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PROJECTIVE DYNAMICS OF HOMOGENEOUS SYSTEMS: LOCAL INVARIANTS, SYZYGIES AND THE GLOBAL RESIDUE THEOREM

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Abstract We give an explicit formula for the projective dynamics of planar homogeneous polynomial differential systems in terms of natural local invariants and we establish explicit algebraic connections (syzygies) between these invariants (leading to restrictions on possible global dynamics). We discuss multidimensional generalizations together with applications to the existence of first integrals and bounded solutions.

Keywords: homogeneous polynomial systems; projective dynamics; syzygies; local invariants; Global Residue Theorem; Euler–Jacobi Theorem

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1. Introduction

1.1. Motivating example

To formulate the goal of this note, let us start with a simple setting where the main ideas are demonstrated clearly. A (real) polynomial $P_m : \mathbb{R}^2 \to \mathbb{R}$ is called m-homogeneous if $P_m(\lambda x, \lambda y) = \lambda^m P_m(x, y)$ for all $x, y \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$ and some integer $m > 0$.

Theorem 1.1. *Let $V := \{P_m, Q_m\}$ be a planar m-homogeneous polynomial vector field. Consider a differential system*

$$\dot{x} = P_m(x, y), \qquad \dot{y} = Q_m(x, y). \tag{1.1}$$

Assume the field V admits exactly $m + 1$ non-proportional complex fixed points (x_i, y_i), $i = 1, \ldots, m + 1$ (i.e. $P_m(x_i, y_i) = x_i$ and $Q_m(x_i, y_i) = y_i$). Suppose (without loss of generality) that the so-called characteristic polynomial

$$S_{m+1}(z) := Q_m(1, z) - zP_m(1, z) \tag{1.2}$$

has degree $m + 1$.

Then, the 'projective' dynamics of (1.1) depending on $z(t) := y(t)/x(t)$ is given explicitly by

$$\dot{z} = \text{const.} \times S_{m+1}(z) \prod_{i=1}^{m+1} (z - z_i)^{\gamma_i}, \quad z_i = y_i/x_i, \tag{1.3}$$

where

$$\gamma_i = (m-1)^2 / \det[D(V - \mathrm{Id})]|_{x=x_i,\, y=y_i}, \tag{1.4}$$

with $D(\cdot)$ denoting the Jacobi matrix and Id *the identity operator.*

It turns out that the numbers x_i, y_i and γ_i appearing in Theorem 1.1 cannot take arbitrary values (cf. [**9**], where the case $m = 2$ was considered).

Theorem 1.2. *Let x_i, y_i and γ_i, $i = 1, \dots, m+1$, be as in Theorem 1.1 and let $m > 1$. Then, the following $m + 1$ identities hold:*

$$\sum_{i=1}^{m+1} \gamma_i + m - 1 = 0, \qquad \sum_{i=1}^{m+1} \gamma_i x_i^{m-j-1} y_i^j = 0, \quad j = 0, \dots, m-1. \tag{1.5}$$

The following example illustrates Theorems 1.1 and 1.2.

Example 1.3. Consider the 2-homogeneous system

$$\dot{x} = P_2(x, y) := x^2 - y^2, \qquad \dot{y} = Q_2(x, y) := 2xy. \tag{1.6}$$

Since (1.6) is a quadratic system in $\mathbb{C}$, it admits the explicit integration

$$x(t) = \frac{x_0 - (x_0^2 + y_0^2)t}{(1 - x_0 t)^2 + y_0^2 t^2}, \qquad y(t) = \frac{y_0}{(1 - x_0 t)^2 + y_0^2 t^2}, \tag{1.7}$$

implying the following projective dynamics

$$z(t) = \frac{y_0}{x_0 - (x_0^2 + y_0^2)t}, \qquad \dot{z} = \frac{y_0(x_0^2 + y_0^2)}{(x_0 - (x_0^2 + y_0^2)t)^2} = \text{const.} \times z^2. \tag{1.8}$$

On the other hand, the fixed points (x_i, y_i) for (1.6) are $(1, 0)$, $(\frac{1}{2}, \frac{1}{2}\mathrm{i})$ and $(\frac{1}{2}, -\frac{1}{2}\mathrm{i})$. Also, $S_3(z) = z(1 + z^2)$ (cf. (1.2)); therefore, $z_1 = 0$, $z_{2,3} = \pm\mathrm{i}$. In addition, $\det(D(V - \mathrm{Id})) = (2x - 1)^2 + 4y^2$ and $\gamma_1 = 1$, $\gamma_{2,3} = -1$ (cf. (1.4)). Therefore (cf. (1.8)),

$$\dot{z} = \text{const.} \times (z - 0)^{1+1} (z - \mathrm{i})^{1+(-1)} (z + \mathrm{i})^{1+(-1)} := \text{const.} \times z^2,$$

