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COVERING THEOREMS FOR UNIVALENT FUNCTIONS MAPPING ONTO DOMAINS BOUNDED BY QUASICONFORMAL CIRCLES

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1. Introduction. Let Γ be a Jordan curve in the extended complex plane **C**. Γ is called a *quasiconformal circle* if it is the image of a circle by a homeomorphism f which is quasiconformal in a neighborhood of that circle. If $q(z_1, z_2)$ is the chordal distance from z_1 to z_2, the chordal cross ratio of a quadruple z_1, z_2, z_3, z_4 in **C** is

$$\chi(z_1, z_2, z_3, z_4) = \frac{q(z_1, z_2)\, q(z_3, z_4)}{q(z_1, z_3)\, q(z_2, z_4)}.$$

Ahlfors [**2**] has shown that a Jordan curve Γ is a quasiconformal circle if and only if

$$\sup \{\chi(z_1, z_2, z_3, z_4) + \chi(z_2, z_3, z_4, z_1)\}$$

is finite, where the supremum is taken over all ordered quadruples on Γ.

Definition 1. For $k \in [0, 1]$, a Jordan curve Γ in **C** is a *k-circle* if

(1) $\quad \chi(z_1, z_2, z_3, z_4) + \chi(z_2, z_3, z_4, z_1) \leqq 1/k$

for all ordered quadruples of points on Γ.

For $k = 0$, condition (1) is vacuous, so a 0-circle is an arbitrary Jordan curve, while if $k > 0$, a k-circle is a quasiconformal circle. Since the chordal cross ratio is invariant under Möbius transformations, it is easily verified that a 1-circle is a Euclidean circle or straight line. Thus as k runs from 0 to 1, the class of k-circles interpolates between arbitrary Jordan curves and the simplest Jordan curves. For each $k \in (0, 1]$, the curve consisting of the two rays $\arg(z) = \pm \arcsin(k)$ is a k-circle.

Aharanov and Kirwan [**1**] solved the following covering problem for the class $\mathscr{C}$ of normalized analytic univalent functions, f, which map $U = \{z : |z| < 1\}$ onto a convex domain. Let $R(\varphi) = \{w : \arg w = \varphi\}$ and let $l(\varphi)$ denote the linear measure of $R(\varphi) \cap f(U)$. What is the minimum of $l(\varphi_1) \cdot l(\varphi_2)$ $(0 \leqq \varphi_1 \leqq \varphi_2 < 2\pi)$ for $f \in \mathscr{C}$? We will consider the same problem for different classes of functions $\mathscr{S}_k$ $k \in [0, 1]$, the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in U such that $f(U)$ is bounded by a k-circle. We note $\mathscr{S}_{k_1} \subset \mathscr{S}_{k_2}$ if $k_2 < k_1$ and the uniform closure of $\mathscr{S}_0$ is the

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full class $\mathscr{S}$. A map $f \in \mathscr{S}$ is in $\mathscr{S}_k$ for some $k > 0$ if and only if f can be extended to a quasiconformal mapping of the whole plane [**7**, p. 98].

If D is a simply connected domain, f analytic and univalent in $U, f(U) = D$ and $f(0) = z_0$, the *inner mapping radius of D at z_0*, $r_D(z_0)$, is defined by

$$r_D(z_0) = |f'(0)|.$$

If D^* is obtained from D by circular symmetrization with respect to a ray from z_0 then $r_D(z_0) \leqq r_{D^*}(z_0)$ [**5**, p. 81] and equality holds if and only if D^* is obtained from D by a rotation around z_0 [**6**].

2. A symmetrization lemma.

LEMMA 1. *Let D be a domain bounded by a k-circle, $0 \in D$, $\infty \in \partial D$. If ∂D contains a point z' with $|z'| = a$ then the circular symmetrization D^* of D with respect to the positive real axis is contained in the domain $D_{k,a} = \{z : |\arg (z + a)| < \pi - \arcsin (k)\}$.*

Proof. For $r > a$ the circle $|z| = r$ separates z' from ∞, hence contains a subarc separating z' from ∞ in $\mathbf{C} - D$. The endpoints α and β of this arc separate z' from ∞ on ∂D. Thus inequality (1) may be applied to the quadruple $\alpha, z', \beta, \infty$. We thus obtain

$$|\alpha - z'| + |z' - \beta| \leqq \frac{1}{k} |\alpha - \beta|.$$

Hence z' must be inside an ellipse with foci at α and β and eccentricity k. If we let $2c = |\alpha - \beta|$ and b be the semi-minor axis of the ellipse, we have $b = (c/k)(1 - k^2)^{1/2}$. In order to satisfy $|\alpha| = |\beta| = r$, $|z'| = a$ and z' inside the ellipse, we must have $(a + b)^2 \geqq r^2 - c^2$ which leads to

