A PERIODIC JACOBI-PERRON ALGORITHM

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In the first part of this paper I shall demonstrate that one irrational root of the algebraic equation

$$(1) \qquad U_n(x) = m(x - D)(x^{n-1} + c_1 x^{n-2} + \ldots + c_{n-2}x + c_{n-1}) - d = 0$$

creates an algebraic number field, out of which $n - 1$ irrationals can be chosen so that they yield a periodic Jacobi–Perron algorithm. The coefficients in (1) are subject to certain restrictions which will be elaborated below.

In the second part we shall show that under these restrictions equation (1) is irreducible. Finally we shall state explicitly a unit of the field $K(w)$ where K is the rational field and w is a positive (irrational) root of (1) satisfying $D < w < D + 1$.

1. In seven previous papers **(1–7)** I have investigated special cases of (1), namely

$$c_i = e_i D^i, \qquad e_i = 0, 1 \qquad (i = 1, 2, \ldots, n - 1),$$

in particular the case $e_i = 1$ $(i = 1, 2, \ldots, n - 1)$, namely

$$m(x^n - D^n) - d = 0,$$

and an analogous equation, namely

$$x^n - D^n + d = 0.$$

I have shown in all these cases that $n - 1$ irrationals, suitably chosen from the algebraic field generated by an irrational root of such an equation, yield a periodic Jacobi–Perron algorithm. I have further shown that certain units of these fields can be calculated with the help of this algorithm, and I have modified the algorithm so that it becomes periodic for every irrational of these fields. The integrity of the numbers of the period was the price paid for this modification.

For the reader who may not be familiar with the Jacobi–Perron algorithm we shall repeat its main essentials **(8)**. Let

$$b_1^{(0)}, b_2^{(0)}, \ldots, b_{n-2}^{(0)}, b_{n-1}^{(0)}$$

be a set of $n - 1$ real numbers. Then we form an infinite number of new sets

$$b_1^{(v)}, b_2^{(v)}, \ldots, b_{n-2}^{(v)}, b_{n-1}^{(v)}$$

by the recursive formulae $(v = 0, 1, \ldots)$

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$$(2)\qquad \begin{aligned} b_{k-1}^{(v+1)} &= \frac{b_k^{(v)} - a_k^{(v)}}{b_1^{(v)} - a_1^{(v)}} \qquad (k = 2, 3, \ldots, n-1), \\ b_{n-1}^{(v+1)} &= \frac{1}{b_1^{(v)} - a_1^{(v)}}, \end{aligned}$$

where $a_i^{(v)} = [b_i^{(v)}]$ $(i = 1, 2, \ldots, n-1)$ is the greatest integer not exceeding $b_i^{(v)}$. If we now define the numbers $A_k^{(v)}$ by the recursive formulae

$$A_k^{(k)} = 1, \qquad A_k^{(v)} = 0 \qquad (k, v = 0, 1, \ldots, n-1; k \neq v),$$

$$(3)\quad A_k^{(v+n)} = A_k^{(v)} + a_1^{(v)} A_k^{(v+1)} + \ldots + a_{n-1}^{(v)} A_k^{(v+n-1)}$$
$$(k = 0, 1, \ldots, n-1; v = 0, 1, \ldots).$$

the ratios $A_k^{(v)} : A_0^{(v)}$ are approximate fractions for the $b_k^{(v)}$, i.e.,

$$(4)\qquad \lim_{v\to\infty} \frac{A_k^{(v)}}{A_0^{(v)}} = b_k^{(0)} \qquad (k = 1, 2, \ldots, n-1).$$

The algorithm is said to be *periodical* if

$$(5)\qquad a_k^{(v+m)} = a_k^{(v)} \qquad (k = 1, 2, \ldots, n-1; v = s, s+1, \ldots);$$

the s lines

$$a_1^{(v)}, a_2^{(v)}, \ldots, a_{n-1}^{(v)} \qquad (v = 0, 1, \ldots, s-1)$$

are then called *pre-period*, and the m lines

$$a_1^{(v)}, a_2^{(v)}, \ldots, a_{n-1}^{(v)} \qquad (v = s, s+1, \ldots, s+m-1)$$

the *period*. The algorithm is said to be *purely periodic* if $s = 0$, i.e. if there is no pre-period. In my earlier papers I constructed such $(n-1)$-tuples $b_i^{(v)}$ in the algebraic field generated by the root of the equation $m(x^n - D^n) - d = 0$ which yield a purely periodic Jacobi–Perron algorithm. For $n = 2$ this algorithm is identical with Euclid's algorithm. We shall therefore exclude the case $n = 2$ from our considerations.