Finally, the following identities hold:

$$\sum_{i=1}^{3} \gamma_i + m - 1 = 1 + (-1) + (-1) + 1 = 0,$$

$$\sum_{i=1}^{3} \gamma_i x_i = 1 + (-1)\tfrac{1}{2} + (-1)\tfrac{1}{2} = 0,$$

$$\sum_{i=1}^{3} \gamma_i y_i = (-1)\tfrac{1}{2}\mathrm{i} + (-1) - \tfrac{1}{2}\mathrm{i} = 0.$$

1.2. General framework behind Theorems 1.1 and 1.2

We now analyse the setting and conclusions of Theorems 1.1 and 1.2.

It follows from Theorem 1.1 that in the considered case the projective dynamics of planar homogeneous ordinary differential equations (ODEs) is *completely* determined by the invariants of exactly two types:

(a) *equilibrium points* $z_i = x_i/y_i$ of the 'projective' polynomial (1.2) (vector invariant);

(b) *Jacobians* of V at the points (x_i, y_i) that are fundamental quantities characterizing the behaviour of the solutions around their singularities.

It should be pointed out that, generally speaking, there are two different sources of zeros of (1.2), namely: *fixed* and *equilibrium* points of V. Being generic, the setting described in Theorem 1.1 allows fixed points of V only. For the same reason, it is assumed that V admits only finitely many fixed points and all of them are not multiple.

On the other hand, the main essence of Theorem 1.2 is as follows: the invariants in question coming together follow a pattern consistent with a specific rule. In the invariant theory, this fact is known as the existence of *syzygy*. Summing up, we arrive at the following.

Problem 1.4. What are the n-dimensional analogues of Theorems 1.1 and 1.2 ($n \geqslant 2$) covering non-generic cases?

Remark 1.5. In particular, Problem 1.4 can be regarded as a classification of singular points of homogeneous maps (including algorithms for the 'singularity resolution' in the case of multiple roots).

The *goal* of this paper is twofold:

(i) to prove analogues of Theorems 1.1 and 1.2 covering non-generic settings (in particular, systems with non-zero equilibria) in the two-dimensional case;

(ii) to extend Theorem 1.2 to the generic setting in the n-dimensional case ($n \geqslant 2$).

The results obtained are illustrated by two examples related to the explicit construction of algebraic first integrals and the existence of bounded solutions to homogeneous systems.

1.3. Tools

The Multidimensional Residue Theory and Euler–Jacobi Formula (see, for example, [**1**, **6**, **7**]) combined with Bézout's Theorem on the number of solutions to homogeneous polynomial systems (see, for example, [**13**]) are the main techniques used in this paper.

1.4. Overview

The paper is organized as follows. In § 2, we give the proofs of Theorems 1.1 and 1.2. In § 3, we extend these theorems to cover non-generic situations. The results obtained are applied to studying the existence of the algebraic first integral of the corresponding m-homogeneous differential systems (see § 3.4). Section 4 is devoted to n-dimensional generalizations. Some conjectures, links and applications are discussed in § 5.

2. The two-dimensional case: generic situation

Proof of Theorem 1.1. Using m-homogeneity (see also (1.2)), it follows from (1.1) that

$$\dot{z} = x^{m-1} S_{m+1}(z). \tag{2.1}$$

Hence,

$$\ddot{z} = x^{2(m-1)} S_{m+1}(z)\{(m-1)P_m(1,z) + S_{m+1}(z)'_z\}. \tag{2.2}$$

Combining (2.1) and (2.2) yields

$$\frac{\ddot{z}}{\dot{z}^2} = \frac{(m-1)P_m(1,z) + S_{m+1}(z)'_z}{S_{m+1}(z)}. \tag{2.3}$$

Multiplying (2.3) by $\dot{z}$ and integrating both parts with respect to t implies

$$\ln(\dot{z}) = \ln(S_{m+1}(z)) + \int \frac{(m-1)P_m(1,z)}{S_{m+1}(z)}\,\mathrm{d}z. \tag{2.4}$$

