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A DILATION AND NORM IN SEVERAL VARIABLE OPERATOR THEORY

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ABSTRACT. For every m-tuple of operators acting on a Hilbert space, it is shown that there exists a common dilation of these operators to m commuting normal operators on some larger Hilbert space. We then introduce a norm on the m-fold cartesian product of $\mathcal{B}(\mathcal{H})$ that is defined to be, for a given m-tuple, the infimum of the joint spectral radii of all joint normal dilations of the m operators. This norm has several good features, one of which is that it is invariant under the passage to adjoints.

Introduction. In the influential paper [12], Paul Halmos proved that every contraction on a Hilbert space has a unitary dilation. This important result was followed by the far reaching Sz.-Nagy Dilation Theorem: every contraction T has a unitary dilation U such that every power T^n of T dilates to the unitary U^n. (In other words, the Sz.-Nagy dilation applies not just to T, but to the discrete semigroup generated by T.) An extension of the Sz.-Nagy dilation to two commuting contractions was carried out by T. Ando [1], but several years later S. K. Parrott [19] demonstrated that the dilation theory of Sz.-Nagy is limited to the cases of one contraction or two commuting contractions. In fact, the theorem of Halmos—that of a dilation of T rather than of powers of T—itself does not extend to the case of three commuting contractions. This is described by Halmos in [13; p. 909] where he presents Parrott's example, as viewed by C. Davis, of three commuting contractions such that if each has an isometric dilation, then the three isometries cannot commute. If one removes the demand that the dilations be isometries, then the arguments of Davis do not apply and indeed the situation changes.

In this paper we show that for every m-tuple of operators on a Hilbert space, one can simultaneously dilate each operator (to the same dilating Hilbert space) so that the dilations are normal, have finite spectrum, and are pairwise commuting. One such dilation is constructed via a polyhedral containment region for the joint numerical range of the original operators. The existence of this dilation leads to the introduction of a joint norm for several operators (commuting or otherwise) having many desirable features. One would like, for example, that a joint norm $\|\cdot\|$ satisfy $\|(A_1,A_2)\| = \|A_1 + iA_2\|$ whenever A_1 and A_2 are hermitian, and that $\|A^*\| = \|A\|$, where A^* is the m-tuple obtained by taking the adjoints of the operators in the m-tuple A; our norm has these properties. In addition, the norm that we introduce on m-tuples is shown to dominate both the Clifford norm of

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an m-tuple, which has recently been employed by Pryde in the study of joint spectral variation [21], and the multivariable norm introduced several years ago by Cho and Takaguchi [7]. It is shown in Theorem 2.2 that all of these norms are, however, equivalent (in the sense that they induce the same topology).

By a *dilation* of an m-tuple $A = (A_1, \ldots, A_m)$ of operators on a Hilbert space $\mathcal{H}$ is meant an m-tuple $T = (T_1, \ldots, T_m)$ of pairwise commuting operators on a Hilbert space $\mathcal{H}'$ and an isometry $V: \mathcal{H} \longrightarrow \mathcal{H}'$ such that $A_i = V^* T_i V$ for each i. (Thus, VV^* is a projection with range $V(\mathcal{H})$ and each A_i has the action $A_i = VV^* T_{i|V(\mathcal{H})}$.) A *normal dilation* of an m-tuple A is a dilation N such that each of the operators N_i is normal; a *hermitian dilation* of hermitian operators is defined in the obvious analogous way. A *normal extension* N of an m-tuple A is a normal dilation via an isometry V such that the subspace $V(\mathcal{H})$ is invariant for each N_j. Using this manner of speaking our theorem is like Halmos's: every m-tuple A has a normal dilation. What differs from the Halmos dilation, however, is the method by which our dilation is constructed, even in the single operator case.

We will make extensive use of the joint spectra and the joint numerical range of several operators. Suppose that $\mathcal{H}$ denotes a Hilbert space, that (x, y) denotes the inner product of two vectors $x, y \in \mathcal{H}$, and that $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. The *joint numerical range* of an m-tuple $A = (A_1, \ldots, A_m)$ of operators on $\mathcal{H}$ is the set $W(A)$ of complex m-tuples of the form $\big((A_1 x, x), \ldots, (A_m x, x)\big)$, where $x \in \mathcal{H}$ is a unit vector. The joint numerical radius $w(A)$ of the m-tuple A is the supremum of the euclidean norms of the complex m-tuples in $W(A)$.