We shall now prove

THEOREM 1. *Let the coefficients of $U_n(x)$ fulfil the conditions*

$$0 < d \leqslant D, \qquad c_i \geqslant 0, \qquad m \geqslant 1.$$

Then $U_n(x)$ has one and only one real positive root in the open interval $(D, D+1)$. It is irrational.

Proof. Since $U_n(D) = -d < 0$ and

$$U_n(D+1) \geqslant m(D+1)^{n-1} - d > 0,$$

$U_n(x)$ has real roots in the interval $(D, D+1)$. Differentiating $U_n(x)$ we obtain

$$\frac{1}{m} U'_n(x) = x^{n-1} + \sum_{i=1}^{n-1} c_i x^{n-1-i} + (x - D)[(n-1)x^{n-1} + \ldots + c_{n-2}].$$

Thus $U_n'(x) > 0$ for $x \geqslant D$. Hence $U_n(x)$ has exactly one real root $w > D$. and $w < D+1$. That w is irrational follows from the fact that $U_n(x)$ is irreducible, as will be proved below.

The main results of this section are expressed in the following theorem.

THEOREM 2. *Let d, D, m, n, c_i be integers satisfying*

$$\begin{cases} m \geqslant 1, \quad d|D, \quad n \geqslant 3, \quad D \geqslant 2(n-1)^2 d, \quad d \geqslant 1, \\ 0 \leqslant c_i \leqslant (D+1)^i, \quad c_i \equiv 0 \pmod{d} \ (i = 1, 2, \ldots, n-1). \end{cases} \tag{6}$$

Put

$$f_{s+1}(w, D, t) = \sum_{i=0}^{t} \binom{s+i}{i} w^{t-i} D^i \qquad (s, t = 0, 1, \ldots), \tag{7}$$

$$f_{s+1}(w, D, 0) = 1, \qquad f_0(w, D, t) = w^t, \qquad f_j(w, D, q) = 0 \text{ if } q < 0. \tag{7a}$$

$$z_s(w, D) = \sum_{i=0}^{n-1-s} c_i f_s(w, D, n-1-s-i) \ (s = 0, 1, \ldots, n-2; c_0 = 1), \tag{8}$$

$$\bar{z}_s = z_s(D, D). \tag{9}$$

Then the Jacobi–Perron algorithm for the $n-1$ numbers

$$z_{n-2}(w, D), \quad z_{n-3}(w, D), \quad \ldots, \quad z_2(w, D), \quad z_1(w, D), \quad z_0(w, D) \tag{10}$$

is purely periodic. The period has the primitive length n, only for $m = d = 1$ is the length 1. The first line of the period has the form

$$\bar{z}_{n-2}, \quad \bar{z}_{n-3}, \quad \ldots, \quad \bar{z}_2, \quad \bar{z}_1, \quad \bar{z}_0. \tag{11}$$

The ith line has the form $(i = 2, 3, \ldots, n)$

$$\bar{z}_{n-2}, \bar{z}_{n-3}, \ldots, \bar{z}_{i-1}, m\bar{z}_{i-2}/d, m\bar{z}_{i-3}/d, \ldots, m\bar{z}_1/d, m\bar{z}_0/d. \tag{11a}$$

If the numbers $A_i^{(v)}$ are defined in (3) *for the Jacobi–Perron algorithm for the numbers* (10), *then*

$$w = \lim_{v \to \infty} (A_1^{(v)}/A_0^{(v)}) - (n-2)D - c_1. \tag{12}$$

The proof will be given in two parts. (a) We shall first prove the formulae

$$\left[\frac{m z_s(w, D)}{d}\right] = \frac{m\bar{z}_s}{d}, \tag{13A}$$

$$[z_s(w, D)] = \bar{z}_s \qquad (s = 0, 1, \ldots, n-2). \tag{13B}$$

(b) Then we shall prove that the Jacobi–Perron algorithm for the numbers (10) is purely periodic and that the period has indeed the form (11). From (b), the formula (12) will easily follow.

