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RESTRICTING AND INDUCING ON INNER PRODUCTS OF REPRESENTATIONS OF FINITE GROUPS

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1. Introduction. Of recent years the author has been interested in developing a representation theory of the algebra of representations [**5**; **6**] of a finite group G, and dually of its classes [**7**]. In this paper Frobenius' Reciprocity Theorem provides a starting point for the introduction of the inverses R^{-1} and I^{-1} of the restricting and inducing operators R and I. The condition under which such inverse operations are available is that *the classes of* G *do not split in the subgroup* $\hat{G}$. When this condition is satisfied the application of these operations to inner products is of interest. In particular we show that

$$R(\{\lambda\} \times \{\mu\}) = R\{\lambda\} \times R\{\mu\} \text{ and } R^{-1}(\{\lambda\} \times \{\mu\}) = R^{-1}\{\lambda\} \times R^{-1}\{\mu\}$$

as one might expect, but more surprisingly:

$$I(\{\hat{\lambda}\} \times \{\hat{\mu}\}) = R^{-1}\{\hat{\lambda}\} \times I\{\hat{\mu}\} \text{ and } I^{-1}(\{\lambda\} \times I\{\hat{\mu}\}) = R\{\lambda\} \times \{\hat{\mu}\}.$$

Ito [**2**] has already observed that the inducing operation holds without the restriction on splitting classes when we write:

$$I(R\{\lambda\} \times \{\hat{\mu}\}) = \{\lambda\} \times I\{\hat{\mu}\}.$$

The obvious illustration of the ideas is obtained by setting $G = S_n, \hat{G} = S_m$ $(m < n)$. One can only hope that Young's machinery will eventually be generalized so that representation theory could become explicit.

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2. Young's raising operator. Long ago Alfred Young [**8**] introduced a raising operator R_{ij}, defined as raising a node from the jth row to the ith row of a diagram $[\lambda]$. If the identity representation of the subgroup S_{λ_i} is denoted by the ith row $[\lambda_i]$ of $[\lambda]$, then the permutation representation of S_n induced by the identity representation of $S_{\lambda_1} \times S_{\lambda_2} \ldots$ is given by [**3**]

$$[\lambda_1] \,.\, [\lambda_2] \ldots = \prod (1 - R_{ij})^{-1}[\lambda]$$

and conversely

$$[\lambda] = \prod (1 - R_{ij})[\lambda_1] \,.\, [\lambda_2] \ldots$$

Note that R_{ij} is applicable subject to its *viability*. If we use the suffix 0 to

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denote a non-row of $[\lambda]$ we may factor $R_{0i} = R_{01}R_{1i}$ to obtain the *restricting operator* which removes a node in all possible ways:

$$R = R_{01} + R_{02} + \ldots = R_{01}(1 + R_{12} + R_{13} + \ldots)$$

and the *inducing operator* which adds a node:

$$I = R_{10} + R_{20} + \ldots = (1 + R_{21} + R_{31} + \ldots)R_{10}.$$

Similarly we can define *lowering* operators S_{ij} applied to the columns instead of the rows. We define the *inverses* of the operators R and I thus:

$$R^{-1} = (1 + R_{12} + R_{13} + \ldots)^{-1}R_{10},\ I^{-1} = R_{01}\,(1 + R_{21} + R_{31} + \ldots)^{-1}$$

It is worth illustrating these operators in the following

Example 2.1. $R[3, 2, 1] = [2^2, 1] + [3, 1^2] + [3, 2]$ and using the above factorization of R^{-1} we have:

$$\begin{aligned} R^{-1}R[3, 2, 1] &= R^{-1}([2^2, 1] + [3, 1^2] + [3, 2]) \\ &= (1 + R_{12} + R_{13})^{-1}([3, 2, 1] + [4, 1^2] + [4, 2]) \\ &= ([3, 2, 1] + [4, 1^2] + [4, 2]) \\ &\quad - ([4, 1^2] + [4, 2] + 2[5, 1]) + (2[5, 1] + 2[6]) - 2[6] \\ &= [3, 2, 1]. \end{aligned}$$

It should be noted that R_{10} is applicable to any $[\lambda]$, though R_{01} is not. Comparable restrictions apply to R_{i0} and R_{0i}; we illustrate other factorizations of R in

Example 2.2.

$$\begin{aligned} R^{-1}[2, 1] &= (1 + R_{12})^{-1}R_{10}[2, 1] = [3, 1] - [4] \\ &= (1 + R_{21})^{-1}R_{20}[2, 1] = [2^2] \\ &= (1 + S_{12})^{-1}S_{10}[2, 1] = [2, 1^2] - [1^4] \end{aligned}$$

and in each case $RR^{-1}[2, 1] = [2, 1]$.