By assumption,

$$S_{m+1}(z) = \prod_{i=1}^{m+1}(z - z_i), \quad \text{where } z_i \neq z_j \text{ if } i \neq j. \tag{2.5}$$

Therefore,

$$\frac{(m-1)P_m(1,z)}{S_{m+1}(z)} = \sum_{i=1}^{m+1} \frac{\gamma_i}{z - z_i}, \tag{2.6}$$

where

$$\gamma_i = \lim_{z \to z_i} \frac{(m-1)P_m(1,z)(z - z_i)}{S_{m+1}(z)}, \quad i = 1, \dots m+1. \tag{2.7}$$

Combining (2.4) with (2.5) and (2.6) yields (1.3).

Let us show that the numbers γ_i satisfy (1.4). We have

$$\det[D(V) - \mathrm{Id}] = \det\begin{pmatrix} \partial_x P_m(x,y) - 1 & \partial_y P_m(x,y) \\ \partial_x Q_m(x,y) & \partial_y Q_m(x,y) - 1 \end{pmatrix}. \tag{2.8}$$

Using the Euler Formula for homogeneous functions, one can exclude partial differentiation with respect to x in (2.8):

$$\partial_x P_m(x,y) = \frac{1}{x}[m P_m(x,y) - y \partial_y P_m(x,y)], \tag{2.9}$$

$$\begin{aligned}\partial_x P_m(x,y)\partial_y Q_m(x,y) &- \partial_y P_m(x,y)\partial_x Q_m(x,y) \\ &= \frac{m}{x}[P_m(x,y)\partial_y Q_m(x,y) - Q_m(x,y)\partial_y P_m(x,y)]. \end{aligned} \tag{2.10}$$

Combining (2.8)–(2.10) with the assumption that $P_m(x_i, y_i) = x_i$ and $Q_m(x_i, y_i) = y_i$ and using the notation $z_i = x_i/y_i$, one obtains

$$\begin{aligned}\det[D(V-\mathrm{Id})]|_{x=x_i,\,y=y_i} &= m[\partial_y Q_m(x_i,y_i) - z_i\partial_y P_m(x_i,y_i)] \\ &\quad - \frac{1}{x_i}[mP_m(x_i,y_i) - y_i\partial_y P_m(x_i,y_i)] - \partial_y Q_m(x_i,y_i) + 1 \\ &= (m-1)[\partial_y Q_m(x_i,y_i) - z_i\partial_y P_m(x_i,y_i) - 1] \\ &= (m-1)x_i^{m-1}[Q_m(1,z_i) - z_iP_m(1,z_i)]'_z \\ &= (m-1)x_i^{m-1}S_{m+1}(z_i)'_z,\end{aligned}$$

i.e.

$$S_{m+1}(z_i)'_z = \frac{\det[D(V-\mathrm{Id})]}{(m-1)x^{m-1}}\bigg|_{x=x_i,y=y_i}. \tag{2.11}$$

Finally, combining (2.11) with (2.7) and using $x_i^{m-1}P_m(1, z_i) = 1$, one obtains

$$\gamma_i = \frac{(m-1)P_m(1,z)}{S_{m+1}(z_i)'_z}\bigg|_{z=z_i} = \frac{(m-1)^2}{\det(D(V-\mathrm{Id}))}\bigg|_{x=x_i,\,y=y_i}, \tag{2.12}$$

and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Take the functions

$$f(z) := \frac{(m-1)P_m(1,z)}{S_{m+1}(z)}, \qquad f_j(z) := \frac{z^j}{S_{m+1}(z)}, \quad j = 0,\dots,m-1$$

(considered as functions in complex variables; (cf. (1.2))). Using the standard formula for residue at infinity (see, for example, [**11**]), one obtains

$$\mathrm{Res}(f(z),\infty) = \mathrm{Res}\left(\frac{-(m-1)P_m(z,1)}{z(zQ_m(z,1)-P_m(z,1))}, 0\right) = m-1, \tag{2.13}$$

$$\mathrm{Res}(f_j(z),\infty) = \mathrm{Res}\left(\frac{z^{m-j-1}}{zQ_m(z,1)-P_m(z,1)}, 0\right) = 0. \tag{2.14}$$

Also, by (2.11), (2.12),

$$\mathrm{Res}(f(z),z_i) = \gamma_i, \qquad \mathrm{Res}(f_j(z),z_i) = \frac{z^j}{S_{m+1}(z)'}\bigg|_{z=z_i} = \frac{\gamma_i x^{m-j-1}y^j}{m-1}. \tag{2.15}$$

Combining (2.13)–(2.15) with the Global Residue Theorem (see, for example, [**11**]) yields syzygies (1.5). □

3. The two-dimensional case: extensions and application

In this section, we consider planar m-homogeneous maps that can have both fixed and equilibrium points. Clearly, if (x_o, y_o) is an equilibrium point of an m-homogeneous map, then so is any point $(\lambda x_o, \lambda y_o)$, $\lambda \in \mathbb{R}$.