We first prove that

$$[w^s] = D^s \qquad (s = 1, 2, \ldots, n-1). \tag{14}$$

By Theorem 1

$$D < w, \qquad D^s < w^s. \tag{14a}$$

We have further, from (1), (6), (14a),

$$w = D + d\Big/\left(m \sum_{i=0}^{n-1} c_i w^i\right) < D + d\Big/\left(\sum_{i=0}^{n-1} c_i D^i\right) \leqslant D + d/D^{n-1}. \tag{14b}$$

We shall prove that

$$w^s < (D + d/D^{n-1})^s < D^s + 1 \qquad (s = 1, \ldots, n-1).$$

We thus have to prove that

$$\sum_{i=1}^{s} \binom{s}{i} (d/D^{n-1})^i D^{s-i} < 1.$$

Put

$$F(i) = \binom{s}{i} (d/D^{n-1})^i D^{s-i}.$$

Then

$$F(i+1)/F(i) = (s-i)dD^{-n}/(i+1) < (n-1)dD^{-n} \leqslant 1: 2D^2(n-1) < 1.$$

Hence

$$\sum_{i=1}^{s} \binom{s}{i} (d/D^{n-1})^i D^{s-i} = \sum_{i=1}^{s} F(i) \leqslant \sum_{i=1}^{s} F(1) = s\binom{s}{1} (d/D^{n-1}) D^{s-1}$$
$$\leqslant s^2 (d/D^{n-1}) D^{n-2} \leqslant (n-1)^2 d/D \leqslant 1/2 < 1.$$

Thus $w^s < D^s + 1$, which, together with (14a), proves (14).

For the proof of Theorem 2 we need some notations. Put

$$f'_s(w, D, t+1) = \frac{f_s(w, D, t+1) - f_s(D, D, t+1)}{w - D} \qquad (s, t = 0, 1, \ldots). \tag{15}$$

The following identities are obvious:

$$f_s'(w, D, t+1) = f_{s+1}(w, D, t), \tag{16}$$

$$f_s(D, D, t+1) = \binom{s+t+1}{s} D^{t+1}, \tag{17}$$

$$f_1(D, D, t) = (t+1)D^t, \qquad f_0(D, D, t) = D^t. \tag{17a}$$

Further put

$$z'_s(w, D) = \frac{z_s(w, D) - z_s(D, D)}{w - D}. \tag{15a}$$

Thus

$$z_s'(w, D) = z_{s+1}(w, D). \tag{16a}$$

By (1)

$$\frac{1}{w-D} = \left(m \sum_{i=0}^{n-1} c_i w^{n-1-i}\right) \Big/ d = \frac{mz_0(w, D)}{d}. \tag{18}$$

We shall now prove (13a). We have to prove that

$$m\bar{z}_s/d < (mz_s(w, D))/d < m\bar{z}_s/d + 1 \qquad (s = 0, 1, \ldots, n-2)$$

or, following (15a), (16a), (18), that

$$0 < z_{s+1}(w, D) < z_0(w, D).$$

The left-hand inequality is obvious because of (14a). For $s = n-2$ we have $z_{n-1}(w, D) = 1 < z_0(w, D)$. We therefore have to prove only that

$$z_{s+1}(w, D) < z_0(w, D) \qquad (s = 0, 1, \ldots, n-3).$$

We prove *a fortiori* (with $c_0 = 1$) that

$$\begin{aligned}
z_{s+1}(w, D) &< \sum_{i=0}^{n-2-s} c_i f_{s+1}(D+1, D+1, n-2-s-i) \\
&= \sum_{i=0}^{n-2-s} c_i \binom{n-1-i}{s+1} (D+1)^{n-2-s-i} \\
&\leqslant \sum_{i=0}^{n-2-s} \binom{n-1-i}{s+1} D^i (D+1)^{n-2-s-i} \\
&< \sum_{i=0}^{n-2-s} \binom{n-1}{s+1} (D+1)^i (D+1)^{n-2-s-i} \\
&= (n-1-s)\binom{n-1}{s+1}(D+1)^{n-2-s} \\
&\leqslant (n-1)(D+1)^{n-2}(n-1)\frac{n-2}{2D}\cdot\frac{n-3}{3D}\cdots\frac{n-s-1}{(s+1)D} \\
&< (n-1)^2(D+1)^{n-2}.
\end{aligned}$$

