.C.
Canada V8W 2Y2