There are several notions of joint spectrum, most of which are discussed in the recent survey [9] of Curto. We will be interested in the following joint spectra. The *joint point spectrum* of the m-tuple A is the set $\sigma_p(A)$ of complex m-tuples $(\lambda_1, \ldots, \lambda_m)$ for which there exists a unit vector $x \in \mathcal{H}$ satisfying $A_i x = \lambda_i x$ for every i. The *joint approximate point spectrum* of the m-tuple A is the set $\sigma_\pi(A)$ of complex m-tuples $(\lambda_1, \ldots, \lambda_m)$ for which there exists a sequence of unit vectors $x_k \in \mathcal{H}$ satisfying $\lim_k \|(A_i - \lambda_i I)x_k\| = 0$ for every i. The joint approximate point spectrum is compact; it is always nonvoid whenever A is an m-tuple of commuting operators [8]. For m-tuples A of commuting operators, $\sigma(A)$ will denote *Taylor's joint spectrum*, which coincides with $\sigma_p(A)$ whenever $\mathcal{H}$ has finite dimension [9; p. 38]. The joint spectral radius $\rho_\pi(A)$ of an m-tuple A is the maximum of the euclidean norms of the elements of $\sigma_\pi(A)$, or is $-\infty$ if $\sigma_\pi(A)$ is void. For m-tuples A of commuting operators, $\rho(A)$ will denote the joint spectral radius of the joint spectrum $\sigma(A)$.

1. The dilation.

THEOREM 1.1. *Suppose that the joint numerical range $W(A)$ of an m-tuple A of hermitian operators $A_i \in \mathcal{B}(\mathcal{H})$ is contained in a simplex $\mathcal{K} \subset \mathbb{R}^m$ with vertices $v_0, \ldots, v_m$, where $v_i = (v_{i1}, \ldots, v_{im})$ for $i = 0, \ldots, m$. Then there exist positive semidefinite contractions $P_0, \ldots, P_m \in \mathcal{B}(\mathcal{H})$ satisfying $P_0 + \cdots + P_m = I$ such that for each $j = 1, \ldots, m$,*

$$A_j = \sum_{i=0}^{m} v_{ij} P_i = \sum_{i=0}^{m} P_i^{1/2} v_{ij} P_i^{1/2} = V^* D_j V,$$

where $D_j = v_{0j}I \oplus \cdots \oplus v_{mj}I$ *acts on* $\oplus_{i=0}^m \mathcal{H}$ *and* $V: \mathcal{H} \longrightarrow \oplus_{i=0}^m \mathcal{H}$ *is the isometry defined by* $Vx = P_0^{\frac{1}{2}}x \oplus \cdots \oplus P_m^{\frac{1}{2}}x$. *Consequently,* A_j *is a compression of* D_j *to the range of* V, *and* $D_1, \ldots, D_m$ *are commuting hermitian operators satisfying*

$$W(D) = \operatorname{conv} \sigma_p(D) = \mathcal{K}.$$

PROOF. Let M be the real matrix with (i,j)-entry $v_{ij} - v_{0j}$, and let γ_{ik} be the (k,i)-entry of M^{-1} (which exists because the vertices of $\mathcal{K}$ are affinely independent). Define

$$P_i = \sum_{k=1}^m \gamma_{ik}(A_k - v_{0k}I) \quad i = 1, \ldots, m$$

$$P_0 = I - \sum_{i=1}^m P_i.$$

Then

$$V^*D_jV = v_{0j}I + \sum_{i,k=1}^m (v_{ij} - v_{0j})\gamma_{ik}(A_k - v_{0k}I) = A_j$$

as required.

Now let $u \in \mathcal{H}$ be a unit vector and write $a_j = (A_ju, u)$, $p_i = (P_iu, u)$, and $a = (a_1, \ldots, a_m)$. Then

$$\sum_{i=1}^m p_i(v_{ij} - v_{0j}) = \sum_{i,k=1}^m (v_{ij} - v_{0j})\gamma_{ik}(a_k - v_{0k}) = a_j - v_{0j},$$

from which we see that $p_1, \ldots, p_m$ are the coordinates of $a - v_0$ relative to the linear basis $\{v_1 - v_0, \ldots, v_m - v_0\}$ of $\mathbb{R}^m$. Because $W(A) \subset \mathcal{K}$, $p_0, p_1, \ldots, p_m$ are in fact the barycentric coordinates of $a \in W(A)$ relative to $v_0, \ldots, v_m$ and so $\sum_{i=0}^m p_i = 1$. From this it is clear that the operators $P_0, \ldots, P_m$ are positive semidefinite contractions.

Finally, because each D_j is a diagonal operator with finite spectrum, the joint range $W(D)$ is closed and is readily seen to coincide with the convex hull of the joint eigenvalues. ■

Although the dilation multiplies $\dim \mathcal{H}$ by $m + 1$, this factor can be reduced to one plus the affine dimension α of the joint range $W(A)$. Indeed, if $\alpha < m$, then $m - \alpha$ of the A_j depend affinely on the remainder [4], and the construction above can be applied to the latter. In the case $\alpha = 1$, no dilation is needed since the A_j are already multiples of I—see [4] for this result.