$$c \geqq k((r^2 - k^2a^2)^{1/2} - a(1 - k^2)^{1/2}).$$

Thus the complement of D^* includes the arc

$$\{z : |z| = r, |\arg (z)| \geqq \pi - \arcsin (k/r((r^2 - a^2k^2)^{1/2} - a(1 - k^2)^{1/2}))\}$$

which is more easily described as

$$\{z : |z| = r, |\arg(z + a)| \geqq \pi - \arcsin (k)\}.$$

3. Covering of radial segments. We first obtain a boundary distortion result.

LEMMA 2. *If D is a domain bounded by a k-circle, $0 \in D$ and $\zeta_1, \zeta_2 \in \partial D$ then*

$$\frac{|\zeta_1 - \zeta_2|}{|\zeta_1|\,|\zeta_2|} \leqq \frac{1}{r_D(0)} \frac{4}{\pi} (\pi - \arcsin k).$$

Proof. The Möbius transformation

(2) $$T(z) = \frac{z}{z - \zeta_1} \frac{\zeta_2 - \zeta_1}{\zeta_2}$$

maps D onto a k-domain D^* and $r_{D^*}(0) = r_{D^*}(T(0)) = r_D(0)|T'(0)|$ so

$$r_{D^*}(0) = \frac{|\zeta_2 - \zeta_1|}{|\zeta_2||\zeta_1|} r_D(0).$$

Since $\infty = T(\zeta_1) \in \partial D^*$ and $1 = T(\zeta_2) \in \partial D^*$, the symmetrization, D^{**}, of D^* is contained in the domain $D_{k,1}$ of Lemma 1. A branch of the function

(3) $$f_k(z) = \left(\frac{1+z}{1-z}\right)^{\frac{2(\pi - \arcsin k)}{\pi}} - 1$$

maps U onto $D_{k,1}$ with $f(0) = 0$ so $r_{D_{k,1}}(0) = |f_k'(0)| = (4/\pi)(\pi - \arcsin k)$. Thus

(4) $$r_D(0) \frac{|\zeta_2 - \zeta_1|}{|\zeta_1|\,|\zeta_2|} = r_{D^*}(0) \leqq r_{D^{**}}(0) \leqq r_{D_{k,1}}(0) = \frac{4(\pi - \arcsin k)}{\pi}$$

We note in passing that Lemma 2 implies the following known covering theorem [**3**, Corollary 2.3].

THEOREM 1. *The Koebe region for the class* $\mathscr{S}_k$ *is a disk of radius* $\pi/4(\pi - \arcsin k)$.

Proof. We must find

$$\inf_{f \in \mathscr{S}_k} \left(\min_{0 \leqq \theta < 2\pi} |f(e^{i\theta})| \right).$$

By composing f with an appropriate Möbius transformation we see that the infimum is attained for the case when $f(U)$ is unbounded. Then we apply Lemma 2 with $D = f(U)$ so that $r_D(0) = 1$ by the normalization of f, and let $\zeta_2 \to \infty$. The function $F_k(z) = f_k(z)/(f_k'(0))$, where $f_k(z)$ is defined by (3), is in $\mathscr{S}_k$ and

$$F_k(-1) = \frac{\pi}{4(\pi - \arcsin k)}.$$

Thus the bound is sharp.

Let $f \in \mathscr{S}_k$, $R(\varphi) = \{w : \arg w = \varphi\}$ and $l(\varphi)$ denote the linear measure of $R(\varphi) \cap f(U)$. We wish to minimize $l(\varphi_1) \cdot l(\varphi_2)$ $(0 \leqq \varphi_1 \leqq \varphi_2 \leqq 2\pi)$ over the class $\mathscr{S}_k$. Equivalently we wish to minimize $l(\varphi) \cdot l(-\varphi)$ for $0 \leqq \varphi \leqq \pi/2$.

THEOREM 2. *Let* $f \in \mathscr{S}_k$ *and* $\varphi \in [0, \pi/2]$. If $\pi/6 \leqq \varphi \leqq \pi/2$ *then*

$$l(\varphi) \cdot l(-\varphi) \geqq \left(\frac{\pi \sin \varphi}{2(\pi - \arcsin k)}\right)^2$$

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while if $0 \leqq \varphi < \pi/6$ *then*

$$l(\varphi) \cdot l(-\varphi) \geqq \left(\frac{\pi}{4(\pi - \arcsin k)}\right)^2.$$

For $\varphi \geqq \pi - \arcsin k/2$, equality is attained only for the function

$$F(z) = T^{-1}(-f_k(iz))$$

where f_k is the function defined in Equation (3) and T is the function (2) with

$$\zeta_1 = \frac{\pi \sin \varphi \, e^{-i\varphi}}{2(\pi - \arcsin k)} \quad \text{and} \quad \zeta_2 = \frac{\pi \sin \varphi}{2(\pi - \arcsin k)} e^{i\varphi}.$$