3.1. Simple fixed and equilibrium points

Theorem 3.1. *Using the notation of Theorem 1.1, assume that V admits exactly k non-proportional complex fixed points (x_i, y_i), $i = 1, \dots, k$. Assume also that V admits exactly $m + 1 - k$ straight lines of equilibrium points $y = z_i x$, $i = k + 1, \dots, m + 1$. Suppose that the polynomial $S_{m+1}(z)$ (see (1.2)) has degree $m + 1$. Then*

(i) *the 'projective' dynamics of (1.1) depending on $z(t) := y(t)/x(t)$ is given explicitly by*

$$\dot{z} = \text{const.} \times S_{m+1}(z) \prod_{i=1}^{k} (z - z_i)^{\gamma_i}, \tag{3.1}$$

where γ_i, $i = 1, \dots, k$, are defined by (1.4),

(ii) *the $m + 1$ syzygies*

$$\sum_{i=1}^{k} \gamma_i + m - 1 = 0, \tag{3.2}$$

$$\sum_{i=1}^{k} \gamma_i x_i^{m-j-1} y_i^j + \sum_{i=k+1}^{m+1} \beta_i x_i^{m-j-1} y_i^j = 0, \quad 0 \leqslant j \leqslant m - 1, \tag{3.3}$$

hold, where γ_i, $i = 1, \dots, k$, are defined by (1.4), and $\beta_i := 1/\operatorname{tr}[D(V)]|_{x=x_i,\, y=y_i}$, $i = k + 1, \dots, m + 1$.

Proof. Up to slight modifications, we follow the same lines as the proof of Theorems 1.1 and 1.2.

(i) Set $\omega(x, y) := \text{GCD}\{P_m(x, y), Q_m(x, y)\}$. Then

$$\{P_m(x, y), Q_m(x, y)\} = \omega(x, y)\{\tilde{P}_m(x, y)\tilde{Q}_m(x, y)\} \tag{3.4}$$

with

$$\omega(x, y) = \prod_{i=k+1}^{m+1} (y - z_i x). \tag{3.5}$$

By assumption, (2.1)–(2.6) are still valid, while (cf. (3.4), (3.5) and (2.4)–(2.6)) (2.7) should be replaced with

$$\gamma_i = \begin{cases} \lim_{z \to z_i} \dfrac{(m-1)P_m(1, z)(z - z_i)}{S_{m+1}(z)}, & i = 1, \dots, k, \\ 0, & i = k + 1, \dots, m + 1. \end{cases} \tag{3.6}$$

Using the same argument as in the proof of Theorem 1.1, one can combine (3.6) with (2.11) and $x_i^{m-1} P_m(1, z_i) = 1$, $i = 1, \dots, k$, to obtain (3.1).

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(ii) To prove the second part of the theorem, consider the functions

$$f(z) := \frac{(m-1)\tilde{P}_m(1,z)}{\tilde{Q}_m(1,z) - z\tilde{P}_m(1,z)}, \qquad f_j(z) := \frac{z^j}{S_{m+1}(z)}, \quad j = 0, \dots, m-1$$

(cf. (1.2), (3.4), (3.5)). Clearly, $\operatorname{Res}(f(z), \infty)$, $\operatorname{Res}(f_j(z), \infty)$, as well as $\operatorname{Res}(f(z), z_i)$ and $\operatorname{Res}(f_j(z), z_i)$ for $i = 1, \dots, k$, are the same as in Theorem 1.1. At the same time, $\operatorname{Res}(f(z), z_i)$ and $\operatorname{Res}(f_j(z), z_i)$ for $i = k+1, \dots, m+1$ should be recalculated. We have

$$\operatorname{Res}(f(z), z_i) = 0, \tag{3.7}$$

$$\operatorname{Res}(f_j(z), z_i) = \lim_{z \to z_i} \frac{z^j (z - z_i)}{S_{m+1}(z)} = \frac{z_i^j}{S_{m+1}(z_i)'_z}, \tag{3.8}$$

where $i = k+1, \dots, m+1$. On the other hand,

$$\begin{aligned} \operatorname{tr}[D(V(x,y))] &= \partial_x P_m(x,y) + \partial_y Q_m(x,y) \\ &= \frac{m}{x} P_m(x,y) - z\partial_y P_m(x,y) + \partial_y Q_m(x,y) \\ &= x^{m-1}[(m-1)P_m(1,z) + S_{m+1}(z_i)'_z]. \end{aligned}$$