COROLLARY 1.2. *Every m-tuple of operators has a normal dilation. More precisely, if* $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$, *then there exist commuting normal operators* $N_1, \ldots, N_m$ *acting on* $\oplus_{i=1}^k \mathcal{H}$ *for some* $k \leq 2m + 1$, *and an isometry* $V: \mathcal{H} \longrightarrow \oplus_{i=1}^k \mathcal{H}$ *such that* $A_i = V^*N_iV$ *for all* $i = 1, \ldots, m$. *Furthermore,* N_i *can be chosen so that there exists a unitary* U *satisfying* $U^*N_iU = \oplus_{j=1}^k \mu_{ij}I$. *As a result, for any* $\gamma_1, \ldots, \gamma_m \in \mathbb{C}$, $\|\sum_{i=1}^m \gamma_iA_i\| \leq \|\sum_{i=1}^m \gamma_iN_i\| \leq \max\{|\sum_{i=1}^m \gamma_i\mu_{ij}| : 1 \leq j \leq k\}$.

PROOF. Express each operator A_i as a sum of its real and imaginary parts and apply Theorem 1.1. ■

We wish to point out that our dilation results extend and are motivated by the insightful observations of B. Mirman [17] and Y. Nakamura [19] concerning numerical ranges and their triangular containment regions.

2. **A joint spectral norm.** It follows from the Halmos dilation theorem that an operator is a contraction if and only if it has a unitary dilation. Consequently, the norm of an operator $A \in \mathcal{B}(\mathcal{H})$ can be achieved via the equality $\|A\| = \inf\{\|N\| : N$ is a normal dilation of $A\}$. Therefore, it is natural to define the *joint norm* of $A_1, \ldots, A_m$ via the infimum of the joint spectral radii of the normal dilations of A.

If $N = (N_1, \ldots, N_m)$ is an m-tuple of commuting normal operators on $\mathcal{H}$, then the joint spectrum $\sigma(N_1, \ldots, N_m)$ is the set of all complex m-tuples of the form $\big(\varphi(N_1), \ldots, \varphi(N_m)\big)$, where φ is an element of the maximal ideal space of the commutative C^*-algebra generated by the elements of N [9; 7.2]. The joint spectral radius $\rho(N_1, \ldots, N_m)$ in fact coincides with the value of $\|\sum_j N_j^* N_j\|^{\frac{1}{2}}$. This follows directly from C^*-algebra theory: if $\{E_{ij}\}_{i,j}$ denote the canonical matrix units for M_m, then

$$\begin{aligned}
\Big\|\sum_{j=1}^m N_j^* N_j\Big\|^{\frac{1}{2}} &= \Big\|\Big(\sum_{j=1}^m N_j \otimes E_{j1}\Big)^* \Big(\sum_{j=1}^m N_j \otimes E_{j1}\Big)\Big\|^{\frac{1}{2}} \\
&= \Big\|\sum_{j=1}^m N_j \otimes E_{j1}\Big\| \\
&= \max\Big\{\Big\|\sum_{j=1}^m \lambda_j E_{j1}\Big\| : (\lambda_1, \ldots, \lambda_m) \in \sigma(N_1, \ldots, N_m)\Big\} \\
&= \max\Big\{\Big(\sum_{j=1}^m |\lambda_j|^2\Big)^{\frac{1}{2}} : (\lambda_1, \ldots, \lambda_m) \in \sigma(N_1, \ldots, N_m)\Big\}.
\end{aligned}$$

DEFINITION. For an arbitrary m-tuple $A = (A_1, \ldots, A_m)$ of operators on $\mathcal{H}$, let $\|A\|$—that is, $\|(A_1, \ldots, A_m)\|$—be given by the quantity

$$\inf\Big\{\Big\|\sum_{j=1}^m N_j^* N_j\Big\|^{\frac{1}{2}} : N \text{ is a normal dilation of } A\Big\}.$$

THEOREM 2.1. *The function $\|\cdot\|$ is a norm on the m-fold cartesian product vector space $\mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H})$. In addition, the norm has the following properties:*

1. *If $m = 1$, then the norm of the 1-tuple $A = (A_1)$ is precisely the norm of the operator A_1.*
2. *If A_1 and A_2 are hermitian operators, then $\|A_1 + iA_2\| = \|(A_1, A_2)\|$.*
3. *$\|(A_1, \ldots, A_m)\| = \|(B_1, \ldots, B_m)\|$, whenever each $B_j \in \{A_j, -A_j, A_j^*, -A_j^*\}$.*

PROOF. To show that $\|\cdot\|$ is norm, the only nontrivial point to be verified is the triangle inequality. To this end, suppose that A and B are m-tuples of operators on $\mathcal{H}$ and suppose that S and T are, respectively, normal dilations of A and B on Hilbert spaces $\mathcal{E}$ and $\mathcal{L}$ via the isometries $V: \mathcal{H} \longrightarrow \mathcal{E}$ and $W: \mathcal{H} \longrightarrow \mathcal{L}$. Let $\eta = \|\sum_j S_j^* S_j\|^{\frac{1}{2}}$ and $\mu =$

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$\| \sum_j T_j^* T_j \|^{\frac{1}{2}}$, the norms of S and T respectively. Consider the m-tuple N of commuting normal operators $\frac{\eta+\mu}{\eta} S_j \oplus \frac{\eta+\mu}{\mu} T_j$ acting on $\mathcal{E} \oplus \mathcal{L}$, and let $X\colon \mathcal{H} \longrightarrow \mathcal{E} \oplus \mathcal{L}$ be the isometry

$$x \longmapsto \sqrt{\frac{\eta}{\eta+\mu}}\, Vx \oplus \sqrt{\frac{\mu}{\eta+\mu}}\, Wx.$$