THEOREM 2.3. *Subject to the same factorization being applied to inverses*

$$R^{-1}R = RR^{-1} = 1 = II^{-1} = I^{-1}I.$$

3. The general theory. Also in paper **[7]** we have expressed Frobenius' Reciprocity Theorem in the matrix form

$$(3.1)\quad \{\lambda\}F = F(F\{\hat{\lambda}\}),\quad F\{\hat{\lambda}\}F' = (F'\{\lambda\})$$

where $\{\lambda\}$, $\{\hat{\lambda}\}$ are the matrices representing the irreducible representations λ of G and $\hat{\lambda}$ of $\hat{G} \subset G$. The matrix F describes the restrictions of the representation $\{\lambda\}$ of G to $\hat{G}$. Using Young's operator theory we can construct a left inverse F_0 for S_n as illustrated in

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Example 3.2. As in [**7**] we take $G = S_4$ with $\hat{G} = S_3$ so that:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \end{bmatrix}, \text{ etc.}$$

Each allowable factorization of R^{-1} provides a choice for a row of F_0. The Young diagram makes these operations explicit in a remarkable way for S_n.

The existence of such a left inverse of F requires that *the rank of F is equal to its width.* As pointed out elsewhere in [**7**], the matrix $\varphi = \chi^{-1}F\hat{\chi}$ describes the splitting of the classes of G in $\hat{G}$, so *a left inverse F_0 of F exists if and only if the classes of G do not split in $\hat{G}$.* In such case:

$$R : F_0\{\lambda\}F = (F\{\hat{\lambda}\}), \quad F\{\hat{\lambda}\}F' = (F'\{\lambda\}) : I$$

with

$$(3.3) \quad \begin{aligned} (F_0(F\{\hat{\lambda}\})) &= \{\hat{\lambda}\} : R^{-1}R = 1 \\ (F'(F_0'\{\hat{\lambda}\})) &= \{\hat{\lambda}\} : II^{-1} = 1 \end{aligned}$$

so that the reciprocity theorem holds not only for R and I but also for R^{-1} and I^{-1}. Correspondingly by multiplication:

$$(3.4) \quad \begin{aligned} F_0(F_0\{\lambda\})F &= \{\hat{\lambda}\} = F_0F\{\hat{\lambda}\}F_0F : RR^{-1} = 1 \\ F_0(F'\{\lambda\})F_0' &= \{\hat{\lambda}\} = F_0F\{\hat{\lambda}\}F'F_0' : I^{-1}I = 1. \end{aligned}$$

We summarize these operations on representations of representations as follows:

$$\begin{aligned} &R : F_0 \ldots F \text{ or } (F), \qquad R^{-1} : F \ldots F_0 \text{ or } (F_0), \\ &I : F \ldots F' \text{ or } (F'), \qquad I^{-1} : F_0 \ldots F_0' \text{ or } (F_0'). \end{aligned}$$

It is worth noting the subtlety of the matrix interpretation of the relation $R^{-1}R = 1$. From (3.1) we have

$$FF_0\{\lambda\}F = F(F\{\hat{\lambda}\}) = \{\lambda\}F = FF_0\{\lambda\}FF_0F$$

so that the best we can say in general, using (3.4), is that

LEMMA 3.5.

$$FF_0\{\lambda\}FF_0 = \{\lambda\}, \quad F\{\hat{\lambda}\}F_0 = (F_0\{\lambda\})$$

as left multipliers of F.

On the other hand, from (3.4)

$$(3.6)\quad FF_0(F'\{\lambda\})F_0'F' = F\{\hat{\lambda}\}F' = (F'\{\lambda\})$$

without the restriction in the lemma. We turn now to an interesting application of these ideas.