Since $P_m(x_i, y_i) = 0$ for $i = k+1, \dots, m+1$, the above formula yields

$$S_{m+1}(z_i)'_z = x_i^{1-m} \operatorname{tr}[D(V(x,y))]|_{x=x_i,\, y=y_i} \tag{3.9}$$

($i = k+1, \dots, m+1$). Combining (3.7)–(3.9) with formulae for $\operatorname{Res}(f, \infty)$, $\operatorname{Res}(f_j, \infty)$, $\operatorname{Res}(f, x_i)$, $\operatorname{Res}(f_j, x_i)$ for $i = 1, \dots, k$ and using the Global Residue Theorem (see, for example, [**11**]) yields statement (ii). □

3.2. Multiple roots

Suppose the polynomial (1.2) admits a root $z_i = y_i/x_i$ of multiplicity $m_i > 1$. Then, the following possibilities may occur:

(I) (x_i, y_i) is a fixed point of V of multiplicity m_i;

(II) (x_i, y_i) is an equilibrium point of V of multiplicity m_i;

(III) (x_i, y_i) is an equilibrium point of V of multiplicity $< m_i$ and, at the same time, (x_i, y_i) is a fixed point of the vector field $(\tilde{P}_m, \tilde{Q}_m)$ (see (3.4) and (3.5)).

The following example illustrates case (III).

Example 3.2. Let $V(x,y) = (xy - y^2, y^2)$. Then, $S_3(z) = z^3$ admits the root $z = 0 = 0/1$ of multiplicity 3. However, $(1,0)$ is the equilibrium point of V of multiplicity 1 only.

In order to study the projective dynamics of system (1.1) as well as to establish the corresponding syzygies in the presence of multiple roots of (1.2), one can follow the same lines as in the proof of Theorems 1.1, 1.2 and 3.1. In particular, (2.1)–(2.4) are still valid, while the next argument essentially depends on the computation of the integral in (2.4). This integral can be easily evaluated by combining the standard fraction decomposition algorithm with the multiple residue formula (see, for example, [**11**]). However, providing a 'smart' interpretation of the quantities obtained in terms of natural invariants of the field V requires additional work. This problem is studied in detail in our forthcoming paper [**10**]. We restrict ourselves below to a statement that does not need any additional explanation (see also [**5**], where a generalization of the Euler–Jacobi Formula was established for *double* stationary points of polynomial (in general, non-homogeneous) fields).

Proposition 3.3.

(i) *Using the notation of Theorem 1.1, assume V admits exactly k non-proportional complex fixed points (x_i, y_i), $i = 1, \dots, k$. Suppose that the characteristic polynomial $S_{m+1}(z)$ (see (1.2)) has degree $m+1$. Assume also that all the equilibria of V (possibly multiple) are of type (II). Then, the projective dynamics of (1.1) is the same as in (3.1) and, in addition, syzygy (3.2) remains unchanged.*

(ii) *Let $z = z_0$ be a root of the characteristic polynomial (1.2) of multiplicity $k+1$. Then, in the formula for the projective dynamics (cf. (3.1)) an additional factor*

$$\exp\left[\frac{p_k(z)}{(z-z_0)^k}\right] \tag{3.10}$$

appears, where $p_k(z)$ is a polynomial of degree k.

3.3. Infinitely many fixed points

Assume that in the decomposition (3.4) (see also (3.5)) one has $\tilde{P}_m(x, y) \equiv x$ and $\tilde{Q}_m(x, y) \equiv y$. Clearly, in this case $S_{m+1}(z) \equiv 0$. Since $\omega(x, y)$ is a polynomial of degree at most $m-1$, the field V may have at most $m-1$ straight lines of equilibria. It remains to observe that any other direction is generated by a fixed point of the field V that determines the obvious dynamics of (1.1).