For each j, $X^* N_j X = V^* S_j V + W^* T_j W = A_j + B_j$ and so N is a normal dilation of $A + B$. Thus,

$$\|A+B\| \leq \Big\|\sum_j N_j^* N_j\Big\|^{\frac{1}{2}} = \max\Big\{\frac{\eta+\mu}{\eta}\Big\|\sum_j S_j^* S_j\Big\|^{\frac{1}{2}}, \frac{\eta+\mu}{\mu}\Big\|\sum_j T_j^* T_j\Big\|^{\frac{1}{2}}\Big\} = \eta+\mu,$$

and consequently

$$\|A+B\| \leq \|A\| + \|B\|.$$

We now prove that the norm has the stated properties. Property (1) follows from the Halmos dilation theorem: there exists a normal dilation N of A such that $\|A_1\| = \|N_1\| = \|N_1^* N_1\|^{\frac{1}{2}}$.

To prove (2), we use the elementary fact that for hermitian operators H_1 and H_2, $\|H_1^2 + H_2^2\|^{\frac{1}{2}} \leq \|H_1 + iH_2\|$ and equality occurs whenever H_1 and H_2 commute. To begin with, if (H_1, H_2) is an arbitrary hermitian dilation of (A_1, A_2) via an isometry V, then

$$\|A_1 + iA_2\| = \|V^*(H_1 + iH_2)V\| \leq \|H_1 + iH_2\| = \|H_1^2 + H_2^2\|^{\frac{1}{2}},$$

which implies that $\|A_1 + iA_2\| \leq \|(A_1, A_2)\|$. On the other hand, the Halmos dilation theorem yields a hermitian dilation (H_1, H_2) of (A_1, A_2) such that $\|A_1 + iA_2\| = \|H_1 + iH_2\|$ and so

$$\|A_1 + iA_2\| = \|H_1^2 + H_2^2\|^{\frac{1}{2}} \geq \|(A_1, A_2)\|.$$

Property (3) is an immediate consequence of the definition of the norm. ∎

REMARK. Suppose that A is an m-tuple of hermitian operators $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$ such that $W(A)$ is contained in a simplex $\mathcal{K}$ with vertices $v_1, \ldots, v_k$. Then

$$\|A\| \leq \max\{\|v_i\| : i = 1, \ldots, k\}.$$

This reason for this is that Theorem 1.1 provides a hermitian dilation H of A such that $\|H\| = \max\{\|v_i\| : i = 1, \ldots, k\}$; hence, the result follows from the fact that $\|A\| \leq \|H\|$. In the case where $m = 2$, $\|(A_1, A_2)\| = \|A_1 + iA_2\| \leq \max\{\|v_i\| : i = 1, \ldots, k\}$ and this is precisely Mirman's Theorem [17].

There are other norms that one can place on m-tuples of operators and we discuss some of them below.

Two very natural norms are introduced in [7] by Cho and Takaguchi and in [18] by Muller and Vasilescu. In [7], the quantity $\| \sum_j A_j^* A_j \|^{\frac{1}{2}}$ is considered as the norm of the m-tuple $A = (A_1, \ldots, A_m)$; in [17], the norm of A is defined to be the norm of the positive operator whose action on the C^*-algebra $\mathcal{B}(\mathcal{H})$ is given by $T \longmapsto \sum_j A_j^* T A_j$. However, these norms can change value when passing from A to the m-tuple A^* of adjoints. (One example of such behavior occurs with isometries $A_1, \ldots, A_m$ having mutually orthogonal ranges.) A second observation is that the norms of [7] and [18] do not have the property that the norm of a hermitian pair (A_1, A_2) is $\|A_1 + iA_2\|$.

Along the lines of [7] and [18], Bunce [6] generalises the norm and the spectral radius of a single operator in the following manner. For an m-tuple A, consider the quantity $\|\sum_{f\in F(k,m)} A_f^* A_f\|^{\frac{1}{2}}$, where $F(k,m)$ is the set of functions $\{1,2,\ldots,k\} \longrightarrow \{1,2,\ldots,m\}$ and $A_f = A_{f(1)}A_{f(2)}\cdots A_{f(k)}$. In addition, it is shown in [6;7] that $\rho_\pi(A) \leq \lim_k \|\sum_{f\in F(k,m)} A_f^* A_f\|^{1/2k} < \infty$, which is nearly an extension of the spectral radius formula to several variables.