But

$$D^{n-1} \leqslant D^{n-1} + \sum_{i=1}^{n-1} c_i D^{n-1-i} = \bar{z}_0 < z_0(w, D).$$

Thus it suffices to prove that

$$(n-1)^2(D+1)^{n-2} < D^{n-1}$$

or

$$(n-1)^2(1+1/D)^{n-2} < D.$$

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We shall prove *a fortiori* that

$$(n-1)^2(1+1/D)^{n-2} < 2(n-1)^2 \leqslant 2(n-1)^2 d \leqslant D.$$

We have only to prove that

$$(1+1/D)^{n-2} < 2$$

or

$$\sum_{i=1}^{n-2} \binom{n-2}{i} D^{-i} < 1.$$

Put

$$F(i) = \binom{n-2}{i} D^{-i} \qquad (i = 1, 2, \ldots, n-2).$$

Then

$$\frac{F(i+1)}{F(i)} = \frac{n-2-i}{(i+1)D} \leqslant \frac{n-3}{2D} < 1.$$

Therefore

$$\sum_{i=1}^{n-2} \binom{n-2}{i} D^{-i} = \sum_{1}^{n-2} F(i) < \sum_{i=1}^{n-2} F(1) = (n-2)^2 D^{-1} < 1.$$

This completes the proof of (13A). Formula (13B) is the special case $m = d$.

We shall now prove that the Jacobi–Perron algorithm for the numbers

$$b_s^{(0)} = z_{n-1-s}(w, D) \qquad (s = 1, 2, \ldots, n-1)$$

is periodic and that the period has indeed the form (11). We have

$$b_1^{(0)} = z_{n-2}(w, D) = \sum_{i=0}^{1} c_i f_{n-2}(w, D, 1-i),$$

$$b_1^{(0)} = f_{n-2}(w, D, 1) + c_1 f_{n-2}(w, D, 0),$$

$$b_1^{(0)} = w + (n-2)D + c_1, \qquad \bar{z}_{n-2} = a_1^{(0)} = (n-1)D + c_1,$$

$$b_1^{(0)} - a_1^{(0)} = w - D, \qquad b_{n-1}^{(1)} = 1/(w-D) = \frac{mz_0(w, D)}{d},$$

$$b_s^{(1)} = (z_{n-2-s}(w, D) - \bar{z}_{n-2-s})/(w-D) = z_{n-s-1}(w, D)$$

$$(s = 1, 2, \ldots, n-2).$$

Using the same method by which the $b_s^{(1)}$ $(s = 1, 2, \ldots, n-1)$ were found, one can easily prove the following Lemma by induction.

LEMMA. *Let* $1 \leqslant t < n$. *Then*

$$b_s^{(t)} = z_{n-s-1}(w, D) \qquad \text{if } s = 1, 2, \ldots, n-t-1,$$

$$b_s^{(t)} = \frac{mz_{n-1-s}(w, D)}{d} \qquad \text{if } s = n-t, n-t+1, \ldots, n-1.$$

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This Lemma implies that

$$b_s^{(n-1)} = \frac{mz_{n-1-s}(w, D)}{d} \qquad (s = 1, 2, \ldots, n-1).$$

Using again the method by which the $b_s^{(1)}$ were found, we derive from the $b_s^{(n-1)}$

$$b_s^{(n)} = z_{n-1-s}(w, D) = b_s^{(0)} \qquad (s = 1, 2, \ldots, n-1).$$

This establishes the structure of the period, as stated in Theorem 2.

Regarding the numbers appearing in the period, we have the following types

Natural numbers of the first kind, namely

$$\begin{aligned} \bar{z}_s &= \sum_{i=0}^{n-1-s} c_i \binom{n-1-i}{s} D^{n-1-s-i} \\ &= \binom{n-1}{s} D^{n-1-s} + \sum_{i=0}^{n-1-s} c_i \binom{n-1-i}{s} D^{n-1-s-i} \\ &\equiv 0 \pmod{d}, \end{aligned}$$

since $d|D$, $d|c_i$ $(i = 1, 2, \ldots, n-1)$.