3.4. Application: existence of first integrals

In this subsection we apply the results obtained above to study algebraic first integrals for planar homogeneous systems.

Definition 3.4. A polynomial $F(x, y)$ is called a *(polynomial) first integral* for (1.1) if

$$P_m(x, y)\partial_x F(x, y) + Q_m(x, y)\partial_y F(x, y) = 0 \quad \text{for all } (x, y) \in \mathbb{R}^2. \tag{3.11}$$

Combining (3.11) with (3.4) yields that, to study the existence of a polynomial first integral, one can assume without loss of generality that the field $V = (P_m, Q_m)$ does not have non-trivial equilibria. Also, since any polynomial is a sum of homogeneous polynomials, without loss of generality, F in (3.11) can be assumed to be M-homogeneous for some M.

Theorem 3.5 (Tsygvintsev [15]). *Assume the field $V = (P_m, Q_m)$ does not have non-trivial equilibria ($m > 1$). Then, (1.1) admits an M-homogeneous polynomial first integral (for some integer M) if and only if the following two conditions are satisfied:*

(i) *all fixed points of V are simple (in particular, the hypotheses of Theorem 1.1 are satisfied);*

(ii) *all γ_i provided by Theorem 1.1 (see (1.4)) are negative rational numbers.*

Proof. Since F is supposed to be M-homogeneous for some M, the Euler Formula implies

$$P_m(x,y)\left[\frac{M}{x}F(x,y) - \frac{y}{x}\partial_y F(x,y)\right] + Q_m(x,y)\partial_y F(x,y) = 0,$$

meaning that, for some M, F must satisfy the following differential equation with respect to $z = y/x$:

$$\frac{F'_z(1,z)}{F(1,z)} = -\frac{MP_m(1,z)}{Q_m(1,z) - zP_m(1,z)} = -\frac{MP_m(1,z)}{S_{m+1}(z)}. \tag{3.12}$$

Thus, the existence of a required first integral essentially depends on the integration of the right-hand side of (3.12).

Assume conditions (i) and (ii) are satisfied. Then, combining (i) with (3.12) and (2.6) yields an explicit formula for $F(1,z)$:

$$F(1,z) = C\prod_{i=1}^{m+1}(z - z_i)^{M\gamma_i/(1-m)}, \tag{3.13}$$

depending on a parameter $M \geqslant 1$. Choosing an appropriate M and using (ii), the existence of a required F now follows from (3.13).

Conversely, assume that there exists a required first integral $F(x,y)$. If (i) is not satisfied, then (see Proposition 3.3 (ii)) at least one non-algebraic factor appears in $F(1,z)$. Also, if (i) is satisfied and (ii) is not satisfied, then, for any M, (3.13) does not allow a polynomial first integral for (1.1). □

4. Projective dynamics in $\mathbb{R}^n$

In this section, we extend Theorem 1.2 to the multidimensional case. Consider the system of polynomial m-homogeneous ODEs:

$$\dot{x}_s = P_s(x), \quad P_s(\lambda x) = \lambda^m P_s(x), \quad s = 1,\dots,n, \tag{4.1}$$

for all $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and some integer $m > 0$ (cf. (1.1)). To begin with, we are interested in a maximal number of non-proportional fixed points of the field $V := (P_1,\dots,P_n)$.

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4.1. Bézout's Theorem

The following standard fact, known as Bézout's Theorem, is a starting point for our discussion.

Theorem 4.1. *Let $f_1, \dots, f_n$ be n complex polynomials of degree $d_1, \dots, d_n$ in n variables having finitely many common zeros. Then, the number N of common zeros of $f_1, \dots, f_n$ satisfies the inequality $N \leqslant d_1, \dots d_n$. The bound is attained by generic polynomials of degree $d_1, \dots, d_n$.*

Corollary 4.2. *Assume P_s is as in (4.1), the system*

$$P_s(x) = x_s, \quad s = 1, \dots n, \tag{4.2}$$

has finitely many complex solutions and the number of these solutions is maximal. Denote by M the number of pairwise non-proportional (non-zero) solutions to (4.2). Then, $M = (m^n - 1)/(m - 1)$.

