DIFFERENCES, DERIVATIVES, AND DECREASING REARRANGEMENTS

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To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction. The decreasing rearrangement of a finite sequence $a_1, a_2, \ldots, a_n$ of real numbers is a second sequence $a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}$, where $\pi(1), \pi(2), \ldots, \pi(n)$ is a permutation of $1, 2, \ldots, n$ and

$$a_{\pi(1)} \geqslant a_{\pi(2)} \geqslant \ldots \geqslant a_{\pi(n)}$$

(**1**, p. 260). The kth term of the rearranged sequence will be denoted by a_k^*. Thus the terms of the rearranged sequence a_k^* correspond to and are equal to those of the given sequence a_k, but are arranged in descending (non-increasing) order.

The equimeasurable decreasing rearrangement of a real-valued measurable function f with domain $[0, b]$ is a second function f^* with domain $[0, b]$ and the same range as f. However f^* is monotonic decreasing (non-increasing) and the measures

$$m(f \geqslant c), \qquad m(f^* \geqslant c)$$

are equal for every real c (**1**, p. 276).

In this paper we establish and study an inequality related to the operation of rearrangement in decreasing order, namely, that the total variation of the sequence or function is in general diminished by such rearrangement. We show that the L^p norm of the difference sequence (or the derivative function) is diminished by this rearrangement operation unless the given sequence or function is already monotonic (or almost everywhere equal to a monotonic function).

We first study finite sequences $\{a_1, a_2, \ldots, a_n\}$, and, with $\Delta a_k = a_{k+1} - a_k$ $(k = 1, \ldots, n-1)$, establish the basic inequality

$$(1) \qquad \sum_{k=0}^{n} |\Delta a_k^*|^p \leqslant \sum_{k=0}^{n-1} |\Delta a_k|^p, \qquad p \geqslant 1.$$

For infinite sequences, a more general definition of the rearranged sum is required as the rearranged sequence may have a different order type such that some terms have no next neighbour. All such terms belong to the set of limit points of the sequence, and in § 3 the rearranged sum is defined and it is shown that these terms do not contribute to the sum. The basic inequality is then established for infinite sequences.

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The basic inequality for functions is

$$\int_a^b |f^{*\prime}(x)|^p\,dx \leqslant \int_a^b |f'(x)|^p\,dx, \qquad p \geqslant 1. \tag{2}$$

This is established for finite intervals (a, b) in § 4. For infinite domains, a generalization is again found to be necessary for the usual definition of the equimeasurable decreasing rearrangement of a function. Such a generalization, involving many-valued functions, together with a notion of asymptotic density, is given in § 5. In § 6 the basic inequality is extended to functions with infinite domain. The analogue for functions of the set of limit points of the sequence is the set of values at which f is asymptotically dense. The contribution of the closed set of asymptotically dense range points to the rearranged sum is shown to be zero.

Finally, in § 7 some results for second and higher order derivatives are presented.

2. Finite sequences. Let $\{a_1, a_2, \ldots, a_n\}$ be a sequence of n real numbers, where n is a fixed positive integer. Let

$$\Delta a_k = a_{k+1} - a_k, \qquad k = 1, \ldots, n-1.$$

With the decreasing rearranged sequence denoted by $\{a_1^*, a_2^*, \ldots, a_n^*\}$ let

$$\Delta a_k^* = a_{k+1}^* - a_k^*.$$

We shall now establish the basic inequality (1) for such sequences. For the proof, we employ the two following lemmas. A term a_k will be called a *local maximum* if

$$a_{k-1} \leqslant a_k \quad \text{and} \quad a_{k+1} < a_k$$

and a *local minimum* if

$$a_{k-1} \geqslant a_k \quad \text{and} \quad a_{k+1} > a_k.$$

LEMMA 2.1. *Removal of a local maximum or local minimum term from* $\{a_1, a_2, \ldots, a_n\}$ *decreases the sum* $\sum|\Delta a_k|^p$, *unless the term is equal to its predecessor in which case the sum is unchanged.*

Proof. Consider the case of a local maximum term a_k. Thus

$$\Delta a_{k-1} = a_k - a_{k-1} \geqslant 0$$

while

$$\Delta a_k = a_{k+1} - a_k < 0.$$

If $a_{k-1} \geqslant a_{k+1}$, then

$$|a_{k-1} - a_{k+1}| \leqslant |a_k - a_{k+1}|;$$

therefore

$$|a_{k-1} - a_{k+1}|^p \leqslant |a_k - a_{k-1}|^p \leqslant |a_{k-1} - a_k|^p + |a_k - a_{k+1}|^p$$

with equality only for $a_{k-1} = a_k$. On the other hand, if $a_{k-1} < a_{k+1}$, then

$$|a_{k-1} - a_{k+1}| < |a_{k-1} - a_k|;$$

therefore

$$|a_{k-1} - a_{k+1}|^p < |a_{k-1} - a_k|^p + |a_k - a_{k+1}|^p.$$

In either case, the contribution to the sum $\sum|\Delta a_k|^p$ after removal of a_k is dominated by the terms present earlier. This proves the lemma for a local maximum term, and a similar proof holds for a local minimum.

LEMMA 2.2. *If a_k, a_{k+1} are consecutive terms ($a_k \neq a_{k+1}$) and c is intermediate to a_k, a_{k+1}, then insertion of c in the sequence diminishes the sum $\sum|\Delta a_k|^p$.*

Proof. Consider, for definiteness, the case

$$a_k < c < a_{k+1}.$$

Set

$$\lambda = \frac{a_{k+1} - c}{a_{k+1} - a_k}, \qquad \mu = \frac{c - a_k}{a_{k+1} - a_k}.$$

Then $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \mu \leqslant 1$, and $\lambda + \mu = 1$. For $p > 1$, it follows that $\lambda^p + \mu^p < 1$, unless $\lambda = 1$ or $\mu = 1$.

Returning to the definitions of λ, μ, we find that

$$|c - a_k|^p + |a_{k+1} - c|^p < |a_{k+1} - a_k|^p.$$

This proves the lemma for the case considered. The opposite case is similar, while the case where the term inserted is equal to a term a_k is trivial.

THEOREM 2.1. *The inequality*

$$\sum_{k=1}^{n-1} |\Delta a_k{}^*|^p \leqslant \sum_{k=1}^{n-1} |\Delta a_k|^p, \qquad p \geqslant 1,$$

is valid, equality holding if and only if the sequence $\{a_1, a_2, \ldots, a_n\}$ is monotonic.

Proof. Let a_T denote an absolute maximum term and a_B an absolute minimum term of the sequence. Thus $a_B \leqslant a_k \leqslant a_T$, $1 \leqslant k \leqslant n$. For definiteness, suppose $1 < T < B < n$, other possibilities can be discussed similarly.

Examine in succession

$$a_{T+1}, a_{T+2}, \ldots, a_k, \ldots$$

and remove the first local maximum term a_k.

By Lemma 2.1, the sum $\sum|\Delta a_n|^p$ is not increased. Then insert any such term again in the position such that $a_T, a_{T+1}, \ldots, a_h, a_k, a_{h+1}$ is monotonic decreasing. This is possible, since a_T is maximal and a_k a local maximum. By Lemma 2.2, the sum $\sum|\Delta a_n|^p$ is again not increased.

Continuing with $a_k, a_{B-1}, a_B, a_{B+1}, \ldots, a_m$, remove and reinsert any other local maximum terms. Then treat the initial segment $a_1, \ldots, a_{T-1}$ of the sequence similarly. The sum decreases if any proper local maximum is encountered.

Again, consider the revised segment $a_{B-1}, a_{B-2}, \ldots, a_l, \ldots$, and remove the first local minimum term encountered. Reinsert in the position such that

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the sequence preceding a_B is monotonic. Continue removal and reinsertion of terms in reverse order to the initial term a_1, and then treat the segment $(a_n, a_{m-1}, \ldots, a_{B+1})$ similarly. As before, the sum $\sum|\Delta a_k|^p$ is decreased whenever a proper minimum term is encountered.

After this process, the only local maximum is a_T and the only local minimum is a_B. The sequence is now composed of three monotonic segments, (a_1, a_T), (a_T, a_B), (a_B, a_n), respectively increasing, decreasing, and increasing.