Natural numbers of the second kind, namely $m\bar{z}_s/d$, which are natural since $d|\bar{z}_s$. By this the pattern of the period is explained.

Taking into account the value of $b_1^{(0)}$, we obtain from (4)

$$\begin{aligned} z_{n-2}(w, D) &= \lim_{v\to\infty}(A_1^{(v)}/A_0^{(v)}), \\ w &= \lim_{v\to\infty}(A_1^{(v)}/A_0^{(v)}) - (n-2)D - c_1. \end{aligned}$$

Theorem 2 is now completely proved.

EXAMPLE.

$$U_5(x) = x^5 - 62x^4 - 124x^3 - 250x^2 - 382x - 130 = 0.$$

This equation has one real root. Since

$$U_5(64) = -2 \leqslant 0, \qquad U_5(65) > 0,$$

we have $D = 64$ and

$$U_5(x) = (x - 64)(x^4 + 2x^3 + 4x^2 + 6x + 2) - 2.$$

Here $n = 5$, $d = 2$, $D = 64 = 2d(n-1)^2$, $m = 1$,

$$c_i \leqslant 64^i, c_i \equiv 0 \pmod{2} \qquad (i = 1, 2, 3, 4).$$

We easily calculate that

$$\begin{aligned} z_0(w, D) &= w^4 + 2w^3 + 4w^2 + 6w, \\ z_1(w, D) &= w^3 + w^2D + wD^2 + D^3 + 2(w^2 + wD + D^2) + 4(w + D) + 6, \\ z_2(w, D) &= w^2 + 2wD + 3D^2 + 2w + 4D + 4, \\ z_3(w, D) &= w + 3D + 2, \end{aligned}$$

$$\bar{z}_0 = 17318272, \qquad \bar{z}_1 = 1073670, \qquad \bar{z}_2 = 24964, \qquad \bar{z}_3 = 258.$$

The period has the following form:

258	24964	1073670	17318272
258	24964	1073670	8659136
258	24964	536835	8659136
258	12482	536835	8569136
129	12482	536835	8569136

2. The main result of this section is the following theorem.

THEOREM 3. *Equation* (1) *is irreducible under the conditions imposed on* m, D, d, *and the* c_i.

Proof. Perron (**9**) calls the equation of degree n

$$\begin{vmatrix} A_0^{(s+t)} - yA_0^{(s)} & A_0^{(s+t+1)} - yA_0^{(s+1)} & \dots & A_0^{(s+t+n-1)} - yA_0^{(s+n-1)} \\ A_1^{(s+t)} - yA_1^{(s)} & A_1^{(s+t+1)} - yA_1^{(s+1)} & \dots & A_1^{(s+t+n-1)} - yA_1^{(s+n-1)} \\ \dots & \dots & \dots & \dots \\ A_{n-1}^{(s+t)} - yA_{n-1}^{(s)} & A_{n-1}^{(s+t+1)} - yA_{n-1}^{(s+1)} & \dots & A_{n-1}^{(s+t+n-1)} - yA_{n-1}^{(s+n-1)} \end{vmatrix} = 0 \tag{19}$$

the "characteristic equation" of a periodic Jacobi–Perron algorithm; the $A_i^{(v)}$ were defined in Formulae (3). The letters s, t denote the lengths of the pre-period and of the period of the algorithm, respectively. In our case, $s = 0$, $t = n$. Our main concern will now be to prove the irreducibility of this equation in our case. For this purpose we shall make use of Perron's theorem (**9**, Satz 13, p. 62). It states:

Suppose for every v *and all* $a_i^{(v)}$ $(i = 1, 2, \dots, n-1)$ *of the period of a* (*periodic*) *Jacobi algorithm the relations*

$$a_{n-1}^{(v)} \geqslant n + a_1^{(v)} + a_2^{(v)} + \dots + a_{n-2}^{(v)} \tag{20}$$

hold. Then the characteristic equation of the algorithm is irreducible.

