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PRONORMALITY IN GENERALIZED *FC*-GROUPS

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Abstract

We extend some results known for *FC*-groups to the class FC^* of generalized *FC*-groups introduced in de Giovanni *et al.* ['Groups with restricted conjugacy classes', *Serdica Math. J.* **28**(3) (2002), 241–254]. The main theorems pertain to the join of pronormal subgroups. The relevant role that the Wielandt subgroup plays in an FC^*-group is pointed out.

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1. Introduction

Recall that an *FC-group* is a group with finite conjugacy classes. The theory of *FC*-groups has been much studied, originally by Baer and Neumann, and it is clearly described in [17]. An important class of generalized *FC*-groups, which we denote by FC^*, was recently introduced in [2]. The class FC^n is defined recursively as follows: FC^0 is the class of finite groups and a group G belongs to the class FC^{n+1} if $G/C_G(\langle x\rangle^G) \in FC^n$ for all x in G. Finally,

$$FC^* = \bigcup_{n\geq 0} FC^n.$$

Clearly FC^1-groups are just the *FC*-groups. It is easy to check that if the nth term of the upper central series of a group G has finite index in G then G is an FC^n-group, in particular every finite-by-nilpotent group is an FC^*-group. It is shown in [15] that several theorems concerning the strong form of residual finiteness for *FC*-groups can be extended to FC^*-groups.

Our object is to extend to the class of FC^*-groups the theory of pronormality developed for *FC*-groups in [4] and in [5].

Let G be a group and let H be a subgroup of G. Following the notation introduced in [4], we will say that an element x of G *pronormalizes* H if the subgroups H and H^x are conjugate in $\langle H, H^x\rangle$. Moreover, H is said to be a *pronormal subgroup* of G

if each element of G pronormalizes H. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups; moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. The concept of 'pronormal subgroup' was introduced by Hall, and the first results on pronormality appeared in a paper by Rose [16]. The study of the property of pronormality has been the object of many investigations by several authors (see, for instance, [1, 5–8, 10, 13, 18]). In particular, the problem of the join of pronormal subgroups has been considered. Here we will show that the product of two pronormal permuting subgroups of a locally soluble FC^*-group is likewise pronormal (see Theorem 2.1), and that the join of any chain of pronormal subgroups of a finite-by-nilpotent group is pronormal (see Theorem 3.7). In order to investigate the pronormality of subgroup H of a group G, the notion of the *pronormalizer* of H in G, that is, the set $P_G(H)$ of the elements of G pronormalizing H, has been introduced (see [5]). Thus a subgroup H of a group G is pronormal in G if and only if $P_G(H) = G$. Also we recall that the intersection $P(G)$ of the pronormalizers of all subgroups of a group G is called the *pronorm* of G. We remark that the pronorm of the alternating group A_5 has order 40 so it is not a subgroup; moreover; in this example the pronormalizer of any subgroup of order 2 is not a subgroup. In Section 2 we will show that the pronorm of a locally soluble FC^*-group is a subgroup of G.

It is easy to check that a subgroup H of a group G is normal if and only if it is both pronormal and subnormal. In particular, every group whose subgroups are pronormal is a T-group (that is a group in which normality is a transitive relation). On the other hand, using some basic properties of T-groups (see [12]), it is shown in [4] that a group whose cyclic subgroups are pronormal is a $\overline{T}$-group (that is, a group whose subgroups are T-groups). It is well known that for a finite group G the property $\overline{T}$ is equivalent to saying that G has all pronormal subgroups (see [11] and [3, Lemma 9]), but this fails to be true for infinite periodic soluble groups (see [9]). More recently the same characterization has been extended to FC-groups (see [4, Theorem 3.9]), and furthermore to FC^*-groups (see [2, Theorem 4.6]). Clearly G is a T-group precisely when $\omega(G) = G$. Here $\omega(G)$ denotes the Wielandt subgroup of G, that is, the intersection of all the normalizers of subnormal subgroups of G. In Section 3 we will show that every FC^*-group either is an FC-group or has a Wielandt subgroup much smaller than G (see Theorem 3.3). In the same section we highlight some other relevant properties of the Wielandt subgroup of an FC^*-group (see Theorems 3.6 and 3.9). Among other results, some interesting properties pertaining to FC^*-groups are obtained.

Most of our notation is standard and can, for instance, be found in [14].