Now remove a_1, thus decreasing the sum. Reinsert a_1 in the segment (a_T, a_B) in such a position that the segment is still monotonic. Since $a_B \leqslant a_1 \leqslant a_T$, this is possible. By Lemma 2.2, the sum again decreases. Similarly remove and reinsert in succession $a_2, a_3, \ldots, a_{T-1}$, and likewise $a_m, a_{m-1}, \ldots, a_{B+1}$. At each step the sum decreases, and finally the sequence becomes rearranged in monotonically decreasing order with $a_T = a_1^*$, $a_B = a_m^*$. This concludes the proof of Theorem 2.1 in the case discussed. The remaining cases are similar.

The following counterexample shows that this basic inequality does not hold for second differences. Let

$$\{a_k\} = \{0, 3, 6, \ldots, 3n, 3n-1, 3n-4, \ldots, 5, 2\}.$$

Then $\sum|\Delta^2 a_k|^p = 4^p + 2^p$ while $\sum|\Delta^2 a_k^*|^p = 2n - 3$. By choice of n the rearranged sum can be made the larger.

3. Infinite sequences. For infinite sequences $\{a_1, a_2, \ldots\}$ or $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ we shall again show that the general inequality

$$\sum|\Delta a_k^*|^p \leqslant \sum|\Delta a_k|^p$$

holds, provided that certain conventions are adopted regarding the rearranged sequence and the sum associated with it.

First, let us remark that the sum $\sum|\Delta a_k|^p$ may be bounded, for unbounded sequences $\{a_k\}$.

Example 3.1. Let $a_k = \log k$ $(k = 1, 2, \ldots)$. Then

$$\Delta a_k = \log(1 + 1/k) < 1/k$$

and

$$\sum|\Delta a_k|^p \ll \sum 1/k^p < \infty, \qquad p > 1.$$

The sequence $\{a_1, a_2, \ldots\}$ may be unbounded above and below or even everywhere dense, while the difference sequence still has finite L^p norm, as the following example shows.

Example 3.2. Consider the sequence

$$0, 1, 1/2, 0, -1/2, -1, -3/2, -2, -7/4, -3/2, -5/4, -1, \ldots$$

containing two terms with unit difference, six terms in arithmetic sequence with common difference, $-\frac{1}{2}$, 20 terms in arithmetic progression with common difference $-\frac{1}{8}$, and so on. The sequence oscillates from 0 to 1 to -2 to 3 to -4 and so on with increasing amplitude. The sequence is unbounded above and below and is in fact everywhere dense. Between n and $-n-1$, for instance, there are $(2n+1)$. 2^{2n} terms a_k each of magnitude $1/2^{2n}$. Their contribution to the L^p norm is

$$\frac{(2n+1)\cdot 2^{2n}}{2^{2np}} = \frac{2n+1}{2^{2n(p-1)}}.$$

Since this is a term of an absolutely and geometrically convergent series, the sum

$$\sum_{k=1}^{\infty} |\Delta a_k|^p$$

is finite for $p > 1$.

Example 3.2 also illustrates another situation that may arise for infinite sequences. Each term is repeated an infinite number of times. If any finite subsequence is rearranged, the coincident terms will not contribute to the sum $\sum|\Delta a_k^*|^p$. We shall thus avoid the difficulty of rearranging sequences with infinite repetitions by counting only distinct terms in the rearranged sequence.

Returning to Example 3.2, let us construct the rearranged sequence, which is the ordered set of terminating binary decimals, and which has the order type η of the unbordered rationals (**2**, p. 71). A typical "leg" (segment) of the given sequence consists of equally spaced terms ranging from $2n-1$ to $-2n$, or from $-2n$ to $2n+1$. Taking the first case for definiteness, we note that all terms in every previous "leg" appear also in this leg and so can be discarded. Thus the rearranged *partial* sequence consists of the given "leg" only. These terms are spaced at intervals $\Delta a_m^* = 1/2^{2n+1}$, and consequently there are $1 + (4n-1)\cdot 2^{2n+1} < 4n\cdot 2^{2n+1}$ such terms. For $p > 1$, we then have

$$\sum_{\text{leg}} |\Delta a_k^*|^p \leqslant \frac{4n\cdot 2^{2n+1}}{2^{(2n+1)p}} = \frac{4n}{2^{(2n+1)(p-1)}}$$
$$\rightarrow 0 \qquad \text{as } n \rightarrow \infty.$$

For Example 3.2, therefore, if the sum for the rearranged sequence is defined as the limit of the sums for the rearranged partial sequences, we have

$$\sum|\Delta a_k^*|^p = 0.$$

As the set of limit points of a general infinite sequence may have a more complicated structure than that of Example 3.1, we shall need to use a more elaborate definition in the general case.

We begin by giving the definition of the contribution $S^*_{(K,L)}$ arising from the interval $a_{m_1} = K \leqslant a \leqslant L = a_{m_2}$ of the range.

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Definition.

$$S^*_{(K,L)} = \liminf \sum_{K<a_k^*\leqslant L} |\Delta a_k^*|^p$$

where the lim inf *is taken over all finite sequences of terms* $\{a_1, a_2, \ldots, a_n\}$ *such that* $K < a_k^* \leqslant L$ *for* $1 < k \leqslant n$ *while* $a_1 = L$ *and* $a^*_{n+1} = K$.

Note that $S^*_{(K,L)}$ is defined only for values of K, L that belong to the range of the given sequence. When we later write $L \to \infty$, it will be understood that for a bounded sequence a limit is taken as the least upper bound is approached. Similarly for lower bounds when we let $K \to -\infty$.

Alternatively, we may consider $S^*_{(K,L)}$ as a function of (K, L) for real K, L, that is, constant in the intervals between terms of $\{a_k\}$, and continuous from the left in K and from the right in L at points a_k of the sequence. The following lemma will show that it is also possible to define $S^*_{(K,L)}$ by continuity from the right or left at a limit point of the sequence.

A rearranged sequence containing convergent infinite subsequences yields a finite contribution to the rearranged sum from every finite interval of the range, and a small contribution from every small interval. This is shown by

LEMMA 3.1. $S^*_{(K,L)} \leqslant (L - K)^p$.

Proof. If the sequence is finite, the result follows at once from Lemma 2.2. It is sufficient to show in general that a single sequence of partial sequences $\{a_k\}$ yields a limit less than the right-hand side. We may assume that this sequence has a single limit point within the interval (K, L). The assertion of the lemma then follows from a limiting form of the inequality,

$$\sum_{k=1}^{n} |\Delta a_k^*|^p \leqslant \left| \sum_{k=1}^{n} |\Delta a_k^*| \right|^p.$$

For the main proof, we require the following additional lemma for finite sequences.

LEMMA 3.2. *For any finite sequence* $\{a_1, a_2, \ldots, a_n\}$, *the sum* $\sum|\Delta a_k|^p$ $(k = 1, \ldots, n - 1)$ *is decreased (not increased) by removing from the sequence all terms exceeding any given real number* a.

Proof. Let $a_h, a_{h+1}, \ldots, a_k$ be any maximal consecutive set of terms exceeding a. By Lemma 2.1, these terms can be dropped from the sum in the order of their magnitude, the largest first, without increase in the sum. Similarly every other consecutive set of terms exceeding a can be dropped. This proves the lemma.

Similarly, all terms less than any given number c can be dropped, without increasing the given sum.

Lemma 3.2 shows that sums of the form

$$\sum_{-K<a_k^*\leqslant L} |\Delta a_k^*|^p$$

increase monotonically with the interval determined by K, L.

Definition. Let E denote the set of limit points of $\{a_k\}$.

LEMMA 3.3. *If the interval (K, L) is contained in E, then* $S^*_{(K,L)} = 0$ $(p > 1)$.

Proof. Given $\epsilon > 0$, we must find a sequence of rearranged partial sums with members ultimately less than ϵ. Given $\eta > 0$, we can select a finite sequence of terms a_k of the given sequence, which span (K, L) with maximal difference less than η. Since $|\Delta a_k^*| < \eta$, the corresponding sum satisfies

$$\sum|\Delta a_k^*|^p < \eta^{p-1}\sum|\Delta a_k^*| \leqslant \eta^{p-1}(L - K).$$

We now choose $\eta < (\epsilon/(L - K))^{1/(p-1)}$ so that the right-hand side is less than ϵ. This completes the proof.

We now give a different definition of $S^*_{(K,L)}$ as a limit, rather than as a limit inferior. It is necessary for this purpose to use partial sequences which do not omit too many terms from any part of the range (K, L). By the *box size* of a subsequence $\{\bar{a}_k^*\}$ we shall denote the least upper bound of the intervals $\Delta\bar{a}_k^*$ which contain at least one element a_k omitted from the subsequence.