On account of (11), (11a) we have to prove that

$$\bar{z}_0 \geqslant n + (\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_{n-2}), \tag{21}$$

$$\begin{cases} (m/d)\bar{z}_0 \geqslant n + (\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_{n-2}), \\ (m/d)\bar{z}_0 \geqslant n + (m/d)(\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_{i-2}) + (\bar{z}_{i-1} + \dots + \bar{z}_{n-2}) \end{cases} \tag{22}$$

$$(i = 3, 4, \dots, n).$$

We shall now state explicitly the numbers $\bar{z}_s$ $(s = 0, 1, \dots, n-2)$:

$$\bar{z}_0 = \sum_{i=0}^{n-1} c_i f_0(D, D, n-1-i) = \sum_{i=0}^{n-1} c_i \binom{n-1-i}{0} D^{n-1-i} = \sum_{i=0}^{n-1} c_i D^{n-1-i},$$

$$\bar{z}_s = \sum_{i=0}^{n-1-s} c_i f_s(D, D, n-1-s-i) = \sum_{i=0}^{n-1-s} c_i \binom{n-1-i}{s} D^{n-1-s-i}.$$

Therefore

$$\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_{n-2} = \sum_{s=1}^{n-2} \sum_{i=0}^{n-1-s} c_i \binom{n-1-i}{s} D^{n-1-s-i}$$
$$= -1 + \sum_{i=0}^{n-2} c_i((D+1)^{n-1-i} - D^{n-1-i}).$$

To prove (21) we have to verify that

$$\sum_{i=0}^{n-1} c_i D^{n-1-i} \geqslant n - 1 + \sum_{i=0}^{n-2} c_i((D+1)^{n-1-i} - D^{n-1-i}).$$

Since $c_{n-1} \geqslant 0$, we have

$$\sum_{i=0}^{n-1} c_i D^{n-1-i} \geqslant \sum_{i=0}^{n-2} c_i D^{n-1-i} \geqslant n - 1 + \sum_{i=0}^{n-2} c_i((D+1)^{n-1-i} - D^{n-1-i}).$$

We thus have to prove that

$$\sum_{i=0}^{n-2} c_i D^{n-1-i} \geqslant n - 1 + \sum_{i=0}^{n-2} c_i((D+1)^{n-1-i} - D^{n-1-i}).$$

Since $c_i \geqslant 0$, it suffices to prove that

(A) $2D^{n-1} \geqslant n + (D+1)^{n-1}$,
(B) $2D^{n-1-i} \geqslant (D+1)^{n-1-i}$ $\quad (i = 1, 2, \ldots, n-3)$,
(C) $2D \geqslant D + 1$.

(C) is obvious, since the condition $D \geqslant 2(n-1)^2 d$ implies that $D \geqslant 8$. We shall now prove (A). Here we have to prove that

$$(1 + 1/D)^{n-1} + nD^{-(n-1)} < 2,$$
$$\frac{n+1}{D^{n-1}} + \sum_{i=1}^{n-2} \binom{n-1}{i} \frac{1}{D^i} \leqslant 1.$$

But for $n \geqslant 3$ we proved in Part 1 that

$$\sum_{i=1}^{n-2} \binom{n-1}{i} \frac{1}{D^i} \leqslant \frac{(n-2)(n-1)}{D}.$$

Furthermore $(n+1)D^{-(n-1)} \leqslant (n+1)D^{-2}$. Thus

$$(n+1)D^{-(n-1)} + \sum_{i=1}^{n-2} \binom{n-1}{i} D^{-i} \leqslant (n-2)(n-1)D^{-1} + (n+1)D^{-2}$$
$$\leqslant \frac{(n-2)(n-1)}{2(n-1)^2} + \frac{n+1}{4(n-1)^4} \leqslant \frac{n-2}{2(n-1)} + \frac{(n-1)^2}{4(n-1)^4}$$
$$\leqslant \frac{n-1}{2(n-1)} + \frac{1}{4(n-1)^2} \leqslant \frac{1}{2} + \frac{1}{16} < 1,$$

which proves (A).

The reader will note that equality holds true only for $n = 2$, $D = 3$, a case which has been eliminated from our considerations.