2. Pronormality in FC^*-groups

Recall that two subgroups H and K of a group G are said to *permute* if $HK = KH$, and this is precisely the condition for HK to be a subgroup. In particular, H and K permute if $H^K = H$.

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A result of Rose [16] shows that the product of two pronormal subgroups H and K of G such that $H^K = H$ is still pronormal in G. More generally, Legovini proved in [10] that in a finite soluble group the product of two pronormal subgroups that permute is also pronormal. Recently, de Giovanni and Vincenzi proved in [5] that in a locally soluble FC-group the product of two pronormal subgroups that permute is also pronormal. The proof of this property makes use of the fact that if G is an FC-group, the factor $G/Z(G)$ is periodic. Even if this property is no longer true for FC^n-groups (examples can be found in [15]), we will extend the above result as follows.

THEOREM 2.1. *Let G be a locally soluble FC^*-group. Let X and Y be pronormal subgroups of G that permute. Then the product XY is a pronormal subgroup of G.*

PROOF. Assume G to be an FC^n-group, where n is a positive integer. Put $H = XY$, and let g be an element of G. In order to prove that H and H^g are conjugate in $\langle H, H^g\rangle$, it can obviously be assumed that $G = H^{\langle g\rangle}\langle g\rangle$. Since g pronormalizes X, there exists $u \in [g, X]$ such that $X^g = X^u$. The element u pronormalizes X, then there exists $v \in [u, X] \leq [[g, X], X]$ such that $X^u = X^v$. Iterating this process, we find an element a in $\gamma_{n+1}(G)$ such that $X^g = X^a$, and similarly $X^{g^{-1}} = X^b$ with b in $\gamma_{n+1}(G)$. With the same argument for Y, we have that $Y^g = Y^c$ and $Y^{g^{-1}} = Y^d$, where c and d are elements of $\gamma_{n+1}(G)$. All the elements a, b, c and d are periodic and lie in the FC-center of G by [2, Theorem 3.2 and Corollary 3.3], so that $N = \langle a, b, c, d\rangle^G$ is finite by Dietzmann's lemma. Remark that for each $z \in \mathbb{Z}$, $X^{g^z} = X^{a_1}$ where $a_1 \in N$, and $Y^{g^z} = Y^{b_1}$ where $b_1 \in N$, thus $H^{\langle g\rangle} \leq NH$. This means that H has finite index in $H^{\langle g\rangle}$. If there exists a positive power of g lying in $H^{\langle g\rangle}$, then $|H^{\langle g\rangle}\langle g\rangle : H^{\langle g\rangle}|$ is finite and H has finite index in $G = H^{\langle g\rangle}\langle g\rangle$. Arguing in the finite soluble factor G/H_G, we have that $H/H_G = (XH_G/H_G)(YH_G/H_G)$ is pronormal in G/H_G by the quoted result of Legovini. Therefore we may assume that g has infinite order and $G = \langle g\rangle \ltimes H^{\langle g\rangle}$. As N is finite, there exist two distinct integers i and j such that $X^{g^i} = X^{g^j}$ and so g^s lies in $N_G(X)$ for some $s \in \mathbb{N}$. Similarly, there exists $t \in \mathbb{N}$ such that g^t lies in $N_G(Y)$. Put $m = st$; it follows that g^m normalizes H. If x is an element of $G = \langle g\rangle \ltimes H^{\langle g\rangle}$, then $x = g^i g^{mj} yh$, where $i = 0, \ldots, m-1$, $j \in \mathbb{Z}$, $y \in N$ and $h \in H$. It follows that $x = g^i y' g^{mj} h$, where y' is a suitable element of the finite normal subgroup N. It turns out that the finite set $T = \{g^i y : i = 0, \ldots, m-1 \text{ and } y \in N\}$ is a left transversal of $\langle g^m\rangle H$ in G, and for each $t \in \mathbb{N}$ we may claim that $\langle g^{mt}\rangle H$ has finite index in G. Put $J_t = (\langle g^{mt}\rangle H)_G$ and consider the finite factor G/J_t. By hypothesis, XJ_t/J_t and YJ_t/J_t are pronormal subgroups of G/J_t and so $HJ_t/J_t = (XJ_t/J_t)(YJ_t/J_t)$ is pronormal in G/J_t so that HJ_t is a pronormal subgroup of G, again by Legovini's result. It follows that $(HJ_t)^g = (HJ_t)^y$, where $y = jz$ with $j \in J_t$ and $z \in \langle H, H^g\rangle$. Since $\langle H, H^g\rangle$ is a subgroup of HN, then $z \in HN \cap \langle H, H^g\rangle = H(N \cap \langle H, H^g\rangle)$. Put $N \cap \langle H, H^g\rangle = \{u_1, \ldots u_s\}$ so that $(HJ_t)^g = (HJ_t)^{u_i}$ for some u_i. For each $r \leq s$, let Ω_r be the subset of $\mathbb{N}$ consisting of all integers n such that $(HJ_n)^g = (HJ_n)^{u_r}$. Clearly, there exists $i \leq s$ such that Ω_i is infinite. The relation