Proof. Let $z_1 = r_1 e^{i\varphi}$, $z_2 = r_2 e^{-i\varphi}$ be points on the boundary of $f(U)$ such that the segments $[0, r_1 e^{i\varphi})$, $[0, r_2 e^{-i\varphi})$ are in $f(U)$. Then $l(\varphi) \cdot l(-\varphi) \geqq r_1 r_2$. By Lemma 2,

$$\frac{|r_1 e^{i\varphi} - r_2 e^{-i\varphi}|}{r_1 r_2} \leqq \frac{4}{\pi}(\pi - \arcsin k).$$

For notational convenience we let $K = (4/\pi)(\pi - \arcsin k)$ and so

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2}{r_1 r_2} \cos 2\varphi \leqq K^2,$$

$$\frac{1}{r_1} \leqq \frac{\cos 2\varphi}{r_2} + \left[\frac{\cos^2 2\varphi}{r_2^2} - \frac{1}{r_2^2} + K^2\right]^{1/2},$$

thus

$$r_1 \geqq \frac{r_2}{\cos 2\varphi + [(Kr_2)^2 - \sin^2 2\varphi]^{1/2}} \tag{5}$$

and

$$l(\varphi) \cdot l(-\varphi) \geqq \frac{r_2^2}{\cos 2\varphi + [(Kr_2)^2 - \sin^2 2\varphi]^{1/2}} = h(r_2). \tag{6}$$

By Theorem 1, $r_2 \geqq 1/K$, so we wish to minimize the function $h(x)$ on the interval $[1/K, \infty)$. Since $\lim_{x \to \infty} h(x) = \infty$ we have a finite minimum. The substitution

$$(Kx)^2 = \zeta^2 - 2\zeta \cos 2\varphi + 1$$

replaces $h(x)$ by $(1/K^2)(\zeta + 1/\zeta - 2\cos 2\varphi)$ which expression we wish to minimize for $\zeta \in [\cos 2\varphi + |\cos 2\varphi|, \infty)$. For $\varphi > \pi/6$, the minimum occurs for $\zeta = 1$ while if $\varphi \leqq \pi/6$ the minimum occurs at $\zeta = \cos 2\varphi + |\cos 2\varphi|$. Thus for $\varphi > \pi/6$ we have

$$l(\varphi) \cdot l(-\varphi) \geqq h\left(\frac{2}{K} \sin \varphi\right) = \frac{4}{K^2} \sin^2 \varphi$$

as claimed. For $\varphi \leqq \pi/6$ we have

$$l(\varphi) \cdot l(-\varphi) \geqq h\left(\frac{1}{K}\right) = \frac{1}{2K^2 \cos 2\varphi}.$$

However, by Theorem 1, we know $l(\varphi)$ and $l(-\varphi)$ are separately greater than or equal to $1/K$ so we have the better estimate

$$l(\varphi) \cdot l(-\varphi) \geqq 1/K^2.$$

In order for the minimum to be attained we must have $r_2 = (2/K) \sin \varphi$ and $r_1 r_2 = h((2/K) \sin \varphi)$ which requires $r_1 = r_2$, and we must have equality at each step in Inequality (4). This occurs if and only if D^* is a rotation of D^{**} and $D^{**} = D_{k,1}$. This will occur if and only if $f(U) = D = T^{-1}(-D_{k,1})$ where T is defined as above. This is possible if and only if $\infty \notin T^{-1}(-D_{k,1})$ or $e^{-2i\varphi} - 1 \notin D_{k,1}$ which means $\pi - \arcsin k \leqq 2\varphi$. In this case, an extremal function F must map U onto the image of U under the map $T^{-1}(-f_k(z))$ hence $F(z) = T^{-1}(-f_k(S(z)))$ for some self map, S, of U. Normalization of F then requires $S(z) = iz$ and the proof is finished.

The inequality (5) which was used in the proof of Theorem 2 has the following interesting interpretation.

THEOREM 3. *If $f \in \mathscr{S}_k$ and $\zeta \in \partial(f(U))$ then $f(U)$ contains the disk with center at $-\zeta/(K^2|\zeta|^2 - 1)$ and radius $K|\zeta|^2/(K^2|\zeta|^2 - 1)$, where $K = 4(\pi -$ arcsin $k)/\pi$.*

Proof. With r_2 constant, $r_1 = r$, and $2\varphi = \theta$, the equation corresponding to inequality (5) is the polar coordinate equation of such a circle.

COROLLARY 1. *If $f \in \mathscr{S}_k$ and $|f(z)| \leqq M$ for all $z \in U$, then $f(U)$ contains a disk, containing the origin, with radius $KM/(K^2M^2 - 1)$ where $K = 4(\pi -$ arcsin $k)/\pi$.*

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