To prove (B) we have to show that

$$(1 + 1/D)^{n-1-i} \leqslant 2 \qquad (i = 1, 2, \ldots, n-3).$$

This inequality was proved in Part 1, too. Thus (21) is completely proved.

To verify (22) we shall prove that

$$(m/d)\bar{z}_0 \geqslant n + m(\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_{n-2}).$$

We thus have to prove that

$$(m/d) \sum_{i=0}^{n-1} c_i D^{n-1-i} \geqslant n + m\left(\sum_{i=0}^{n-2} c_i((D+1)^{n-1-i} - D^{n-1-i}) - 1\right),$$

and shall show that

$$(m/d) \sum_{i=0}^{n-1} c_i D^{n-1-i} \geqslant (m/d) \sum_{i=0}^{n-2} c_i D^{n-1-i}$$

$$\geqslant n + m\left(\sum_{i=0}^{n-2} c_i((D+1)^{n-1-i} - D^{n-1-i}) + c_{n-2}\right).$$

Here, too, we shall proceed in steps:

(A) $(m/d)c_{n-2} D \geqslant mc_{n-2}$,
(B) $(m/d)D^{n-1} \geqslant n + m((D+1)^{n-1} - D^{n-1})$,
(C) $(m/d)c_i D^{n-1-i} \geqslant mc_i((D+1)^{n-1-i}) - D^{n-1-i}) \qquad (i = 1, \ldots, n-3)$.

In (A) the equality sign holds for $c_{n-2} = 0$; otherwise $D > d$.

In (B) we have to prove that

$$D^{n-1} \geqslant nd/m + d((D+1)^{n-1} - D^{n-1}),$$

and prove *a fortiori* that

$$(d+1)D^{n-1} \geqslant nd + d(D+1)^{n-1},$$

$$d + 1 \geqslant ndD^{-(n-1)} + d + d \sum_{i=1}^{n-1} \binom{n-1}{i} D^{-i},$$

$$1 \geqslant ndD^{-(n-1)} + d \sum_{i=1}^{n-1} \binom{n-1}{i} D^{-i},$$

$$1 \geqslant ndD^{-2} + d(n-1)^2 D^{-1} \geqslant ndD^{-(n-1)} + d \sum_{i=1}^{n-1} \binom{n-1}{i} D^{-i}.$$

We thus have to prove that

$$1 \geqslant ndD^{-2} + d(n-1)^2 D^{-1},$$

and prove *a fortiori* that

$$1 \geqslant (n-1)^2 dD^{-1} + d(n-1)^2 D^{-1} = 2d(n-1)^2 D^{-1},$$
$$D \geqslant 2d(n-1)^2.$$

In (C) the equality sign holds for $c_i = 0$. Otherwise we have to prove

$$D^{n-1-i} \geqslant d((D+1)^{n-i-1} - D^{n-1-i}),$$

and the proof of this inequality is completely analogous to the proof in case (B). By this means (22) is proved completely.

We shall now proceed to prove the following theorem.

THEOREM 4. *If the characteristic equation of the Jacobi–Perron algorithm formed by the root w of equation* (1) *and its functions $z_s(w, D)$ is irreducible, so is equation* (1).

Proof. We shall again make use of a fundamental theorem of Perron (**9**, Satz 15, p. 16) which states:

If the characteristic equation of a periodic Jacobi algorithm is irreducible, then the n numbers

$$1,\quad b_1^{(0)},\quad b_2^{(0)},\quad \ldots,\quad b_{n-1}^{(0)}$$

cannot satisfy a linear homogeneous relation with rational coefficients.

We first note that we can express the w^i $(i = 1, 2, \ldots, n-1)$ as linear forms of the $b_i^{(0)}$ with rational (and even integer) coefficients. From (7), (7a), (8) we find, using the formula

$$b_i^{(0)} = z_{n-1-i}(w, D),$$

$$w = b_1^{(0)} - (n-2)D - c_1,$$

$$w_2^{(0)} = b_2^{(0)} - ((n-3)D + c_1)b_1^{(0)} + \left(\binom{n-2}{2}D^2 + \binom{n-2}{1}Dc_1 + c_1^2 - c_2\right),$$

$$\begin{aligned} w^3 = b_3^{(0)} &- ((n-4)D + c_1)b_2^{(0)} \\ &+ \left(\binom{n-3}{2}D^2 + \binom{n-3}{1}Dc_1 + c_1^2 - c_2\right)b_1^{(0)} \\ &- \left(\binom{n-2}{3}D^3 + \binom{n-2}{2}D^2c_1 \right. \\ &+ \left.\binom{n-2}{1}D(c_1^2 - c_2) + c_1^3 - 2c_1c_2 + c_3\right), \end{aligned}$$

. . . .