$$H \subseteq \bigcap_{n\in\Omega_i} HJ_n \subseteq \bigcap_{n\in\Omega_i} \langle g^{mn}\rangle \ltimes H \subseteq H$$

holds and hence all these terms coincide. Then

$$H^g = \Big(\bigcap_{n\in\Omega_i} HJ_n\Big)^g = \bigcap_{n\in\Omega_i} (HJ_n)^g = \bigcap_{n\in\Omega_i} (HJ_n)^{u_i}$$
$$= \Big(\bigcap_{n\in\Omega_i} \langle g^{mn}\rangle \ltimes H\Big)^{u_i} = H^{u_i},$$

and hence H is a pronormal subgroup of G. □

Following [4], we will also consider the *cyclic pronorm* of G, that is, the subset $P_{\mathfrak{C}}(G)$ consisting of the elements of G that pronormalize every cyclic subgroup of G. It is proved that the pronorm and the cyclic pronorm of a group G are subgroups when G is a polycyclic group or a locally soluble FC-group (see [4, Corollary 4.7 and Theorem 4.9]). In the following we will show that also in the universe of locally soluble FC^*-groups both the pronorm and the cyclic pronorm of a group G are subgroups.

THEOREM 2.2. *Let G be a locally soluble FC^*-group. Then the cyclic pronorm $P_{\mathfrak{C}}(G)$ of G is a subgroup of G.*

PROOF. Assume G to be an FC^n-group, where n is a positive integer. Let X be a finite subset of $P_{\mathfrak{C}}(G)$ and let a be an element of G. Since $\langle X, a\rangle^G/Z_n(\langle X, a\rangle^G)$ is finite (see [2, Lemma 3.7]), it follows that $\gamma_{n+1}(\langle X, a\rangle^G)$ is finite. Therefore $\gamma_{n+1}(\langle X, a\rangle)$ is a finite soluble group and so is polycyclic. On the other hand, the nilpotent factor $\langle X, a\rangle/\gamma_{n+1}(\langle X, a\rangle)$ is also polycyclic so that $\langle X, a\rangle$ is polycyclic and $P_{\mathfrak{C}}(\langle X, a\rangle)$ is a subgroup of $\langle X, a\rangle$ (see [4, Corollary 4.7]). Clearly X is contained in $P_{\mathfrak{C}}(\langle X, a\rangle)$, thus every element of $\langle X\rangle$ pronormalizes the subgroup $\langle a\rangle$. The arbitrary choice of the element a in G shows that $P_{\mathfrak{C}}(G)$ is a subgroup of G. □

LEMMA 2.3. *Let G be a group, and let K be a subgroup of G. Let Ω be a chain of normal subgroups of K such that $K = \bigcup_{H\in\Omega} H$. Let x be an element of G such that $x = ky$, where $k \in K$ and $y \in G$ pronormalizes every element of Ω. If the subgroup $[K, x]$ is finite, then both x and y pronormalize K.*

PROOF. Let H be an element of Ω; then $H^y = H^z$ where $z \in [H, y]$. Put $[K, x] = \{y_1, \dots, y_t\}$ and observe that $[H, y]$ is a subgroup of $K[K, x]$, so $z = k_H y_i$ with $k_H \in K$ and $y_i \in [K, x]$. For each $i \le t$, let Ω_i be the subset of Ω consisting of all subgroups $H \in \Omega$ such that $H^y = H^{k_H y_i}$ where k_H is a suitable element of K, and so

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_t.$$

For each $i \le t$, put

$$K_i = \bigcup_{H\in\Omega_i} H$$

so that $K_i^y = K_i^{k_H y_i}$. If h, j are indices such that K_h is not contained in K_j, there exists an element $\overline{H}$ of Ω_h which is not contained in K_j. By hypothesis Ω is a chain, so that