4. Restricting inner products. It is worth remarking that inner product multiplication, denoted here by $\times$, simply amounts to ordinary multiplication of the $\{\lambda\}$ and $\{\mu\}$. Thus from (3.1) we have immediately that

$$(4.1)\quad \{\lambda\} \times \{\mu\}F = \{\lambda\}F(F\{\hat{\mu}\}) = F(F\{\hat{\lambda}\})(F\{\hat{\mu}\})$$

which we can write in operator form:

$$(4.2)\quad R(\{\lambda\} \times \{\mu\}) = R\{\lambda\} \times R\{\mu\}.$$

(4.2) is valid whether or not the classes of G split in $\hat{G}$. However, with this assumption, we can use the Lemma to write

$$(4.3)\quad F_0\{\lambda\} \times \{\mu\}F = F_0\{\lambda\}F \times F_0\{\mu\}F.$$

Once again we have

$$(4.4)\quad F\{\lambda\} \times \{\mu\}F_0 = F\{\lambda\}F_0 \times F\{\mu\}F_0$$

so that

$$(4.5)\quad R^{-1}(\{\lambda\} \times \{\mu\}) = R^{-1}\{\lambda\} \times R^{-1}\{\mu\}.$$

Thus we can write

$$F_0F\{\lambda\} \times \{\mu\}F_0F = F_0F\{\lambda\}F_0F \times F_0F\{\mu\}F_0F$$

and using (4.3)

$$FF_0\{\lambda\} \times \{\mu\}FF_0 = FF_0\{\lambda\}FF_0 \times FF_0\{\mu\}FF_0$$

so that both RR^{-1} and $R^{-1}R$ factor similarly over the inner product.

In illustrating these processes we recall the general expression for an inner product [**3**]

$$(4.6)\quad (\hat{\lambda} \uparrow G) \times \mu = ((\mu \downarrow \hat{G}) \times \hat{\lambda}) \uparrow G$$

which yields for S_n

$$\begin{aligned}(4.7)\quad [\alpha] \times [\beta] &= \prod (1 - R_{ij})[\alpha_1] \, . \, [\alpha_2] \ldots \times [\beta] \\ &= \prod (1 - R_{ij})([\beta] \downarrow \hat{G}) \uparrow S_n,\end{aligned}$$

with $\hat{G} = S_{\alpha_1} \times S_{\alpha_2} \times \ldots$ In particular we have

Example 4.8.

$$\begin{aligned} R([4,1] \times [3,2]) &= R([4,1] + [3,2] + [3,1^2] + [2^2,1] \\ &= ([4] + [3,1]) \times ([3,1] + [2^2]) \\ R^{-1}([4,1] \times [3,2]) &= ([5,1] - [6]) \times ([4,2] - [5,1] + [6]) \\ &= [3^2] + [3,2,1] + 2[5,1] - [4,2] - 2[6] \end{aligned}$$

with

$$RR^{-1}([4,1] \times [3,2]) = [4,1] + [3,2] + [3,1^2] + [2^2,1]$$
$$= [4,1] \times [3,2]$$

as expected. These relations can be written using representation matrices as in Example 3.2, but we leave this to the reader.

5. Inducing on inner products. For any subgroup $\hat{G} \subset G$ in which the classes of G do not split

$$F\{\hat{\lambda}\} \times \{\hat{\mu}\}F' = F\{\hat{\lambda}\}F_0 \times F\{\mu\}F'$$

so that we have the somewhat surprising result

$$(5.1) \quad I(\{\hat{\lambda}\} \times \{\hat{\mu}\}) = R^{-1}\{\hat{\lambda}\} \times I\{\hat{\mu}\} = R^{-1}\{\mu\} \times I\{\hat{\lambda}\}$$

which we illustrate in the following

Example 5.2. Utilizing Example 4.8 we have

$$\begin{aligned} I([4,1] \times [3,2]) &= ([5,1] + [4,2] + [4,1^2]) \times ([4,2] - [5,1] + [6]) \\ &= ([5,1] - [6]) \times ([4,2] + [3^2] + [3,2,1]) \\ &= [5,1] + 2[4,2] + 2[4,1^2] + [3^2] + 3[3,2,1] \\ &\qquad + [3,1^3] + [2^3] + [2^2,1^2]. \end{aligned}$$