In a rather different direction, Pryde [21] uses the norm of the Clifford operator induced by an m-tuple to prove some spectral variation results analogous to those established by Bauer and Fike. Recall that the Clifford algebra $\mathbb{R}_{(m)}$ is a 2^m-dimensional complex associative algebra generated by the forms $e_1, \ldots, e_m$ subject to the relations

1. $e_j^2 = -1$ for all j, where 1 is the unit of $\mathbb{R}_{(m)}$;
2. $e_j e_k = -e_k e_j$ whenever $j \neq k$;
3. $e_S = e_{j_1} \cdots e_{j_n}$, whenever $S = \{j_k : 1 \leq n\}$ with $j_1 < j_2 < \cdots < j_n$ (if S is the null set, then $e_S = 1$);

By using all subsets $S \subset \{1, \ldots, n\}$, $\mathbb{R}_{(m)}$ is spanned by the 2^m elements e_S, the set of which form an orthonormal basis under the inner product

$$\Big(\sum_S \lambda_S e_S, \sum_S \mu_S e_S\Big) = \sum_S \lambda_S \mu_S^*,$$

where ζ^* denotes the complex conjugate of ζ. The involution * on $\mathbb{R}_{(m)}$ is defined so that e_S^* is equal to one of e_S or $-e_S$, whichever satisfies $e_S^* e_S = e_S e_S^* = 1$; in particular, $e_j^* = -e_j$.

The operator algebra $\mathcal{B}(\mathcal{H}) \otimes \mathbb{R}_{(m)}$ can be identified with a subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathbb{R}_{(m)})$ as follows. Suppose that we have 2^m operators A_S on $\mathcal{H}$ indexed by the subsets $S \subset \{1, \ldots, m\}$. The operator $\sum_S A_S \otimes e_S$ is the unique operator on $\mathcal{H} \otimes \mathbb{R}_{(m)}$ whose action on elements of the form $\sum_T x_T \otimes e_T$ is $\sum_{S,T} A_S x_T \otimes e_S e_T$. Thus, $I \otimes 1$ is the identity operator and has norm 1. In particular, if $A = (A_1, \ldots, A_m)$, then the Clifford operator $\mathrm{Cliff}(A)$ on $\mathcal{H} \otimes \mathbb{R}_{(m)}$ is the operator given by $i \sum_j A_j \otimes e_j$. We have

$$(\dagger) \qquad \mathrm{Cliff}(A)^* \, \mathrm{Cliff}(A) = \sum_j A_j^* A_j \otimes (1) - \sum_{j<k} (A_j^* A_k - A_k^* A_j) \otimes e_j e_k$$

and

$$(\dagger^*) \qquad \mathrm{Cliff}(A) \, \mathrm{Cliff}(A)^* = \sum_j A_j A_j^* \otimes (1) - \sum_{j<k} (A_j A_k^* - A_k A_j^*) \otimes e_j e_k$$

Evidently, $\mathrm{Cliff}(A)$ is normal whenever A consists of commuting normal operators.

THEOREM 2.2. *For every m-tuple $A = (A_1, \ldots, A_m)$, the following statements are true.*

1. $\rho_\pi(A) \leq w(A) \leq \|\sum_j A_j^* A_j\|^{\frac{1}{2}} \leq \|\mathrm{Cliff}(A)\| \leq \sqrt{2}\|A\| \leq \sqrt{2}(r + 2s)w(A)$, where r and s are the number of normal and nonnormal A_i respectively. In fact,

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$r + 2s$ can be replaced by k if we know that the real and imaginary parts of A_i can be partitioned into k subsets, each of which consists of commuting hermitian operators.

2. *If $A_1, \ldots, A_m$ are commuting, then $\rho(A) = \rho_\pi(A) \leq w(A)$.*
3. *If $A_1, \ldots, A_m$ are commuting normals, then $\rho(A) = \|A\|$.*
4. *If A_1 and A_2 are hermitian, then $\|\operatorname{Cliff}(A)\| = \|A\| = \|A_1 + iA_2\|$.*

PROOF. If λ is a joint approximate eigenvalue of the m-tuple A, then λ is in the closure of the joint numerical range of A and $\rho_\pi(A) \leq w(A)$. Suppose that $x \in \mathcal{H}$ is an arbitrary unit vector. Note that $x \otimes 1$ is a unit vector in $\mathcal{H} \otimes \mathbb{R}_{(m)}$. Then

$$\sum_j |(A_j x, x)|^2 \leq \sum_j \|A_j x\|^2 = \Big(\sum_j A_j^* A_j x, x\Big) \leq \Big\|\sum_j A_j^* A_j\Big\|$$

and

$$\begin{aligned}\Big\|\sum_j A_j^* A_j x\Big\|^2 &\leq \Big\|\Big(\sum_j A_j^* A_j \otimes (+1)\Big)(x \otimes 1)\Big\|^2 + \Big\|\sum_{j<k}(A_j^* A_k - A_k^* A_j) \otimes e_j e_k (x \otimes 1)\Big\|^2 \\ &= \Big\|\Big(\sum_j A_j^* A_j \otimes (+1)\Big)(x \otimes 1) \\ &\qquad - \sum_{j<k}(A_j^* A_k - A_k^* A_j) \otimes e_j e_k (x \otimes 1)\Big\|^2 \quad \text{(Pythagoras Theorem)} \\ &= \|\operatorname{Cliff}(A)^* \operatorname{Cliff}(A)(x \otimes 1)\|^2 \leq \|\operatorname{Cliff}(A)^* \operatorname{Cliff}(A)\|^2,\end{aligned}$$