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every element of Ω_j is contained in $\overline{H}$, and hence also in K_h. Thus K_j is contained in K_h, and the finite set $\{K_1, \ldots, K_t\}$ is a chain. On the other hand,

$$K = \bigcup_{H \in \Omega} H = \langle K_1, \ldots, K_t \rangle,$$

so that $K = K_n$ for some $n \leq t$. Since every element of Ω is a normal subgroup of K, we have that

$$K^x = K^y = \bigcup_{H \in \Omega_n} H^{k_H y_n} = \bigcup_{H \in \Omega_n} H^{y_n} = K^{y_n},$$

where y_n is an element of $[K, x]$ and so x pronormalizes K. On the other hand, $y_n \in \langle K, K^x \rangle = \langle K, K^y \rangle$ and so also y pronormalizes K. □

LEMMA 2.4. *Let G be a locally soluble FC^n-group. Then every subgroup K of G is a pronormal subgroup of $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$.*

PROOF. Let K be a subgroup of G. Let

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_n \leq K_{n+1} \leq \cdots \leq K_\gamma = K$$

be an ascending characteristic series with abelian factors of the locally soluble FC^n-group K (see [2, Theorem 3.9]). Assume that the lemma is false, so that some K_α is not a pronormal subgroup of $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$, and α can be chosen to be smallest with respect to this condition. Let $x = ky$, where $k \in K$ and $y \in P_{\mathfrak{C}}(G)$, be an element of $\gamma_{n+1}(KP_{\mathfrak{C}}(G))$ which does not pronormalize K_α. As G is an FC^n-group, the subgroup $[K, x]$ is finite (see [2, Theorem 3.2 and Corollary 3.3]). If $\alpha = 1$, then K_α is abelian and so there exists an ascending normal series of K_α, say

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_\mu = K_\alpha,$$

whose factors are cyclic. Since K_α is a normal subgroup of K, if x does not pronormalize K_α, then y does not even pronormalize K_α. Let δ be the first ordinal such that y does not pronormalize K_δ. It is obvious that $\delta > 1$. If δ were a limit ordinal, then $H_\delta = \bigcup_{\beta<\delta} H_\beta$ and H_β is pronormalized by y for each $\beta < \delta$. As the subgroup $[H_\delta, x]$ is finite, it follows by Lemma 2.3 that y pronormalizes H_δ, a contradiction. Hence δ cannot be a limit ordinal and y pronormalizes $H_{\delta-1}$. Since the factor $H_\delta/H_{\delta-1}$ is cyclic, there exists an element h_δ of H_δ such that $H_\delta = H_{\delta-1}\langle h_\delta \rangle$. We can observe that $H_{\delta-1}$ and $\langle h_\delta \rangle$ are subgroups of G such that $H_{\delta-1}^{\langle h_\delta \rangle} = H_{\delta-1}$; moreover, $P_{\mathfrak{C}}(G)$ is a normal subgroup of G such that $\langle h_\delta \rangle$ is pronormal in $\langle h_\delta \rangle P_{\mathfrak{C}}(G)$ and the element y of $P_{\mathfrak{C}}(G)$ pronormalizes $H_{\delta-1}$, so that y pronormalizes H_δ (see [4, Lemma 2.2]). This contradiction shows that $\alpha > 1$. If α is not a limit ordinal, we can assume that x pronormalizes $K_{\alpha-1}$. Since the factor $K_\alpha/K_{\alpha-1}$ is abelian, there exists a chain of ascending normal subgroups of K_α whose factors are cyclic, that is,

$$H_0 = K_{\alpha-1} \leq H_1 \leq \cdots \leq H_\gamma = K_\alpha.$$

Let λ be the smallest ordinal such that y does not pronormalize H_λ. By assumption, $\lambda > 0$. Hereafter, we can argue as above to get a contradiction both in the case that λ

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is a limit and not. Finally, assume that α is a limit; then $K_\alpha = \bigcup_{\delta<\alpha} K_\delta$ and K_δ is pronormalized by x for each $\delta < \alpha$. Because of K_δ is a normal subgroup of K for every $\delta < \alpha$, it follows that K_δ is pronormalized by y for each $\delta < \alpha$. The subgroup $[K_\alpha, x]$ is finite, and a new application of Lemma 2.3 yields that x pronormalizes K_α. This contradiction proves the lemma. □

THEOREM 2.5. *Let G be a locally soluble FC^*-group. Then the pronorm $P(G)$ of G is a subgroup of G.*

PROOF. Assume G to be an FC^n-group, where n is a positive integer. Let K be a subgroup of G. By Lemma 2.4, $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$ is contained in $P_{KP_{\mathfrak{C}}(G)}(K)$. Moreover, $K\gamma_{n+1}(KP_{\mathfrak{C}}(G))$ is a subnormal subgroup of $KP_{\mathfrak{C}}(G)$; it follows that $P_{KP_{\mathfrak{C}}(G)}(K)$ is a subgroup of $KP_{\mathfrak{C}}(G)$ (see [5, Theorem 2.2]). We can observe that $P_{KP_{\mathfrak{C}}(G)}(K) = P_G(K) \cap KP_{\mathfrak{C}}(G)$.