Again referring to (3.1) we have

$$\begin{aligned} (5.3) \quad I(\{\hat{\lambda}\} \times R\{\mu\}) &= F\{\hat{\lambda}\} \times (F\{\hat{\mu}\})F' = F(F\{\hat{\mu}\}) \times \{\hat{\lambda}\}F' \\ &= \{\mu\}F\{\hat{\lambda}\}F' = I\{\hat{\lambda}\} \times \{\mu\} \end{aligned}$$

for *any* subgroup $\hat{G}$ of G. This version of (5.1) is attributed by Lomont [**2**, p. 226] to Ito, but no specific reference is given. It follows without difficulty that

$$(5.4) \quad I^{-1}I(\{\lambda\} \times \{\mu\}) = I^{-1}I\{\lambda\} \times RR^{-1}\{\mu\} = \{\lambda\} \times \{\mu\}.$$

Reversing the order of the operators I and I^{-1} is more difficult:

$$\begin{aligned} (5.5) \quad I^{-1}(\{\mu\} \times I\{\hat{\lambda}\}) &= F_0\{\mu\} \times (F'\{\lambda\})F_0' \\ &= F_0 \,.\, \{\mu\} \times FF_0(F'\{\lambda\})F_0'F' \,.\, F_0' \\ &= F_0 \,.\, FF_0\{\mu\}FF_0 \times FF_0(F'\{\lambda\})F_0'F' \,.\, F_0' \\ &= R\{\mu\} \times I^{-1}(I\{\hat{\lambda}\}) \\ &= R\{\mu\} \times \{\hat{\lambda}\}. \end{aligned}$$

We use Lemma 3.5 and (3.6) again to yield:

$$\begin{aligned} (\{\mu\} \times (F'\{\lambda\})) &= \{\mu\} \times FF_0(F'\{\lambda\})F_0'F' \\ &= FF_0\{\mu\}FF_0 \times FF_0(F'\{\lambda\})F_0'F' \\ &= R^{-1}R\{\mu\} \times II^{-1}(F'\{\lambda\}), \end{aligned}$$

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so that, reversing the order of the factors we obtain

$$(5.6)\quad II^{-1}(I\{\hat{\lambda}\} \times \{\mu\}) = II^{-1}(F'\{\lambda\}) \times R^{-1}R\{\mu\}$$
$$= I\{\hat{\lambda}\} \times \{\mu\}.$$

Note that we are using here the multiplications in (3.4) rather than (3.3) so that reversing the order of I and I^{-1} is the cause of the trouble.

6. Operations on classes. In order to complete the story we write the corresponding operations on classes of G and $\hat{G}$ as developed in [**7**]. Using the same notation, we have

$$\varphi = \chi^{-1}F\hat{\chi}, \quad \Omega = \Gamma'\varphi\Gamma'^{-1} \text{ with } \varphi_1 = \hat{\chi}^{-1}F'\chi, \quad \Omega_1 = \Gamma'\varphi_1\Gamma'^{-1}$$

so that we may calculate the inverses to satisfy the equations

$$\Omega_0\Omega = 1 \text{ and } \Omega_1\Omega_{10} = 1.$$

Corresponding to the relations in § 3 we write:

$$R : \Omega_0\{C\}\Omega = (\{\hat{C}\}\Omega'), \quad \Omega\{\hat{C}\}\Omega_1 = (\{C\}\Omega_1') : I$$

with

$$(6.1)\quad \begin{aligned} ((\{C\}\Omega')\Omega_0') &= \{C\} : R^{-1}R = 1,\\ ((\{C\}\Omega_{10}')\Omega_1') &= \{C\} : II^{-1} = 1. \end{aligned}$$

Here the reciprocity theorem is complicated by transformation but does relate Ω' to Ω_1' and their inverses Ω_0' to Ω_{10}'. Again,

$$(6.2)\quad \begin{aligned} \Omega_0(\{C\}\Omega_0')\Omega &= \{\hat{C}\} = \Omega_0\Omega\{\hat{C}\}\Omega_0\Omega : RR^{-1} = 1,\\ \Omega_0(\{C\}\Omega_1')\Omega_{10} &= \{\hat{C}\} = \Omega_0\Omega\{\hat{C}\}\Omega_1\Omega_{10} : I^{-1}I = 1, \end{aligned}$$

so that

$$R : \Omega_0 \ldots \Omega \text{ or } (\Omega'), \quad R^{-1} : \Omega \ldots, \Omega_0 \text{ or } (\Omega_0'),$$
$$I : \Omega \ldots \Omega_1 \text{ or } (\Omega_1'), \quad I^{-1} : \Omega_0 \ldots \Omega_{10} \text{ or } (\Omega_{10}'),$$

as before, with corresponding formulae for operations on products of classes which we shall not consider in detail.

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