LEMMA 3.4. *If $\{\bar{a}^*_{1(n)}, \bar{a}^*_{2(n)}, \ldots\}$ is a subsequence with partial sequences $\{\bar{a}^*_{1(n)}, \bar{a}^*_{2(n)}, \ldots, \bar{a}^*_{n(n)}\}$ whose box sizes tend to zero (as $n \to \infty$), then*

$$S^*_{(K,L)} = \lim_{n\to\infty}\sum_{k=1}^{n} |\Delta\bar{a}^*_{k(n)}|^p.$$

Proof. Since, by definition,

$$S^*_{(K,L)} \leqslant \lim_{n\to\infty}\sum_{k=1}^{n} |\Delta\bar{a}^*_{k(n)}|^p,$$

it is sufficient to show that for n sufficiently large, the part sum on the right side exceeds $S^*_{K,L}$ by less than a given positive ϵ. Let the box size of the sequence $\{\bar{a}^*_{(1n)}, \bar{a}^*_{2(n)}, \ldots, \bar{a}^*_{n(n)}\}$ be η, and choose n so large that

$$\eta < (\tfrac{1}{2}\epsilon/(L - K))^{1/(p-1)}.$$

Suppose now that any other finite subsequence $\{\bar{\bar{a}}_k^*\}$ is given, and denote by $\{a'_k{}^*\}$ the two sequences combined into one rearranged sequence. Let $\bar{S}^*_{(K,L)}$, $\bar{\bar{S}}^*_{(K,L)}$, and $S'^*_{(K,L)}$ denote the values of the corresponding sums, as originally defined. Since $\{\bar{\bar{a}}_k^*\}$ is arbitrary, we can choose it so that

$$\bar{\bar{S}}^*_{(K,L)} \leqslant S^*_{(K,L)} + \tfrac{1}{2}\epsilon.$$

By Lemma 2.2, the insertion of further terms diminishes the sum and thus

$$S'^*_{(K,L)} \leqslant \bar{\bar{S}}^*_{(K,L)}.$$

We next show that the difference $\bar{S}^*_{(K,L)} - S'^*_{(K,L)}$ does not exceed $\tfrac{1}{2}\epsilon$. For every additional term falls into a box of width at most η. The contribution

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from a typical box interval is diminished at most by the amount η_j^p, where the box width $\eta_j \leqslant \eta$. The sum total of such diminutions is at most

$$\sum_j \eta_j^p \leqslant \eta^{p-1} \sum \eta_j \leqslant \eta^{p-1}(L - K) \leqslant \tfrac{1}{2}\epsilon,$$

in view of the choice of η and the non-overlapping of the box intervals. Finally,

$$\bar{S}^*_{(K,L)} \leqslant S'^*_{(K,L)} + \tfrac{1}{2}\epsilon \leqslant \bar{\bar{S}}^*_{(K,L)} + \tfrac{1}{2}\epsilon,$$

by Lemma 2.2, so that

$$\bar{S}^*_{(K,L)} \leqslant S^*_{(K,L)} + \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = S^*_{(K,L)} + \epsilon.$$

Hence,

$$S^*_{(K,L)} = \lim_{\eta \to 0} \bar{S}^*_{(K,L)}$$

and this completes the proof of the lemma.

Now we define the sum S^* for the entire sequence.

Definition. $S^* = \lim_{K\to-\infty, L\to+\infty} S^*_{(K,L)}$. For simplicity we write $K \to -\infty$, $L \to +\infty$, where $K \to \text{glb } a_k$, $L \to \text{lub } a_k$ if the sequence is bounded above or below. By monotone convergence (**4**, p. 29) the sum is independent of the order in which the two limits are taken. The resulting sum may be finite or it may diverge to plus infinity.

THEOREM 3.1. *If* $\Delta a_k = a_{k+1} - a_k$, *and*

$$S = \sum_{k=1}^{\infty} |\Delta a_k|^p < \infty,$$

then S^* *exists and* $S^* \leqslant S$.

Proof. Rearrange the first n consecutive terms of $\{a_k\}$ as $\{a^*_{1(n)}, a^*_{2(n)}, \ldots, a^*_{n(n)}\}$, and denote the rearranged sum by $S^*_{(n)}$. By Theorem 2.1, we have

$$S^*_{(n)} \leqslant S_{(n)} \leqslant S,$$

where $S_{(n)}$ denotes a partial sum of S. Next choose an interval (K, L) where $K = a_{m_1} < L = a_{m_2}$. Since the partial sum for (K, L) contains only a part of the contributions to $S^*_{(n)}$, we have

$$S^*_{(n)(K,L)} \equiv \sum_{K < ak(n) \leqslant L} |\Delta a^*_{k(n)}|^p \leqslant S^*_{(n)} \leqslant S.$$

By definition, the limit inferior $S^*_{(K,L)}$ satisfies

$$S^*_{(K,L)} \leqslant \lim_{n\to\infty} S^*_{(n)(K,L)} \leqslant S.$$

Finally, we let $K \to -\infty$, $L \to \infty$, and we obtain

$$S^* = \lim_{\substack{K\to-\infty \\ L\to\infty}} S^*_{(K,L)} \leqslant S.$$

This concludes the proof of Theorem 3.1.

THEOREM 3.2. *If* $\{a_k\}$ *is everywhere dense, then* $S^* = 0$.

Proof. By Lemma 3.3, $S^*_{(K,L)} = 0$ for every finite (K, L) so the result follows at once (cf. Example 3.2).

THEOREM 3.3. *If* $\{a_k^*\}$ *has one of the order types, corresponding to the positive integers* $(I+)$, *the negative integers* $(I-)$, *or the integers* (I), *then* S^* *is respectively equal to the corresponding infinite sum*

$$\sum_{k=1}^{\infty} |\Delta a_k^*|^p, \qquad \sum_{k=-\infty}^{\infty} |\Delta a_k^*|^p, \quad \text{or} \quad \sum_{k=-\infty}^{\infty} |\Delta a_k^*|^p.$$

The proofs are immediate and will be omitted.

We extend the result of Theorem 3.3 to the most general case as follows. The set E of limit points of $\{a_1, a_2, \ldots\}$ is closed, so that the complement CE is a sum of open intervals O_j. In each open interval O_j there will in general fall terms a_k in one of four possible ways:

(1) A finite number of distinct terms a_k lie in O_j.

(2) The upper end point b_j of O_j is the only limit point of the terms a_k in O_j.

(3) The lower end point of a_j is the only limit point of the terms a_j in O_j.

(4) Both a_j and b_j are limit points of terms a_k in O_j.

In these four cases the rearranged sum $S^*_{(a_j,b_j)}$ arising from O_j has one of the forms

$$\sum_{\text{finite}}, \qquad \sum_{k=-\infty}^{0}, \qquad \sum_{k=1}^{\infty}, \qquad \sum_{k=-\infty}^{\infty} |\Delta a_k^*|^p$$

respectively. In these cases the value of $S^*_{(a_j,b_j)}$ is defined at limit points by continuity from within the interval.

We shall write

$$S_j^* \equiv S^*_{(a_j,b_j)} = \sum_{\Delta a_k^* \subset O_j} |\Delta a_k^*|^p$$

where the summation ranges over all intervals lying in O_j, in any of the four cases listed above.

THEOREM 3.4. *Let* E *be the limit set and* O_j *the complementary open intervals. Then*

$$S^* = \sum_j S_j^* = \sum_j \sum_{\Delta a_k^* \subset O_j} |\Delta a_k^*|^p \leqslant S.$$

Proof. The proof is immediate, from the definition of $S^*_{(K,L)}$. It is necessary only to note that the sum over j yields a result independent of the order of summation, since all summands are non-negative.

In all results of this section, the strict inequality holds unless the given sequence $\{a_k\}$ is monotonic. In particular, the strict inequality holds if there is more than one limit point.

Up to this stage we have tacitly assumed that the given sequence $\{a_k\}$ has the order type ω of the natural numbers (**2**, p. 57). Together with the assumption $S < \infty$ this implies that there are at most two open sets O_j in the complement of the limit set E. For we have the following property of E.

LEMMA 3.5. *If $\{a_k\}$ has the order type ω or $^*\omega$ and if*

$$S = \sum |\Delta a_k|^p < \infty,$$

then the limit set E is connected.