On the other hand,

$$P(G) = \bigcap_{K \le G} P_G(K) = \Big(\bigcap_{K \le G} P_G(K)\Big) \cap P_{\mathfrak{C}}(G) = \bigcap_{K \le G} (P_G(K) \cap P_{\mathfrak{C}}(G))$$

and

$$P_G(K) \cap P_{\mathfrak{C}}(G) = P_G(K) \cap KP_{\mathfrak{C}}(G) \cap P_{\mathfrak{C}}(G) = P_{KP_{\mathfrak{C}}(G)}(K) \cap P_{\mathfrak{C}}(G).$$

It follows that $P(G)$ is a subgroup of G because it is an intersection of subgroups. □

3. The Wielandt subgroup of an FC^*-group

It has already been noted (see [2, Theorem 4.6]) that any FC^*-group with the property T is an FC-group. Here we will show that if G is an FC^*-group that is not an FC-group, then the index $|G : \omega(G)|$ is infinite.

The first results of this section are two basic properties of FC^n-groups pertaining to the terms of the successive normal closures series.

Recall that, if H is an arbitrary subset of G, *the series of successive normal closures* of H in G is the descending series $\{H^{G,\alpha}\}$ between H and G defined inductively by

$$H^{G,0} = G, \quad H^{G,\alpha+1} = H^{H^{G,\alpha}} \quad \text{and} \quad H^{G,\lambda} = \bigcap_{\beta<\lambda} H^{G,\beta}$$

where α is an ordinal and λ a limit ordinal. It is well known that this is the fastest descending series whose terms all contain H, and it is easy to show by induction for every positive integer n that $H^{G,n} = H[G,_n H]$. For the convenience of the reader we remark that the class of FC^n-groups satisfies Max locally and also that every periodic and finitely generated FC^n-group is finite (see [2, Theorem 3.6]).

LEMMA 3.1. *Let G be an FC^n-group. Then $\langle x\rangle^{G,n}$ is finitely generated for every $x \in G$.*

PROOF. Let x be an element of G. Since the factor group $\langle x\rangle^G/Z_{n-1}(\langle x\rangle^G)$ is finitely generated (see [2, Lemma 3.7]), then $\langle x\rangle^G \le YZ_{n-1}(\langle x\rangle^G)$, where Y is a finitely

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generated subgroup of G containing x. We will show that $\langle x\rangle^{G,m} \le YZ_{n-m}(\langle x\rangle^G)$ for each $m \in \{1, \ldots, n\}$.

By contradiction let r be the minimum positive integer less then n such that $\langle x\rangle^{G,r}$ is not contained in $Z_{n-r}(\langle x\rangle^G)$. Clearly $r > 1$, so that

$$\begin{aligned}\langle x\rangle^{G,r} &\le \langle x\rangle^{\langle x\rangle^{G,r-1}} \le \langle x\rangle^{YZ_{n-(r-1)}(\langle x\rangle^G)} \le (\langle x\rangle^Y)^{Z_{n-(r-1)}(\langle x\rangle^G)}\\ &\le \langle x\rangle^Y[\langle x\rangle^Y, Z_{n-(r-1)}(\langle x\rangle^G)] \le \langle x\rangle^Y[\langle x\rangle^G, Z_{n-(r-1)}(\langle x\rangle^G)]\\ &\le \langle x\rangle^Y Z_{n-r}(\langle x\rangle^G) \le YZ_{n-r}(\langle x\rangle^G).\end{aligned}$$

From this contradiction, it follows that $\langle x\rangle^{G,n}$ is a subgroup of Y. As Y satisfies Max, it follows that $\langle x\rangle^{G,n}$ is finitely generated. □

A result of Polovickiĭ [14] states that a group G is an FC-group if and only if $\langle x\rangle^G$ is finite-by-cyclic for every $x \in G$. For FC^n-groups we have the following result.