from which it follows that $w(A) \leq \|\sum_j A_j^* A_j\|^{\frac{1}{2}} \leq \|\operatorname{Cliff}(A)\|$. Next, if N is an arbitrary normal dilation of A via an isometry V, then $\operatorname{Cliff}(N)$ is a (possibly non-normal) dilation of $\operatorname{Cliff}(A)$ via the isometry $V \otimes 1$ and hence $\|\operatorname{Cliff}(A)\| \leq \|\operatorname{Cliff}(N)\|$. Because

$$\begin{aligned}\|\operatorname{Cliff}(N)\| &\leq \|\operatorname{Cliff}(N)^* \operatorname{Cliff}(N) + \operatorname{Cliff}(N) \operatorname{Cliff}(N)^*\|^{1/2} \\ &= \Big\|2\sum_j N_j^* N_j \otimes 1\Big\|^{1/2} = \sqrt{2}\Big\|\sum_j N_j^* N_j\Big\|^{1/2},\end{aligned}$$

it follows that $\|\operatorname{Cliff}(A)\| \leq \sqrt{2}\|A\|$. Finally, express the m-tuple A as the sum of m-tuples B_i, where B_i consists of A_i in the i-th position and zero elsewhere. If A_i is normal, then $\|B_i\| = w(B_i) \leq w(A)$. This proves that $\|A\| \leq \sum_i \|B_i\| \leq (r+2s)w(A)$, where r and s are the number of normal and nonnormal A_j respectively. If the real and imaginary parts of the operators A_j can be partitioned into k subsets, each consisting of commuting hermitian operators, then for the tuples $C_1, \ldots, C_k$ obtained by this partition, each C_i consist of commuting hermitians and $\|A\| \leq \sum_i \|C_i\| = \sum_i w(C_i) \leq kw(A)$. This completes the proof of statements (1) and (3).

In the case that A is a commuting m-tuple, the joint spectral radius $\rho(A)$ coincides with $\rho_\pi(A)$ [8] and so (2) follows from what we have just proved.

In [21; 7.1], Pryde computes the Clifford norm in the case where $A = (A_1, A_2)$. If A is hermitian, then $\|\operatorname{Cliff}(A)\| = \|A_1 + iA_2\|$, whence $\|\operatorname{Cliff}(A)\| = \|A\|$ by Theorem 2.1.

In general, $\|\operatorname{Cliff}(A)\|$ is the maximum of $\|A_1 + iA_2\|$ and $\|A_1 - iA_2\|$. By taking A_1 and A_2 to be the identity, we see that $\sqrt{2} = \|\operatorname{Cliff}(A)\| = \sqrt{2}\|A\|$, and so the inequality $\|\operatorname{Cliff}(A)\| \leq \sqrt{2}\|A\|$ is sharp. ■

Toeplitz and subnormal operators have natural normal dilations and extensions. The following examples show that in the several variable theory these same dilations and extensions can be used in computing the norms of tuples of Toeplitz operators and subnormal m-tuples.

EXAMPLE 1. If A is a subnormal m-tuple, that is A is an m-tuple of commuting subnormal operators that possess a joint normal extension, and if N is the minimal normal extension of A, then $\rho(A) = \|A\| = \|N\|$.

PROOF. As N is an extension of A, certainly $\|A\| \leq \|N\|$. But Putinar's spectral inclusion theorem (see [9; §7]) has that $\sigma(N) \subset \sigma(A)$ and that $\sigma(A)$ is contained within the polynomial-convex hull of $\sigma(N)$, and therefore $\|N\| = \rho(N) \leq \rho(A) \leq \|A\| \leq \|N\|$. ■

EXAMPLE 2. Let S^1 denote the unit circle in $\mathbb{C}$, $C(S^1)$ the continuous maps $S^1 \longrightarrow \mathbb{C}$, and $H^2(S^1)$ the Hardy space of S^1. If $T_{\phi_1}, \ldots, T_{\phi_m}$ are Toeplitz operators on $H^2(S^1)$ with symbols $\phi_j \in C(S^1)$, then

$$\|(T_{\phi_1}, \ldots, T_{\phi_m})\| = \max_{|\zeta|=1}\Big(\sum_{j=1}^{m} |\phi_j(\zeta)|^2\Big)^{\frac{1}{2}}.$$

PROOF. The m-tuple $M = (M_{\phi_1}, \ldots, M_{\phi_m})$ of multiplication operators on $L^2(S^1) \supset H^2(S^1)$ with symbols $\phi_j \in C(S^1)$ is a normal dilation of the Toeplitz tuple $T = (T_{\phi_1}, \ldots, T_{\phi_m})$. Hence,

$$\|T\| \leq \|M\| = \rho(M) = \max_{|\zeta|=1}\Big(\sum_{j=1}^{m} |\phi_j(\zeta)|^2\Big)^{\frac{1}{2}}.$$