Proof. If E is not connected, there are at least two components of E, separated by an interval of positive length that contains no limit points. Hence there is at least one gap, or interval of positive length containing no points a_k of the sequence, in this interval. If the gap width is g, then since there are limit points on both sides of the gap, the series $S = \sum |\Delta a_k|^p$ contains indefinitely many terms each at least g^p. This contradicts the hypothesis $S < \infty$, and completes the proof.

From this lemma we can now deduce

COROLLARY 3.5. *If $\{a_k\}$ has the order type ω or $^*\omega$ and S is finite, the summation over j on the left in the formula of Theorem* 3.4 *involves at most two terms.*

Note that the three possible cases of zero, one, and two terms can all arise: no terms if $\{a_k\}$ is everywhere dense on the real axis, one term if the limit set E is empty (limit points only at infinity), and two terms if E is a single point or a closed interval.

Remark. It will be shown below that, in particular, a sequence of order type $^*\omega + \omega$ involves at most three terms.

Consider now enumerable ordered sequences $\{a_k\}$ of a more general type, namely ordered sums represented by series whose terms are ω or $^*\omega$ or n. Such sequences have cuts or gaps (**2**, p. 74) as well as the jumps from one element to its neighbour on which the differences Δa_k are based. (A decomposition of the sequence into two non-empty subsequences is called a cut if exactly one of the subsequences has a border element and a gap if neither subsequence has a border element.)

The following example of order type $\omega + {}^*\omega$ shows that the sum S must be augmented by difference terms arising from the limit sets of the subsequences of type ω or $^*\omega$.

Example 3.3. Consider the sequence with one gap,

$$\{a_k\} = \{11, 10\tfrac{1}{2}, 10\tfrac{1}{4}, \ldots, 10 + 2^{-n}, \ldots, -10 - 2^{-n}, \ldots, -10\tfrac{1}{2}, -11\}.$$

For this monotonic sequence our definitions yield

$$S^* = \frac{2}{2^p - 1} + 20^p.$$

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In forming the sums $S = \sum |\Delta a_k|^p$ for sequences that include cuts or gaps we therefore adjoin a term which is the pth power of the distance between the corresponding limit sets (or border elements). If the limit set is empty (sequence tends to infinity), then no term need be added. If a limit set E_k of an ω-sequence is an interval, the distance is reckoned as the shortest distance.

We may summarize this convention as follows: at every cut or gap, the limit sets on either side should be adjoined, if necessary, to form border elements. The sum S so formed will be denoted by the symbol

$$\sum_k{}' |\Delta a_k|^p.$$

The result of Theorem 3.1 can now be extended to such sums.

THEOREM 3.6. *Let the sequence $\{a_k\}$ have the order type*

$$\ldots + \mu_{(1)} + \mu_{(0)} + \mu_{(1)} + \ldots + \mu_{(m)} + \ldots$$

where each order type μ_l is either null, finite, ω, or $^*\omega$; and let the total number of entries ω or $^*\omega$ be denoted by N. Let the augmented sum

$$S = \sum_{k'}{}' |\Delta a_k|^p = \ldots + \sum_{(-1)} |\Delta a_k|^p + d_0{}^p + \sum_{(0)} |\Delta a_k|^p + d_1{}^p + \ldots$$

be finite. Then

(a) *S^* exists and $S^* \leqslant S$.*

(b) *E has at most N connected components.*

(c) *S^* has the form*

$$S^* = \sum_j S_j{}^* = \sum_j \sum_{\Delta a_k{}^* \subset O_j} |\Delta a_k{}^*|^p$$

where the summation over j contains at most $N + 1$ terms.

Proof. We use Theorem 3.1 for each of the subsequences of type ω or $^*\omega$. By the *span* of a subsequence, we shall mean the least closed interval of the real axis containing every term of the subsequence. Clearly the limit set of the subsequence is contained in the span.

For a typical subsequence of order type, say $\mu_{(m)}$, we find a connected limit set $E_{(m)} = [e_m, f_m]$ and sums $S^*{}_{(m)j}$ $(j = 1, 2)$ associated with open sets $O_{(m)j}$. By Theorem 3.1,

$$S^*{}_{(m)1} + S^*{}_{(m)2} \leqslant S_{(m)} = \sum\nolimits_{(m)} |\Delta a_k|^p.$$

Now sum over m, observing the overlap of any of the limit sets $E_{(m)}$. The combined limit set $E = \bigcup_{(m)} E_{(m)}$ clearly has at most N components; let O_j $(j = 1, \ldots, J \leqslant N + 1)$ be the complementary open sets. Let $S^*{}_{(m)(j)}$ denote the sum of the contributions to $S^*{}_{(m)1} + S^*{}_{(m)2}$ arising from O_j; in this reckoning proportionate parts of any term $|\Delta a_k|^p$ overlapping the end points of O_j are to be included in $S^*{}_{(m)(j)}$. Thus

$$S^*{}_{(m)(j)} = \sum\nolimits_{(m)} \alpha\, |\Delta a_k{}^*|^p$$

where $\alpha = 0$ if $\{\Delta a_k^*) \cap O_j = \emptyset$, $\alpha = 1$ if $\{\Delta a_k^*\} \subset O_j$, and α denotes the fraction of $\{\Delta a_k^*\}$ contained in O_j otherwise.

Consider first the case of an interval O_j that is bounded. Three cases can now arise. In the first case, there is an h such that O_j lies within the span of $\{a_k\}_{(\mu h)}$. Then we have $S_j^* \leqslant S^*_{(h)(j)}$, which can be shown as follows. The terms of other sections of the entire sequence falling in O_j subdivide further so that by repeated applications of Lemma 2.2 we see that the contributions to S_j^* from any such interval are dominated by those of $S^*_{(h)(j)}$. For the end point or "bordering" intervals which may be subdivided by the limit sets, say into part intervals of lengths l_1 and l_2, we have a contribution to S_j^* of the form l_1^p. Since $p \geqslant 1$, and

$$l_1^p \leqslant l_1(l_1 + l_2)^{p-1} = \alpha(l_1 + l_2)^p,$$

with $l_1 = \alpha(l_1 + l_2)$, we again find that the contribution to S_j^* is dominated by the corresponding term in $S^*_{(m)(j)}$. This shows that in the first case

$$S_j^* \leqslant S^*_{(h)(j)} \leqslant \sum_{(m)} S^*_{(m)(j)}.$$

In the second case, there is no spanning interval intersecting O_j. Since O_j is bounded above and below by limit sets, there must be at least one limit term, or border term d_h^p, referring to an interval enclosing O_j. Clearly, in this case

$$S_j^* = |O_j|^p \leqslant d_h^p.$$

The third case is a combination of the first two, with O_j being partially covered by spanning intervals. Each of the two (or three) subintervals of O_j so determined contributes a rearranged sum which may again be denoted by S_j^* (with suppression of the two (or three) new labels) and which is again subject to one of the two foregoing types of estimate.

Any unbounded intervals O_j remain to be considered. For these, however, only the first case will arise, since the range of the entire sequence is bounded by the full set of spanning intervals.

We can now carry out the summation over m and j, and we find that

$$\begin{aligned} S^* = \sum_j S_j^* &\leqslant \sum_{(m)} \sum_{(j)} S^*_{(m)(j)} + \sum_{(m)} d^p_{(m)} \\ &\leqslant \sum_{(m)} \sum_{j=1}^{2} S^*_{(m)j} + \sum_{(m)} d^p_{(m)} \\ &\leqslant \sum_{(m)} S_{(m)} + \sum_{(m)} d^p_{(m)} \\ &= S. \end{aligned}$$

This completes the proof of part (a) of the theorem. Parts (b) and (c) are immediate consequences of the proof.

Note that if the given enumerable sequence has c cuts and g gaps, then

$$N = c + 2g + (1 - f) + (1 - l),$$

where f (or l) $= 1$ if there is a first (or last) element and zero otherwise.

4. Rearrangements of functions. Let f be a function measurable on the interval $a \leqslant x \leqslant b$. The equimeasurable decreasing rearrangement f^* is conventionally defined as follows (**1**, p. 276). Let $M(y)$ be the measure of the set upon which $f(x) > y$. Then f^* is the translated inverse function of M; $f^*(a + x) = M^{-1}(x)$. Note that f^* is monotonic decreasing, and that the measure of the set upon which $f^*(x) > y$ is also $M(y)$. In particular, if f^p is integrable, then

$$\int_a^b |f^*(x)|^p \, dx = \int_a^b |f(x)|^p \, dx.$$

We shall prove in this section that the inequality

$$\int_a^b |f^{*\prime}(x)|^p \, dx \leqslant \int_a^b |f'(x)|^p \, dx, \qquad p \geqslant 1,$$

holds for functions f satisfying the fundamental formula of calculus. The inequality is to be interpreted in the following sense:

(1) If $f' \in L^p$, then $f^{*\prime} \in L^p$ and the inequality holds.