COROLLARY 3.2. *Let G be an FC^n-group. Then $\langle x\rangle^{G,n}$ is finite-by-cyclic and* $\mathrm{Aut}\langle x\rangle^{G,n}$ *is finite for every $x \in G$.*

PROOF. Since $\langle x\rangle^{G,n} = \langle x\rangle[G,_n \langle x\rangle]$, we have to show that $[G,_n \langle x\rangle]$ is finite. By definition, $[G,_n \langle x\rangle]$ is contained in the periodic subgroup $\gamma_{n+1}(G)$ of G (see [2, Corollary 3.3]). On the other hand, $\langle x\rangle^{G,n}$ is finitely generated by Lemma 3.1, so that it satisfies Max. It follows that the subgroup $[G,_n \langle x\rangle]$ is periodic and finitely generated, so that it is finite. In addition, it is easy to show that if H is a finite-by-cyclic group, then Aut H is finite. □

THEOREM 3.3. *Let G be an FC^*-group in which the Wielandt subgroup $\omega(G)$ has finite index. Then G is an FC-group.*

PROOF. Let G be an FC^n-group, where n is a positive integer. Let x be an element of G. Since $\langle x\rangle^{G,n}$ is a subnormal subgroup of G, it follows that

$$\omega(G)\langle x\rangle^{G,n} \le N_G(\langle x\rangle^{G,n})$$

and so $|G : N_G(\langle x\rangle^{G,n})| < \infty$. By Corollary 3.2, $N_G(\langle x\rangle^{G,n})/C_G(\langle x\rangle^{G,n})$ is finite and so $|G : C_G(\langle x\rangle^{G,n})|$ is also finite. This proves that x is an FC-element. □

We remark that there are easy examples of infinite FC-groups with trivial center in which normality is a transitive relation. Thus an FC^*-group whose Wielandt subgroup has finite index may be not center-by-finite. A well-known theorem of Neumann states that a group G is center-by-finite if and only if every subgroup has finitely many conjugates. It follows that if G is nilpotent, then $Z(G)$ has finite index in G if and only if $\omega(G)$ has finite index in G. This characterization can easily be extended to finite-by-nilpotent groups. We will show this equivalence as an application of Theorem 3.3.

COROLLARY 3.4. *Let G be a finite-by-nilpotent group. If $|G : \omega(G)|$ is finite, then $G/Z(G)$ is finite.*

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PROOF. By hypothesis, $G/Z_n(G)$ is finite for some positive integer n, in particular G is an FC^*-group, and hence it is an FC-group by Theorem 3.3. On the other hand, $Z_n(G) \cap \omega(G)$ is a Dedekind group, so that G is abelian-by-finite and hence $G/Z(G)$ is finite. □

Recall that a subgroup H of a group G is said to be *ascendant* if there is an ascending series between H and G. Following the notation introduced in [5], we will denote by $\tau(G)$ the intersection of all the normalizers of ascendant subgroups of G. Clearly every subnormal subgroup is also ascendant, so that for any group G the subgroup $\tau(G)$ is contained in $\omega(G)$. Moreover, if G is a polycyclic-by-finite group, ascendant and subnormal subgroups of G coincide, and hence $\tau(G) = \omega(G)$. It was proved by de Giovanni and Vincenzi in [5] that $\tau(G)$ and $\omega(G)$ also coincide in an FC-group. This property also holds for FC^*-groups.

LEMMA 3.5. *Let G be an FC^*-group. Then $\omega(G) = \tau(G)$.*

PROOF. Let G be an FC^n-group, where n is a positive integer. Let x be an element of $\omega(G)$, and let H be any ascendant subgroup of G. For every element $h \in H$, put $N = \langle h, x\rangle^G$. By [2, Lemma 3.7] the factor group $N/Z_n(G)$ is finite, thus $(H \cap N)Z_n(G)$ is subnormal in N and $H \cap N$ is subnormal in G. It follows that x normalizes $H \cap N$, and hence h^x belongs to H. The arbitrary choice of h in H yields $x \in N_G(H)$. The lemma is proved. □

In order to study the properties of a special subclass of FC^*-groups, the class of finite-by-nilpotent groups, we introduce for any group G two descending normal series related to the subgroups $\omega(G)$ and $\tau(G)$.