Let $\mathcal{T}$ denote the C^*-algebra of operators on $H^2(S^1)$ generated by all Toeplitz operators with continuous symbol, and let $K\big(H^2(S^1)\big)$ denote the compact operators on the Hardy space. By Coburn's theorem [10; 7.23], there is a unital $*$-homomorphism $\varrho\colon \mathcal{T} \longrightarrow C(S^1)$ such that

$$(0) \longrightarrow K\big(H^2(S^1)\big) \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow (0)$$

is an exact sequence for which the map $\xi\colon C(S^1) \longrightarrow \mathcal{T}$ given by $\xi(\phi) = T_\phi$ is a cross section (*i.e.* $\varrho\big(\xi(\phi)\big) = \phi$ for all $\phi \in C(S^1)$). Hence, $\varrho(T) = \big(\varrho(T_{\phi_1}), \ldots, \varrho(T_{\phi})\big)$ is an m-tuple of elements in the commutative C^*-algebra $C(S^1)$ and has joint spectrum $\sigma\big(\varrho(T)\big) = \sigma(M)$. However, if $\lambda \in \sigma\big(\varrho(T)\big)$, then there exists a character ψ on the unital C^*-algebra $\mathcal{A}$ generated by $T_{\phi_1}, \ldots, T_{\phi_m}$ that annihilates every compact operator in $\mathcal{A}$ and that sends the tuple T to the tuple λ. But this implies that there exists a sequence of unit vectors $x_k \in H^2(S^1)$ converging weakly to zero and such that $\lim_k \|T_{\phi_j}x_k - \psi(T_{\phi_j})x_k\| = 0$ for every j [5; 2.1]. Hence, $\sigma(M) \subset \sigma_\pi(T)$ and so

$$\max_{|\zeta|=1}\Big(\sum_{j=1}^{m} |\phi_j(\zeta)|^2\Big)^{\frac{1}{2}} \leq \rho_\pi(T) \leq \|T\|. \qquad ■$$

We do not know whether the infimum in our definition of joint norm can always be achieved by the spectral radius of some normal dilation (as is the case in one-variable). If the infimum can be achieved, it would be of interest to know if in the case of matrices $A_1, \ldots, A_m$ one of the normal dilations N satisfying $\|N\| = \|A\|$ occurs using matrices $N_1, \ldots, N_m$.

3. **Joint approximate eigenvalues.** If A is an m-tuple of arbitrary operators and if $\lambda \in \sigma_p(A)$, then there is a unit vector $x \in \mathcal{H}$ that is a joint eigenvector of the operators A_j. Therefore, with respect to the decomposition $\mathcal{H} = \operatorname{span}\{x\} \oplus \{x\}^{\perp}$, each A_j has the form

$$\begin{pmatrix} \lambda_j & * \\ 0 & * \end{pmatrix}.$$

If λ is a joint reducing eigenvalue, which is to say that both $A_j x = \lambda_j x$ and $A_j^* x = \lambda_j^* x$ hold, then the matrix above is in fact a block-diagonal matrix; such occurs, for example, if each A_j is hyponormal (*i.e.* $A_j^* A_j - A_j A_j^* \geq 0$).

The purpose of the first proposition is to clarify the structural meaning of a joint approximate eigenvalue, a joint reducing approximate eigenvalue, and of an element in the closure of the joint numerical range. (A reducing approximate eigenvalue of A is a complex m-tuple λ for which $\lim_n \|(A_j - \lambda_j I)x_n\| = \lim_n \|(A_j - \lambda_j I)^* x_n\| = 0$ for some sequence of unit vectors $x_n \in \mathcal{H}$.) The result is a special case of a more general theorem of Hadwin [11].

DEFINITION. An m-tuple of operators A_j is (jointly) approximately equivalent to an m-tuple of operators T_j if there exist unitaries U_n such that $\lim_n \|U_n^* A_j U_n - T_j\| = 0$ for every j. If $U_n = U_m$ for all m, n, then A is said to be unitarily equivalent to T. We denote approximate equivalence by $A \sim_a T$.

REMARK. The joint approximate point spectrum and the closure of the joint numerical range are invariant under approximate equivalence; thus, it makes no difference if we replace A by some $T \sim_a A$ when dealing with these sets.

THEOREM 3.1. *For every m-tuple $A = (A_1, \ldots, A_m)$, the following statements are true.*

1. *λ is in the closure of $W(A)$ if and only if $A \sim_a T$ and each T_j has the form*
$$T_j = \begin{pmatrix} \lambda_j & * \\ * & * \end{pmatrix}.$$
2. *λ is a joint approximate eigenvalue of A if and only if $A \sim_a T$ and each T_j has the form*
$$T_j = \begin{pmatrix} \lambda_j & * \\ 0 & * \end{pmatrix}.$$
3. *λ is a joint reducing approximate eigenvalue of A if and only if $A \sim_a T$ and each T_j has the form*
$$T_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & * \end{pmatrix}.$$

PROOF. All three statements have similar proofs and are applications of Propositions 3.1 and 3.2 of Hadwin [11] to the cases at hand; thus, only (2) will be demonstrated here.