(2) If $f^{*\prime} \in L^p$, then the integral of f'^p is at least equal to that of $f^{*\prime p}$, and may be infinite.

The proof will be given by means of certain piecewise linear approximating functions $p_n(x)$, which are defined by subdividing the range of f. Our first lemma is a proof of the inequality for functions of this special type.

We assume that the number of subdivision points is finite.

LEMMA 4.1. *Let $p(x)$ be a piecewise linear function on (a, b), and let $p^*(x)$ be its decreasing rearrangement. Then $p^*(x)$ is also piecewise linear, and*

$$\int_a^b |p^{*\prime}(x)|^p \, dx \leqslant \int_a^b |p'(x)|^p \, dx, \qquad p \geqslant 1,$$

with equality only if $p(x)$ is monotonic.

Note that the derivatives of $p(x)$ and $p^*(x)$ exist almost everywhere.

Proof. Let $x_1, \ldots, x_m$ be the subdivision points of $p(x)$ and let $y_1, \ldots, y_m$ be the values of $p(x)$ at these subdivision points, ranged in decreasing order:

$$y_1 > y_2 \ldots > y_m.$$

Consider the range $y_k > p(x) > y_{k+1}$. If the graph of $p(x)$ traverses this range once only, then the portion of the graph $p^*(x)$ in this range is a straight line with negative slope having the same absolute value. Thus

$$|p^{*\prime}(x)| \leqslant |p'(x)|$$

in this set.

If the graph of $p(x)$ traverses this range two or more times, then the portion of the graph of $p^*(x)$ for this range is again a straight-line segment with negative slope, but the length of the corresponding domain (x-interval) is the sum of the lengths of the various x-intervals for the graph of $p(x)$. Since

$$\text{run} = \text{rise/slope},$$

and since the rise $y_k - y_{k+1}$ is the same for all the x-intervals involved, we find that

$$\frac{1}{|p^{*\prime}(x)|} = \sum_{k=1}^{q} \frac{1}{|p'(x_k)|},$$

where q is the number of x-intervals for the graph of $p(x)$, and x_k is a typical point of the kth interval. Also x^* denotes a point of the interval for $p^*(x)$. Thus

$$|p^{*\prime}(x^*)| \leqslant \min_{1 \leqslant k \leqslant q} |p'(x_k)|$$

with equality only for the case of a single interval. The result now follows immediately.

A piecewise linear function $p(x)$ will be called a PL approximation to a function $f(x)$ if the corners (subdivision points) of the graph of $p(x)$ lie on the graph of $f(x)$. That is, the graph of $p(x)$ is formed by joining a finite number of points of the graph of $f(x)$ by straight-line segments.

LEMMA 4.2. *If $p(x)$ is a* PL *approximation to $f(x)$ on $[a, b]$, then*

$$\int_a^b |p'(x)|^p \, dx \leqslant \int_a^b |f'(x)|^p \, dx,$$

with equality only if $f(x) = p(x)$ almost everywhere.

Proof. Let x_k, x_{k+1} be consecutive subdivision points of $p(x)$, and note that

$$f(x_k) = p(x_k), \qquad f(x_{k+1}) = p(x_{k+1}),$$

$$p'(x) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}, \qquad x_k < x < x_{k+1}.$$

Considering now only the interval $[x_k, x_{k+1}]$, we have

$$\int_{x_k}^{x_{k+1}} |p'(x)|^p \, dx = \frac{|f(x_{k+1}) - f(x_k)|^p}{|x_{k+1} - x_k|^{p-1}}.$$

By hypothesis the fundamental formula of calculus holds for f. Thus

$$|f(x_{k+1}) - f(x_k)| = \left| \int_{x_k}^{x_{k+1}} 1 . f'(s) \, ds \right|$$
$$\leqslant \left(\int_{x_k}^{x_{k+1}} |f'(s)|^p \, ds \right)^{1/p} \left(\int_{x_k}^{x_{k+1}} 1^q \, ds \right)^{1/q},$$

by Holder's inequality, where $p \geqslant 1$, and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Thus

$$|f(x_{k+1}) - f(x_k)|^p \leqslant \int_{x_k}^{x_{k+1}} |f'(s)|^p \, ds \, |x_{k+1} - x_k|^{p-1}$$

and hence

$$\int_{x_k}^{x_{k+1}} |p'(x)|^p \, dx \leqslant \int_{x_k}^{x_{k+1}} |f'(x)|^p \, dx.$$

Note that equality holds for Hölder's relation only if $f'(s)$ is proportional to 1, that is, constant.

The conclusion of the lemma is now obtained by summation over the subintervals.

Consider the rearranged function $f^*(x)$, which is monotonic and so has a derivative almost everywhere (**4**, p. 358).

LEMMA 4.3. *Let $x_1, x_2, \ldots$ denote a sequence dense in $[a, b]$, and let $p_n^*(x)$ be the* PL *approximation to $f^*(x)$ with subdivision points $x_1, \ldots, x_n$. Then the sequence of derivatives $p_n^{*\prime}(x)$ tends to $f^{*\prime}(x)$ almost everywhere in $[a, b]$.*

Proof. Let E denote the set

$$E = \{x | \, a \leqslant x \leqslant b, \quad f^{*\prime}(x) \text{ exists}, \; x \neq x_k\},$$

which clearly has measure $(b - a)$. We shall show that the result holds in E.

For $x \in E$,

$$f^{*\prime}(x) = \lim_{h \to 0} \frac{f^*(x + h) - f^*(x)}{h}$$

exists. Given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that

$$f^*(x + h) - f^*(x) = h[f^{*\prime}(x) + \eta],$$

where $|\eta| < \epsilon$ whenever $|h| < \delta(\epsilon)$.

Let ξ_n denote the least member of the sequence $x_1, \ldots, x_n$ that exceeds x, and suppose n is so large that $\xi_n - x < \delta(\epsilon)$. Also let ζ_n denote the greatest member of $x_1, \ldots, x_n$ that does not exceed x, and suppose n so large that $x - \zeta_n < \delta(\epsilon)$. Since

$$p_n^*(\zeta_n) = f^*(\zeta_n), \qquad p_n^*(\xi_n) = f^*(\xi_n),$$

therefore $p_n^*(\xi_n) - f^*(x) = (\xi_n - x)[f^{*\prime}(x) + \eta_1]$, where $|\eta_1| < \epsilon$. Also

$$p_n^*(\zeta_n) - f^*(x) = (x - \zeta_n)[f^{*\prime}(x) + \eta_2],$$

where $|\eta_2| < \epsilon$. Therefore, by subtraction,

$$p_n^*(\xi_n) - p_n^*(\zeta_n) = (\xi_n - \zeta_n)[f^{*\prime}(x) + \eta_3],$$

where

$$\eta_3 = \frac{(\xi_n - x)\eta_1 + (x - \zeta_n)\eta_2}{\xi_n - \zeta_n}.$$

It follows that

$$|\eta_3| \leqslant \max(|\eta_1|, |\eta_2|) < \epsilon.$$

Dividing by $\xi_n - \zeta_n$ in the previous formula, we obtain

$$p_n^{*\prime}(x) = \frac{p_n^*(\xi_n) - p_n^*(\zeta_n)}{\xi_n - \zeta_n} = f^{*\prime}(x) + \eta_3.$$

Since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} p_n^{*\prime}(x) = f^{*\prime}(x), \qquad x \in E.$$

This completes the proof of Lemma 4.3.

THEOREM 4.1. *If $f(x)$ has domain $[a, b]$, and the equimeasurable decreasing rearrangement of $f(x)$ is denoted by $f^*(x)$, then*

$$\int_a^b |f^{*\prime}(x)|^p\, dx \leqslant \int_a^b |f'(x)|^p\, dx, \qquad p \geqslant 1,$$

with equality only when $f(x)$ is monotonic.

Proof. As in the proof of Lemma 4.3, we choose a dense sequence x_k^* in $[a, b]$, and a sequence of PL approximations $p_n^*(x)$ to $f^*(x)$ in $[a, b]$. Let

$$y_k = f^*(x_k) = p_n^*(x_k), \qquad k, n = 1, 2, \ldots .$$

Now let $z_{k1}, z_{k2}, \ldots$ be the points of $[a, b]$ such that

$$f(z_{kl}) = y_k.$$

We define a corresponding PL approximation $p_n(x)$ to $f(x)$, based upon the set of all the subdivision points z_{kl}, where $k \leqslant n$.