Let G be a group. The *lower Wielandt series* of G is the descending normal series whose terms $\omega_\alpha(G)$ are defined inductively by positions

$$\omega_0(G) = G, \quad \omega_{\alpha+1} = \bigcap_{K \in \Omega_\alpha(G)} \omega(K),$$

where $\Omega_\alpha(G)$ is the set of all subgroups of G containing $\omega_\alpha(G)$, and

$$\omega_\lambda(G) = \bigcap_{\beta<\lambda} \omega_\beta(K)$$

if λ is a limit ordinal. The last term of the lower Wielandt series of G will be denoted by $\bar{\omega}(G)$. The *lower τ-series* of G is the descending normal series obtained by replacing in the above definition the Wielandt subgroup $\omega(X)$ by the subgroup $\tau(X)$ for each group X. The last term of the lower τ-series of G will be denoted by $\bar{\tau}(G)$. Clearly $\omega_1(G) = \omega(G)$ and $\tau_1(G) = \tau(G)$. The last term of the lower τ-series of G is a characteristic subgroup of G. It has been proved (see [4, Theorem 4.6]) that for hyperabelian periodic groups, $P_{\mathfrak{C}}(G) = \tau(G)$. Application of Lemma 3.5 yields that for FC^*-groups $\bar{\tau}(G) = \bar{\omega}(G)$.

THEOREM 3.6. *Let G be a periodic FC^*-group. If G contains a locally nilpotent normal subgroup N such that the factor G/N is locally nilpotent, then $\tau(G) = \bar{\tau}(G) = \omega(G)$.*

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PROOF. Let K be a subgroup of G. In order to prove that the theorem is true, we show that $K \cap \tau(G)$ is contained in $\tau(K)$.

Put $L = K \cap \tau(G)$, and let X be any ascendant subgroup of K. Since N is a locally nilpotent FC^*-group, it is hypercentral (see [2, Theorem 3.9]), so that $Y = X \cap N$ is an ascendant subgroup of G and L is contained in the normalizer of Y. The factor group X/Y is isomorphic with a subgroup of G/N, so that it is hypercentral and has a unique Sylow p-subgroup X_p/Y, for every prime p. In particular, X_p is ascendant in K. In order to prove that L normalizes X, it is enough to show that L is contained in $N_G(X_p)$ for all p, so we may assume that X/Y is a p-group for some prime p. Let $N = U \times V$, where U is a p-group and V has no elements of order p, and write $\bar{G} = G/V$. Since $\bar{N}$ is a p-subgroup of $\bar{G}$ such that $\bar{G}/\bar{N}$ is locally nilpotent, it follows that $\bar{G}$ has a unique Sylow p-subgroup $\bar{P}$. Moreover, $\bar{X}$ is a p-subgroup of $\bar{G}$, so that $\bar{X} \leq \bar{P}$ and so $\bar{X}$ is ascendant in $\bar{G}$. It follows that $\bar{X}$ is normalized by $\bar{L}$, and hence $L \leq N_G(XV)$. Therefore LY/Y lies in the normalizer of $(XV \cap K)/Y = X(V \cap K)/Y$. On the other hand, the ascendant subgroup X/Y is a Sylow p-subgroup of $X(V \cap K)/Y$, so that X/Y is characteristic in $X(V \cap K)/Y$ and L is contained in $N_G(X)$. This proves that $K \cap \tau(G)$ is contained in $\tau(K)$, for all subgroups K of G. Now, let K be any subgroup of G containing $\tau(G)$; then $\tau(G)$ is contained in $\tau(K)$, so that $\tau_2(G) = \tau(G)$, and hence $\bar{\tau}(G) = \tau(G)$. □

The last part of this section is devoted to the study of the behavior of the Wielandt subgroup of a finite-by-nilpotent group. We shall prove that for this special subclass of FC^*-groups the join of a chain of pronormal subgroups is likewise pronormal; moreover, if G is also metanilpotent, then the pronorm and the Wielandt subgroup of G coincide.