By induction we obtain a relation,

$$(23)\qquad w^i = b_i^{(0)} - s_1 b_{i-1}^{(0)} + s_2 b_{i-2}^{(0)} + \ldots + (-1)^{i-1}s_{i-1}b_1^{(0)} + (-1)^i s_i \qquad (i = 1, 2, \ldots, n-1),$$

where the s_k $(k = 1, 2, \ldots, i)$ are natural numbers. Now, if (1) were reducible, w would be the root of an equation

$$(24)\qquad w^m + t_1 w^{m-1} + \ldots + t_{m-1}w + t_m = 0,$$

where the t_i $(i = 1, 2, \ldots, m)$ are rationals and $m \leqslant n - 1$. Substituting the values from (23) for w^i, we would get a linear homogeneous relation between the $b_i^{(0)}$ with integer coefficients, contrary to our assumption. By this, Theorem 4 is proved. This completes the proof of Theorem 3.

Using the numerical example of Part 1, it is easily seen that the necessary conditions for the irreducibility of the characteristic equation are satisfied in this case, namely

$$17{,}318{,}272 > 1{,}098{,}897, \qquad 8{,}659{,}136m > 1{,}098{,}897,$$
$$8{,}123{,}301m > 25{,}227, \qquad 8{,}110{,}819m > 263, \qquad 8{,}110{,}561m > 5.$$

Having thus proved that $K(w)$ is an algebraic field of degree n, I want to point out that it is possible to give a unit of this field in a most simple form;

$$e = m(w - D)^n/d \tag{25}$$

is such a unit. To prove this, we have to show that e and e^{-1} are integers in the field $K(w)$. We have, from (25),

$$e = m(w^n + DF(w))/d,$$

where $F(w)$ is a polynomial in w with integer coefficients. Furthermore by (1),

$$mw^n = [(D - c_1)w^{n-1} + (Dc_1 - c_2)w^{n-2} + (Dc_{n-2} - c_{n-1})w + Dc_{n-1}]m + d,$$
$$mw^n = dF_1(w).$$

Thus $e = F_1(w) + m(D/d)F(w)$, which shows that e is an algebraic integer in $K(w)$.

Also from (1),

$$\frac{1}{e} = \frac{d}{m(w - D)^n} = \frac{dm^{n-1}}{\left(\dfrac{d}{w^{n-1} + c_1 w^{n-2} + \ldots + c_{n-2} w + c_{n-1}}\right)^n}$$
$$= \frac{(w^{n-1} + c_1 w^{n-2} + \ldots + c_{n-2} w + c_{n-1})^n m^{n-1}}{d^{n-1}}.$$

Put

$$dF_2(w) = c_1 w^{n-2} + \ldots + c_{n-2} w + c_{n-1};$$

then

$$\frac{1}{e} = \frac{(w^{n-1} + dF_2(w))^n m^{n-1}}{d^{n-1}}.$$

But

$$(w^{n-1} + dF_2(w))^n = \sum_{i=0}^{n} \binom{n}{i} (w^{n-1})^{n-i} d^i F_2{}^i(w).$$

Now $i \geqslant 0$ implies that $(n-1)(n-i) \geqslant n(n-1-i)$ or

$$(n-1)(n-i) = n(n-1-i) + r, \qquad r \geqslant 0.$$

Hence

$$\begin{aligned}(w^{n-1})^{n-i}d^i &= w^{(n-1)(n-i)}d^i = w^{n(n-1-i)}w^r d^i \\ &= (dF_1(w))^{n-1-i}w^r d^i \\ &= d^{n-1}w^r(F_1(w))^{n-1-i} = d^{n-1}F_3(w), \text{ say.}\end{aligned}$$

Thus $1/e$ also is an algebraic integer.

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