Consider the equimeasurable decreasing rearrangement of $p_n(x)$, which is the decreasing piecewise linear function taking the value y_k at the subdivision point $x = M(y_k)$ where $M(y_k)$ is the measure of the set

$$\{x | f(x) > y_k\}.$$

Since $f^*(x)$ is the equimeasurable decreasing rearrangement of $f(x)$, we have

$$m\{x| f^*(x) > y_k\} = M(y_k) = x_k^*.$$

The piecewise linear decreasing function with range subdivided at $y_1, \ldots, y_n$ and domain subdivided at $x_1^*, \ldots, x_n^*$ has already been denoted by $p_n^*(x)$. Thus the * notation is justified by the fact that the PL approximation of the rearrangement is the rearrangement of the PL approximation. This is a general property valid provided that the subdivisions are taken with respect to the *range* of the functions involved.

From Lemma 4.3 we now conclude that

$$p_n^{*\prime}(x) \to f'(x), \qquad n \to \infty,$$

almost everywhere in $[a, b]$. Since $|p_n^*(x)|^p \geqslant 0$, we may conclude by Fatou's lemma (**4**, p. 346) that

$$\int_a^b |f^{*\prime}(x)|^p \, dx \leqslant \lim_{n\to\infty} \int_a^b |p_n^{*\prime}(x)|^p \, dx.$$

By Lemma 4.1 we see that the integral on the right is dominated by the corresponding integral for $|p_n'(x)|^p$, and hence that the right-hand side is not greater than

$$\lim_{n\to\infty} \int_a^b |p_n'(x)|^p \, dx.$$

Since $p_n(x)$ is a PL approximation to $f(x)$ on $[a, b]$, we now find from Lemma 4.2 that this latter expression does not exceed

$$\int_a^b |f'(x)|^p \, dx,$$

which integral is independent of n. This completes the general proof of Theorem 4.1. Whenever $f(x)$ is not monotonic, it is easy to find a sequence $p_n(x)$ having the same property, uniformly, and thus, by Lemma 4.1, to establish the strict inequality.

Our hypothesis that f satisfies the fundamental formula of calculus is satisfied, for example, if f is absolutely continuous in $a \leqslant x \leqslant b$, or if (**4**, p. 368) $f'(x)$ exists everywhere and is finite and integrable. It would be interesting to have a proof for functions of bounded variation, which are not covered by our present hypothesis.

5. Decreasing rearrangements of functions on an infinite interval. Whereas the decreasing equimeasurable rearrangement of a function defined over a finite interval (a, b) is well determined, there are functions defined over the infinite interval $(0, \infty)$ for which the usual definition of the decreasing rearrangement yields a trivial result. In this section we shall generalize the construction of such decreasing rearrangements of functions, obtaining in general a many-valued decreasing function that happens to be single valued in all cases where the usual definition applies non-trivially. In the following section we shall show that the fundamental inequality of this paper holds for such generalized decreasing rearrangements, where integration is taken over all branches of the many-valued function.

The portion of the range of f covered by these branches corresponds in the case of sequences (§ 3) to the portion of the range wherein the sequence is not dense. To the dense portion of the range of the sequence corresponds the remaining portion of the range of the function which has a property now called "asymptotic dense."

Given a measurable function f with domain an infinite interval such as $(0, \infty)$ or $(-\infty, \infty)$, we shall say that f is *asymptotically dense* at the value

b of its range if the measure of the set

$$\{x|\ |f(x) - b| \leqslant \epsilon\}$$

is infinite for every $\epsilon > 0$.

It is easily shown that the set of points b of the range at which f is asymptotically dense is closed.

If f is asymptotically dense for some $b > 0$, then $m\{x|\ f(x) > 0\} = \infty$ and it follows that according to the usual definition the equimeasurable decreasing rearrangement of f is constant on an infinite interval. Thus the behaviour of f in any other portion of the range is obscured. To preserve the behaviour of f in all such intervals where the decreasing rearrangement is not trivially constant, we consider the open set of range points wherein f is not asymptotically dense.

Being open, this set consists of open intervals O_j. Select any such open interval O_j; if it is bounded, choose its mid-point and if unbounded choose any one fixed interior point, and call this point b_j.

In effect, we now apply a conventional and suitable rearrangement process to $f - b_j$, and then translate the rearranged graph upward by b_j. An equivalent description of the rearrangement process for this interval, or branch, is as follows. Given any $b \in O_j$, $b > b_j$, let $a = -m(x|\ b > f(x) \geqslant b_j)$; and then let the point (a, b) be adjoined to the graph of f^*. Similarly, for $b \in O_j$, $b < b_j$, let $a = m(x|\ b_j > f(x) \geqslant b)$, and adjoin (a, b) to the graph of f^*. If f is constant on a set of finite measure, the graph of f^* should also include a segment of corresponding length parallel to the x-axis.

For $b \in O_j$, the graph so obtained defines a single-valued function f_j^* with range O_j and domain D_j a connected subset of the real numbers that includes the origin. The domain D_j may have finite or infinite length, and in general depends on the interval index j.

The function f_j^* is non-increasing and equimeasurable with f in the range interval O_j.

We now define the generalized equimeasurable decreasing rearranged function f^* as the (in general many-valued) function with graph the (countable) union of the graphs of f_j^* for all j. The domain of f^* is the union of the domains of all f_j^* while the range of f^* is the open subset of the range of f in which f is not asymptotically dense.

If f is nowhere asymptotically dense, then the graph of f^* consists of one branch, so that f^* is single valued.

Observe that a periodic function is everywhere asymptotically dense in its own range. By inserting periodic components at periodic positions in a given infinite domain, it is possible to construct a function that is asymptotically dense in any given countable set of closed intervals.

6. Unbounded domains. The preceding results can be extended to functions f defined on infinite intervals such as $(0, \infty)$ or $(-\infty, \infty)$. However, we encounter certain complications analogous to those of infinite sequences in § 3. It will therefore be advantageous to define the rearrangement of an unbounded function in the more general sense just described in § 5. The result is, as before, the integral I^* associated with the rearranged function existing in all cases and satisfying the inequality $I^* \leqslant I$.

For our first lemma, we show that as long as the range of a function f is not increased, then the integral $\int |f^{*\prime}(x)|^p\, dx$ is a non-increasing set functional of the domain of f.

LEMMA 6.1. *Let f_1 have range R_1 and domain D_1. Let f_2 be the restriction of f_1 to a domain $D_2 \subset D_1$, such that the range of f_2 is also R_1. Let f_1^* and f_2^* denote the equimeasurable decreasing rearrangements of f_1 and f_2. Then for $p \geqslant 1$*

$$\int_{D_1} |f_1^{*\prime}(x)|^p\, dx \leqslant \int_{D_2} |f_2^{*\prime}(x)|^p\, dx.$$

Proof. As in § 4, Lemma 4.3, let $[x_1, x_2, \ldots]$ be a sequence of subdivision points dense in D_1, and let $p_n^*(x)$, $q_n^*(x)$ respectively be the PL approximations to $f_1^*(x)$, $f_2^*(x)$ with subdivision points $[x_1, \ldots, x_n]$. (For q_n^* we may omit the subdivision points not in D_2.)

Observe that $f_2^*(x) \leqslant f_1^*(x)$ and consequently $q_n^*(x) \leqslant p_n^*(x)$, for $x \in D_2$. Thus the graphs of f_1^* and of p_n^* are more "stretched out" parallel to the x-axis than those of f_2^* and q_n^*. Given any Δy, the corresponding $\Delta_1 x$ for f_1^* is not less than the $\Delta_2 x$ for f_2^*. Consider any sufficiently small interval of the range R_1 containing in its interior no vertices of p_n^* or q_n^*. The contribution from q_n^* may be written as

$$|q_n^{*\prime}(x)|^p \Delta_2 x = |\Delta y/\Delta_2 x|^p \Delta_2 x = |\Delta y|^p/|\Delta_2 x|^{p-1},$$

while the contribution for p_n^* is

$$|p_n^{*\prime}(x)|^p \Delta_1 x = |\Delta y/\Delta_1 x|^p \Delta_1 x = |\Delta y|^p/|\Delta_1 x|^{p-1},$$

where $\Delta_2 x \leqslant \Delta_1 x$. Since $p \geqslant 1$, the latter does not exceed the former.