THEOREM 3.7. *Let G be a finite-by-nilpotent group, and let Ω be a chain of pronormal subgroups of G. Then $\bigcup_{H \in \Omega} H$ is a pronormal subgroup of G.*

PROOF. Since G is a finite-by-nilpotent group, there exists a finite normal subgroup N of G such that the factor G/N is nilpotent of class n (n positive integer). In particular, $\gamma_{n+1}(G) = \{z_1, \ldots, z_t\}$ is finite. Let x be an element of G and let H be an element of Ω. Since x pronormalizes H, there exists $y \in [x, H]$ such that $H^x = H^y$. The element y also pronormalizes H, so there exists $z \in [y, H] \leq [[x, H], H]$ such that $H^x = H^z$. Iterating this process, it follows that $H^x = H^{z_i}$ where z_i is a suitable element of $\gamma_{n+1}(G) \cap \langle H, H^x \rangle$. For each $i = 1, \ldots, t$, let Ω_i be the subset of Ω consisting of all subgroups $H \in \Omega$ such that $H^x = H^{z_i}$ and so $\Omega = \Omega_1 \cup \cdots \cup \Omega_t$. Put $K = \bigcup_{H \in \Omega} H$ and for each $i = 1, \ldots, t$ put $K_i = \bigcup_{H \in \Omega_i} H$, so that $K_i^x = K_i^{z_i}$. If i, j are indices such that K_i is not contained in K_j, there exists an element H of Ω_i which is not contained in K_j, so that every element of Ω_j is contained in H, and hence also in K_i. Thus K_j is contained in K_i, and the finite set $\{K_1, \ldots, K_t\}$ is a chain. On the other hand, $K = \bigcup_{H \in \Omega} H = \langle K_1, \ldots, K_t \rangle$, so that $K = K_i$ for some $i \leq t$, and $K^x = K^{z_i}$. □

LEMMA 3.8. *Let G be a finite-by-nilpotent group, and let K be a locally soluble subgroup of G. If N is a normal subgroup of G such that X is pronormal in XN for every cyclic subgroup X of K, then K is a pronormal subgroup of KN.*

PROOF. Let G be an FC^n-group, where n is a positive integer. Let

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_{\omega+(n+1)} = K$$

be an ascending characteristic series with abelian factors of K of length at most $\omega + (n-1)$ (see [2, Theorem 3.9]). Assume that the lemma is false, so that it follows that K_δ is not pronormal in $K_\delta N$ for some ordinal δ, and δ can be chosen to be the smallest with respect to this condition. Let x be an element of N which does not pronormalize K_δ. If $\delta = 1$, then K_δ is abelian. The ordered set $\mathfrak{L}$ consisting of all subgroups of K_δ pronormalized by x is inductive by Theorem 3.7, thus by Zorn's lemma $\mathfrak{L}$ contains a maximal element M. If $K_\delta \neq M$, we may consider $y \in K_\delta \backslash M$, and the subgroup $\langle y \rangle$ is pronormal in $\langle y \rangle N$ by hypothesis, so that x also pronormalizes $\langle y \rangle M$ by [4, Lemma 2.2]. This contradiction shows that $\delta > 1$. If δ were a limit ordinal, then $K_\delta = \bigcup_{\beta<\delta} K_\beta$ and K_β is pronormalized by x for each $\beta < \delta$, and a new application of Theorem 3.7 yields that x pronormalizes K_δ. Hence δ cannot be a limit ordinal and x pronormalizes $K_{\delta-1}$. We can again use Theorem 3.7 to show that the ordered set $\mathfrak{L}$, consisting of all subgroups of K_δ containing $K_{\delta-1}$ and pronormalized by x, is inductive. Let M be a maximal element of $\mathfrak{L}$. Clearly M is a normal subgroup of K_δ, and for every element y of K_δ we have that $\langle y \rangle M$ is pronormalized by x from [4, Lemma 2.2]. Therefore $M = K_\delta$, a contradiction. □

It has already been remarked in [2] that the Wielandt subgroup and the pronorm coincide for polycyclic groups with nilpotent commutator subgroup and for periodic groups with nilpotent and finite commutator subgroup. We can extend these results as follows.

THEOREM 3.9. *Let G be a soluble finite-by-nilpotent group. Then $P(G) = P_{\mathfrak{C}}(G)$. Moreover, if G is periodic and metanilpotent, then $P(G) = \omega(G)$.*

PROOF. Let G be an FC^n-group where n is a positive integer. By Theorem 2.2, the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a normal subgroup of G. Let K be any subgroup of G. Then X is pronormal in $XP_{\mathfrak{C}}(G)$ for every cyclic subgroup X of K, and K is pronormal in $KP_{\mathfrak{C}}(G)$ by Lemma 3.8. Therefore $P(G) = P_{\mathfrak{C}}(G)$. Suppose now that G is periodic so that $P_{\mathfrak{C}}(G) = \bar{\tau}(G)$ (see [4, Theorem 4.6]), and moreover if G is metanilpotent then $\bar{\tau}(G) = \omega(G)$ by Theorem 3.6. The proof is complete. □

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