Proof. By Theorem 4.1, (4.1) and the maximality condition, there are m^n simple solutions to (4.2) (including constant multiples). Next, if p is a *non-zero* solution to (4.2), then so is $\exp(2k\pi \mathrm{i}/(m-1))p$, $k = 1, \dots, m-1$, meaning that any non-zero solution to (4.2) gives rise to a set of $m - 1$ *proportional* solutions and the result follows. □

4.2. Euler–Jacobi Formula

The following statement is a particular case of the Global Residue Theorem in the complex projective space presented in [**1**] (see also [**3**, **6**, **7**]).

Theorem 4.3. *Let $F(x) = (F_1(x) \cdots F_n(x))$ be a polynomial complex vector field with $\deg F_s = d_s$, $s = 1, \dots, n$. Denote by $\mathrm{Sol}(F)$ the set of common roots of F_s, $s = 1, \dots, n$, and assume (cf. Theorem 4.1) that $\mathrm{Sol}(F)$ contains exactly $d_1 \cdots d_n$ elements (in particular, any $x \in \mathrm{Sol}(F)$ is simple and, therefore, the Jacobian of F is non-zero for all $x \in \mathrm{Sol}(F)$). Then, for every complex polynomial Q with $\deg Q < \sum_s d_s - n$, one has the following Euler–Jacobi Formula:*

$$\sum_{a \in \mathrm{Sol}(F)} \frac{Q(a)}{\det[D(F)]_{x=a}} = 0, \tag{4.3}$$

where $D(\cdot)$ denotes the Jacobi matrix.

Remark 4.4. It is easy to see that the first syzygy in (1.5) is a consequence of (4.3) for $F = V - \mathrm{Id}$ and $Q \equiv 1$: the constant polynomial.

4.3. Syzygies in the multidimensional case

With Corollary 4.2 and Theorem 4.3 in hand, we are now in a position to generalize Theorem 1.2 to the multidimensional case.

Theorem 4.5. *Let P_s, $s = 1,\dots,n$, be as in (4.1), $m > 1$, $V = (P_1,\dots,P_n)$. Set $F := V - \mathrm{Id}$ and $\mathrm{Sol}(F) := F^{-1}(0)$. Assume $\mathrm{Sol}(F)$ is finite and contains a maximal number of elements (in particular, any $x \in \mathrm{Sol}(F)$ is simple, the Jacobian of F is different from zero for all $x \in \mathrm{Sol}(F)$, and there are exactly $M := (m^n - 1)/(m-1)$ pairwise non-proportional (non-zero) elements $\xi_1,\dots,\xi_M$ in $\mathrm{Sol}(F)$ (cf. Corollary 4.2)). Set*

$$\gamma_q = (m-1)^2/\det[D(F)]|_{x=\xi_q}, \quad q = 1,\dots,M$$

(cf. (1.4)). Then, the following syzygies between ξ_q and γ_q hold:

$$\sum_{q=1}^{M} \gamma_q + (-1)^n(m-1) = 0, \tag{4.4}$$

$$\sum_{q=1}^{M} \gamma_q \xi_q^{\alpha} = 0, \quad |\alpha| = s(m-1),\ 0 < s < n \tag{4.5}$$

(here $\alpha = (\alpha_1,\dots,\alpha_n)$ denotes a multindex, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $x \in \mathbb{C}^n$).

Proof. As explained in the proof of Corollary 4.2, $\mathrm{Sol}(F)\setminus\{0\}$ splits into M symmetric groups, say, $\mathcal{B}_1,\dots,\mathcal{B}_M$, and any $\mathcal{B}_q = \{\xi_q^k\}$ consists of the elements

$$\xi_q^k = \exp\left(\frac{2k\pi \mathrm{i}}{m-1}\right)\xi_q, \quad k = 0,\dots,m-1,\ q = 1,\dots,M.$$

Clearly, for any representative $\xi_q \in \mathcal{B}_q$,

$$\det[D(F)]_{x=\xi_q} = \det[D(F)]_{x=\xi_q^k}, \quad k = 0,\dots,m-1. \tag{4.6}$$

Also, since $m > 1$, one has $\det[D(F)]_{x=0} = \det[D(-\mathrm{Id})] = (-1)^n$. Hence, for any polynomial $Q\colon \mathbb{C}^n \to \mathbb{C}$, one obtains

$$\frac{Q(0)}{\det[D(F)]_{x=0}} = (-1)^n Q(0). \tag{4.7}$$

To establish (4.4), it suffices to apply Theorem 4.3 with the constant polynomial $Q(x) \equiv m-1$ taking into account (4.6) and (4.7).