Summing over the range, we find that

$$\int_{D_1} |p_n^{*\prime}(x)|^p\, dx \leqslant \int_{D_2} |q_n^{*\prime}(x)|^p\, dx.$$

Now let $n \to \infty$. As in Lemma 4.3 we find that $p_n^{*\prime}(x)$ tends to $f_1^{*\prime}(x)$ almost everywhere in D_1, while $q_n^{*\prime}(x)$ tends to $f_2^{*\prime}(x)$ almost everywhere in D_2. By the general convergence theorem of Lebesgue (**4**, p. 345) we therefore find the inequality stated in Lemma 6.1.

Observe that whereas $\int |f^{*\prime}(x)|^p\, dx$ is a non-increasing functional of the domain (the range being fixed), the integral $\int |f'(x)|^p\, dx$ is a non-decreasing functional under the same conditions. In the particular case $f(x) = f^*(x)$ the integral must then be constant. But in this case $f(x)$ is monotonic and if the range is held fixed f must be constant in the additional intervals of the domain.

The next lemma shows that the integral considered is a monotonic increasing set functional of the range of the function.

LEMMA 6.2. *Let f_1 have domain D_1 and range $R_2 \subset R_1$. Let f_2 be the restriction of f_1 to the domain D_2 determined by the requirement that the range R_2 of f_2 be a given subset of R_1. Then*

$$\int_{D_2} |f_2^{*\prime}(x)|^p dx \leqslant \int_{D_1} |f_1^{*\prime}(x)|^p \, dx.$$

The proof of this lemma is immediate since the extension of the range introduces new elements into the domain while leaving unchanged those already present.

Given the function $f\colon x \to f(x)$ with (infinite) domain D, and range R, let the restriction of f to the interval $a \leqslant x \leqslant b$ be denoted by $f_{a,b}$, and let the further restriction of $f_{a,b}$ to the range $K < f < L$ be denoted by $f_{a,b;K,L}$.

Note that $f_{a,b;K,L}$ is a proper restriction of $f_{a,b}$ only if the range of $f_{a,b}$ is not included in the interval $K \leqslant f \leqslant L$.

The equimeasurable decreasing rearrangements of these restricted functions, each on their own domain, will be denoted as before by asterisks, thus $f^*_{a,b}$; $f^*_{a,b;K,L}$.

We now define

$$I^* = \lim_{\substack{L\to\infty \\ K\to-\infty}} \liminf_{\substack{(a,b) \\ b\to\infty \\ a\to-\infty}} \int_a^b |f^{*\prime}_{a,b;K,L}(x)|^p \, dx.$$

We shall show that I^* is in general less than or equal to

$$I = \int_D |f'(x)|^p \, dx.$$

THEOREM 6.1. *If $I < \infty$, then I^* exists and $I^* \leqslant I$, with equality if and only if f is almost everywhere equal to a monotonic function.*

Proof. For each $f_{a,b;K,L}$ we have from § 4, Theorem 4.1 the inequality

$$\int_a^b |f^{*\prime}_{a,b,K,L}(x)|^p \, dx \leqslant \int_a^b |f'_{a,b;K,L}(x)|^p \, dx.$$

Since $f_{a,b;K,L}$ is a restriction of f, the integral on the right does not exceed the corresponding integral I for the function f. Thus

$$\int_a^b |f^{*\prime}_{a,b;K,L}(x)|^p \, dx \leqslant I.$$

Take any closed interval $[K, L]$ properly contained within R. For (a, b) sufficiently large the domain of $f_{a,b}$ will include the domain specified by $[K, L]$ as a subset. Thus for any larger values of (a, b) we find by Lemma 6.1

that the integral on the left above is a non-increasing set function of the domain (a, b). Thus, as (a, b) tends to the full domain D, the limit

$$I^*_{K,L} = \liminf_{(a,b)\to D} \int_a^b |f^{*\prime}_{a,b;K,L}(x)|^p \, dx$$

exists.

By the preceding inequality we also have $I^*_{K,L} \leqslant I$.

By Lemma 6.2 it is easily shown that $I^*_{K,L}$ is a non-decreasing set function of the range interval specified by (K, L). Since $I^*_{K,L}$ is bounded above by I, we see that as (K, L) tends to the full range R the limit

$$I^* = \lim_{(K,L)\to D} I^*_{K,L}$$

exists and is less than or equal to I. This proves the main statement of the theorem.

If f is almost everywhere equal to a monotonic function, then the equality sign will clearly hold. If f does not have this property, then the strict inequality can be established for a suitable finite restriction of f, by the methods of § 4. Considering separately the other portions of the domain and range for the given function f, the overall result with the strict inequality can then be verified.

For sequences we found that the limit sets make no contribution to the rearranged sum. Likewise, for functions, we shall now show that the asymptotically dense portions of the range make no contribution to I^*.

THEOREM 6.2. *If the interval* (K, L) *is contained in that portion of the range wherein* f *is asymptotically dense, then*

$$\lim_{(a,b)\to\infty} \int_a^b |f^{*\prime}_{a,b;K,L}(x)|^p \, dx = 0, \qquad p > 1.$$

Proof. The function f^* is monotonic and hence of bounded variation on (a, b). By **(3**, p. 15**)** f^* can be expressed as the sum $\phi^* + s^*$ of a continuous non-increasing function ϕ^* and a non-increasing saltus function s^* both of bounded variation. Since $s^*(a) - s^*(b) \geqslant 0$,

$$0 \leqslant \phi^*(a) - \phi^*(b) \leqslant \phi^*(a) + s^*(a) - \phi^*(b) - s^*(b)$$
$$= f^*(a) - f^*(b),$$

we have

$$|\phi^*(b) - \phi^*(a)| \leqslant |f^*(b) - f^*(a)|.$$

Also

$$f^{*\prime}(x) = \phi^{*\prime}(x)$$

almost everywhere on (a, b).

Therefore (**3**, 11.54, p. 361),

$$\int_a^b |f^{*\prime}(x)|\, dx = -\int_a^b f^{*\prime}(x)\, dx = -\int_a^b \phi^{*\prime}(x)\, dx$$
$$\leqslant \phi^*(a) - \phi^*(b) = |\phi^*(b) - \phi^*(a)|.$$

(Note that we are applying Titchmarsh's result to a non-increasing function and so must observe a change of sign.)

Collecting these results, we have

LEMMA 6.3.

$$\int_a^b |f^{*\prime}(x)|\, dx \leqslant |f^*(b) - f^*(a)|.$$

Returning to the proof of Theorem 6.2, we note that

$$\int_a^b |f^{*\prime}(x)|^p\, dx \leqslant \max_{a\leqslant x\leqslant b} |f^{*\prime}(x)|^{p-1} \int_a^b |f^{*\prime}(x)|\, dx$$
$$\leqslant \max_{a\leqslant x\leqslant b} |f^{*\prime}(x)|^{p-1} |f^*(b) - f^*(a)|.$$

The second factor on the right is bounded by the range interval $|L - K|$. We shall show also that by choosing a and b large enough, the first factor on the right can be made arbitrarily small.

Suppose, on the contrary, that for each (a, b) and for any given $\epsilon > 0$, there is an x such that

$$|f^{*\prime}(x)| \geqslant \epsilon.$$

Then there must exist an interval (x_1, x_2) (containing x, in general) such that the chord of the curve $y = f^*(x)$ meeting the curve at x_1 and x_2 has the slope of magnitude exceeding $\frac{1}{2}\epsilon$. Thus there is a closed interval $(f^*(x_1), f^*(x_2))$ of the range to which corresponds an x-interval of measure at most

$$2\epsilon^{-1} |f^*(x_1) - f^*(x_2)|.$$

As the interval (a, b) tends to infinity, say through a sequence of values (a_n, b_n), the magnitudes of the derivatives of the functions $f^*_{a,b}$ at any place in the range will decrease or at least not increase. Thus let E_n be the closed set of range points such that

$$|f^{*\prime}_{a,b;K,L}(x)| \geqslant \epsilon.$$

Then the sequence E_n is decreasing and, by hypothesis, each E_n is non-empty. Hence the intersection E of all E_n exists and is non-empty. Let the value $f(x_0)$ be any element of E.

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Consider the branch of $f^*(x)$ formed by commencing at $f(x_0)$. The slope at $(0, f(z_0))$ of this branch is a negative and increasing set function of (a, b) that is less than $-\frac{1}{2}\epsilon$ for every (a, b) of the given sequence tending to D. Thus in the limit this slope exists and is less than $-\frac{1}{2}\epsilon$: $f^{*\prime}(0) < -\frac{1}{2}\epsilon$. Therefore there is an η-interval, where $\eta > 0$, about $f^*(0) = f(x_0)$ such that the f^* chord of this interval has slope less than $-\frac{1}{2}\epsilon$. Finally, there exists an interval about $f^*(0)$ such that the corresponding x-measure is finite. Therefore $f(x)$ was not asymptotically dense at $f(x_0)$. But this contradicts our hypothesis and completes the proof of the theorem.

To derive the explicit form of our integral inequality, we return to the definition of I^* and observe that the contributions of the asymptotically dense subsets of the range have been shown to vanish. Consider now the intervals O_j determined as in § 5 by the generalized decreasing rearrangement f^*. If O_j is $K < f < L$, then we shall show that

$$\liminf_{a,b\to\infty} \int_a^b |f^{*\prime}_{a,b;K,L}(x)|^p \, dx = \int_{D_j} |f_j^{*\prime}(x)|^p \, dx.$$

This result follows at once from the fact that $f^*_{a,b;K,L}$ is equimeasurable to a "finite" non-increasing function $f^*_{j;a,b}$ that is an approximation to f_j^* in the range interval O_j. As $(a, b) \to \infty$, $f^*_{j;a,b}(x) \to f_j(x)$ for every $x \in D_j$ while $f^{*\prime}_{j;a,b}(x)$ tends monotonically to $f_j^{*\prime}(x)$. This completes the proof of the relation above.

Taking the limit as $(K, L) \to \infty$, we obtain a summation over the range intervals O_j, and so find that

$$I^* = \sum_{j=1}^{\infty} \int_{D_j} |f_j^{*\prime}(x)|^p \, dx.$$

This yields the final form of the integral inequality, which we state as

THEOREM 6.3. *If f has generalized equimeasurable decreasing rearrangement f^* with branches f_j^* and domains D_j,* then

$$\sum_{j=1}^{\infty} \int_{D_j} |f_j^{*\prime}(x)|^p \, dx \leqslant \int_{-\infty}^{\infty} |f'(x)|^p \, dx,$$

equality holding only if f is almost everywhere monotonic, in which case f^* is single valued.

As in § 3 we have restrictions on the number of terms in the summation over j on the left side.

LEMMA 6.4. *If the domain of f is $[0, \infty)$, and if*

$$\int_0^{\infty} |f'(x)|^p \, dx < \infty,$$

then the set E of asymptotic density of f is connected.

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Proof. If E is not connected, there exists a closed interval of positive length l separating points of E, and a sequence of intervals (c_n, d_n) $(n = 1, 2, \ldots)$ such that in each the graph of f crosses this same interval. By Lemma 4.2, we have

$$\frac{l^p}{|d_n - c_n|^{p-1}} \leqslant \int_{c_n}^{d_n} |f'(x)|^p \, dx, \qquad n = 1, 2, 3, \ldots .$$

Summing over n, we find that

$$l^p \sum_{n=1}^{\infty} \frac{1}{|d_n - c_n|^{p-1}} \leqslant \int_0^{\infty} |f'(x)|^p \, dx < \infty .$$

If $p = 1$, this is already a contradiction since $\sum_{n=1}^{\infty} 1$ is not finite. If $p > 1$, then $|d_n - c_n|^{-1}$ must tend to zero as $n \to \infty$. But then the lengths $d_n - c_n$ of the intervals tend to infinity with n. This contradicts our hypothesis that the separating range interval contains no points of asymptotic density, thus implying that the measure of the corresponding domain is finite.

COROLLARY 6.4. *If the domain of f is semi-infinite* $([0, \infty))$, *then there are at most two domains D_j in Theorem* 6.3.

This result corresponds to Corollary 3.5 for sequences and its proof is now immediate from Lemma 6.4. Likewise, we have a result for the doubly infinite range $(-\infty, \infty)$, for which the set E of asymptotic density has at most two connected components.

COROLLARY 6.5. *If the domain of f is doubly infinite* $(-\infty, \infty)$, *then there are at most three domains D_j in Theorem* 6.3.

Example 6.1. Let

$$f(x) = 10 \tanh x + \sin \log(1 + x^2) , \qquad -\infty < x < \infty .$$

Then the asymptotically dense set consists of the intervals $[-11, -9]$ and $[9, 11]$, and there are three intervals D_j.

Analogues for functions of the generalized order types considered in § 3 would involve many-valued functions f. As these seem less natural than in the case of sequences, we omit further consideration of them.

7. Inequalities for higher derivatives. Although rearrangement inequalities do not hold for second differences of sequences, as shown in § 2, there are inequalities for second and higher order derivatives of functions. We conclude by discussing these under hypotheses of ample differentiability.

The basic relation giving the slope value for a rearranged functions is, as foreshadowed in § 4, the "harmonic sum" relation

$$1/|f^{*\prime}(x)| = \sum_{x_k} 1/|f'(x_k)|, \tag{7.1}$$

where the summation runs over all roots x_k of the equation $f(x_k) = f^*(x)$. From this relation follows

(7.2) $$|f^{*\prime}(x)| \leqslant \min_{x_k} |f'(x_k)|,$$

which could be used to prove the basic integral inequality under less general conditions than those of § 4. The equality of measures in the original and the rearranged graphs can be heuristically represented by the relation

(7.3) $$dx = \sum_k dx_k,$$

where the summation is again taken over the roots x_k of $f(x_k) = f^*(x)$. Our results will be derived formally from (7.1) and (7.3), and their signed analogues, namely

(7.4) $$1/f^{*\prime}(x) = \sum_k \pm 1/f'(x_k)$$

(7.5) $$dx = \sum \pm dx_k,$$

where a negative sign appears in all terms for which $f'(x_k) > 0$.

Differentiating (7.4) with respect to x, we find that

(7.6) $$\frac{f^{*\prime\prime}(x)}{[f^{*\prime}(x)]^2} = \sum_k \pm \frac{f''(x_k)}{[f'(x_k)]^2} \frac{dx_k}{dx}.$$

From the relation

(7.7) $$\pm\frac{dx_k}{dx} = \frac{dy/dx}{\pm dy/dx_k} = \frac{f^{*\prime}(x)}{f'(x_k)},$$

we can then obtain the pointwise equation

(7.8) $$f^{*\prime\prime}(x)/[f^{*\prime}(x)]^3 = \sum_k f''(x_k)/[f'(x_k)]^3.$$

The strongest integral relation obtainable from (7.6) has the form

(7.9) $$\int_a^b \frac{|f^{*\prime\prime}(x)|}{[f^{*\prime}(x)]^2} dx = \int_a^b \frac{|f''(x')|}{[f'(x')]^2} dx'.$$

From (7.2) and (7.6) we have

$$|f^{*\prime\prime}(x)|\, |dx| \leqslant \sum_k |f''(x_k)|\, |dx_k|,$$

and since $|dx| = \sum_k |dx_k|$, it follows from (**1**, p. 26, Theorem 16) that

(7.10) $$|f^{*\prime\prime}(x)|^p\, |dx| \leqslant \sum |f''(x_k)|^p\, |dx_k|, \qquad p \geqslant 1.$$

Integration over the domain yields

THEOREM 7.1. *The basic inequality for second derivatives*

(7.11) $$\int_a^b |f^{*\prime\prime}(x)|^p\, dx \leqslant \int_a^b |f''(x)|^p\, dx, \qquad p \geqslant 1$$

holds for $f \in C^2[a, b]$.

By successive differentiations a sequence of inequalities involving higher-order derivatives can be established. For order higher than two, however, lower-order terms will also be present. Let

$$Q_n(f, x) = f'(x)\left(\frac{1}{f'(x)}\frac{d}{dx}\right)^{n-1}\frac{1}{f'(x)}. \tag{7.12}$$

Then

$$\int_a^b |Q_n(f^*, x)|^p\, dx \leqslant \int_a^b |Q_n(f, x)|^p\, dx, \qquad p \geqslant 1 \tag{7.13}$$

$n = 1, 2, 3, \ldots$.

8. Conclusion. Extension of these inequalities to cover functions of bounded variation would be desirable. The basic property of the norms used is convexity and hence extensions to more general norms can be undertaken. In several dimensions a variety of analogous inequalities can be established. I am indebted to Professors P. G. Rooney and P. Scherk for their helpful suggestions in connection with this topic.

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