To establish (4.5), take a monomial $Q(x) = x^\alpha$. Then

$$\sum_{k=1}^{m-1}\left[\exp\left(\frac{2k\pi \mathrm{i}}{m-1}\right)x\right]^{\alpha} = (m-1)\delta_{|\alpha|}x^\alpha, \tag{4.8}$$

where

$$\delta_{|\alpha|} = \begin{cases} 1, & |\alpha| \equiv 0 \pmod{(m-1)}, \\ 0, & |\alpha| \not\equiv 0 \pmod{(m-1)}. \end{cases}$$

Finally, if $|\alpha|$ is as in (4.5), one can apply Theorem 4.3 with $Q(x) = x^\alpha$, and (4.5) follows immediately from (4.7) and (4.8). □

5. Concluding remarks

There is a vast amount of literature devoted to homogeneous polynomial differential systems and we are far from having given a full roundup of the current state of the art in this topic. We emphasize that, to the best of our knowledge, Theorems 1.1 and 1.2 (see also Theorems 3.1 and 4.5) have not appeared before (at least, in the formulations presented). In fact, these theorems are intimately connected to idempotents and nilpotents in non-associative algebras underlying multi-linear vector fields (see [**2**]) and related Peirce numbers (see [**9**]).

Also (cf. [**4**]), it is our belief that a homogeneous dynamics with complex fixed or stationary points and complex local invariants (eigenvalues of Jacobian matrices at these points) is behind families of decaying spirals, cycles or growing spirals, depending upon whether the signs of real parts of such invariants are negative, zero or positive.

In addition, if for a real fixed point the corresponding invariant is also real, then the sign of this invariant is responsible for the existence of bounded non-zero solutions to homogeneous systems (cf. [**2**,**8**,**12**]). For example, consider the Lotka–Volterra equations in $\mathbb{R}^n$:

$$\dot{x_i} = P_i(x), \quad P_i(x) = x_i\biggl(x_i + \sum_{j\neq i} a_{ij}x_j\biggr), \quad i = 1, 2, \ldots, n. \tag{5.1}$$

By direct computation, all the basis vectors $e_1, e_2, \ldots, e_n$ are fixed points of the field $V = (P_1, \ldots, P_n)$ and any coordinate plane $\mathbb{R}[e_i, e_j]$ generated by e_i and e_j is an invariant set of (5.1). Moreover, if $a_{ij}, a_{ji} \neq 1$ and $a_{ij}a_{ji} \neq 1$, then V restricted to $\mathbb{R}[e_i, e_j]$ admits the additional fixed point

$$p_{ij} := \frac{1 - a_{ij}}{1 - a_{ij}a_{ji}} e_i + \frac{1 - a_{ji}}{1 - a_{ij}a_{ji}} e_j.$$

Denote by F^{ij} the restriction of $F := V - \mathrm{Id}$ to $\mathbb{R}[e_i, e_j]$ and set

$$\gamma_i := \frac{1}{\det\,[F^{ij}]}\bigg|_{x=e_i}, \qquad \gamma_j := \frac{1}{\det\,[F^{ij}]}\bigg|_{x=e_j}, \qquad \gamma_{ij} := \frac{1}{\det\,[F^{ij}]}\bigg|_{x=p_{ij}}. \tag{5.2}$$

Remark 5.1. One can show (cf. [**9**,**14**]) that integral curves in $\mathbb{R}[e_i, e_j]$ are adjoined to the rays (generated by the fixed points) at the origin if and only if the corresponding number in (5.2) is positive. Moreover, two rays with positive numbers (5.2) give rise to an elliptic sector.

By straightforward computations,

$$\gamma_i = \frac{1}{a_{ji} - 1}, \qquad \gamma_j = \frac{1}{a_{ij} - 1}, \qquad \gamma_{ij} = \frac{1 - a_{ij}a_{ji}}{(1 - a_{ij})(1 - a_{ji})}. \tag{5.3}$$

Proposition 5.2. *Given system (5.1), assume there exist i, j such that two of the three numbers in (5.3) are positive. There then exists a non-zero bounded solution to (5.1) in $\mathbb{R}[x_i, x_j]$.*

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Observe that syzygy (1.5) does not permit all the numbers in (5.3) to be positive simultaneously. We believe that syzygies are behind any restrictions on possible global homogeneous dynamics. In particular, the well-known classifications of phase portraits of homogeneous systems in the two-dimensional case up to orbital topological equivalence are fully compatible with Theorem 1.2.

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