Let $X = \{1, \ldots, m\}$ and let $f: X \longrightarrow \mathbb{C}$ and $g: X \longrightarrow \mathcal{B}(\mathcal{H})$ be the functions $f(j) = \lambda_j$ and $g(j) = A_j$. By hypothesis, there exist unit vectors $x_n \in \mathcal{H}$ for which $\|g(j)x_n - f(j)x_n\| \longrightarrow 0$ for each j. On the C^*-algebra generated by $g(X)$, let φ_n be the vector state $\varphi_n(T) = (Tx_n, x_n)$. By the weak*-compactness of the state space, there is a subsequence of these vector states that is weak*-convergent to a state φ on $C^*\big(g(X)\big)$. Furthermore, because λ_j is a joint approximate eigenvalue, the state φ satisfies $\varphi(A_j) = \lambda_j$ and $\varphi(A_j^*A_j) = \varphi(A_j)^*\varphi(A_j)$ for each j. That is, $f(x) = \varphi\big(g(x)\big)$ and $\varphi\big(g(x)^*g(x)\big) = f(x)^*f(x)$ for every $x \in X$. By [11; 3.1, 3.2], f is the restriction of a function h approximately unitarily equivalent to g and so there exist unitaries U_n having the properties claimed in the statement of the proposition. ∎

Theorem 3.2 below extends two results known to hold in the case of a single operator. The first statement extends the corresponding one-variable result of Wintner [24]. The second statement concerns the fact, for example, that every point on the unit circle is a reducing approximate eigenvalue of the unilateral or bilateral shift operator W (indeed, as is well-known, $W \sim_a e^{i\theta} \oplus W$ for every $\theta \in \mathbb{R}$.) It seems an interesting problem to determine whether it is true in general that every $\lambda \in \sigma_\pi(A)$ on the topological boundary of $W(A)$ is actually a reducing approximate eigenvalue.

THEOREM 3.2. *For every m-tuple A, $\rho_\pi(A) = \|A\|$ if and only if $w(A) = \|A\|$. Furthermore, every λ in the closure of $W(A)$ satisfying $\|\lambda\| = \|A\|$ is a joint reducing approximate eigenvalue, which is to say that $A = (A_1, \ldots, A_m)$ is approximately equivalent to some $T = (T_1, \ldots, T_m)$ with $T_j = \lambda_j \oplus B_j$ for each j.*

PROOF. Of course it is true that $w(A) = \|A\|$ whenever $\rho_\pi(A) = \|A\|$. Conversely, suppose that λ is in the closure of the joint numerical range of A and is such that $\|\lambda\| = w(A)$. By Theorem 3.1, there exist unitaries U_n and operators T_j such that $T_j = \lim_n U_n^*A_jU_n$ for each j and $\lambda \in W(T)$. Thus, there is a unit vector $x \in \mathcal{H}$ for which $\lambda_j = (T_jx, x)$ for all j. From

$$\|A\|^2 = \|T\|^2 = w(T)^2 = \sum_j |(T_jx, x)|^2 \leq \sum_j \|T_jx\|^2 \leq \sum_j \|T_j^*T_j\|^2 \leq \|T\|^2$$

and the Cauchy-Schwarz inequality we may conclude that $|(T_jx, x)| = \|T_jx\|$ and hence $T_jx = \lambda_jx$ for each j. Thus $\lambda \in \sigma_\pi(T) = \sigma_\pi(A)$ and $\rho_\pi(A) = \|A\|$.

Let λ^* denote the complex conjugation of the vector λ and observe that $\|\lambda^*\| = w(A^*)$ and $\lambda_j^* = (T_j^*x, x)$ for each j. The argument above shows that λ^* is a joint eigenvalue of T corresponding to the joint eigenvector x. Hence, λ is a joint reducing eigenvalue of T and, by Theorem 3.1, λ must be a joint reducing approximate eigenvalue of A. ∎

COROLLARY 3.3. *If $A = (A_1, \ldots, A_m)$ is such that*

$$\sup_{\|x\|=1} \Big(\sum_j |(A_jx, x)|^2\Big)^{\frac{1}{2}} = \|\sum_j A_j^*A_j\|^{\frac{1}{2}},$$

then $\sigma_\pi(A)$ is not the empty set. In fact, A has a joint reducing approximate eigenvalue.

PROOF. The proof of Theorem 3.2 remains valid when $\|A\|$ is replaced by $\|\sum_j A_j^* A_j\|^{\frac{1}{2}}$ (or by any norm dominating this latter one). ∎

4. **Multiparameter spectral theory.** In this section we are concerned with pencils of the form

$$L_i(\lambda) = A_i + \sum_{j=1}^{m} B_{ij}\lambda_j, \quad \lambda \in \mathbb{C}^m,$$

where the A_i and B_{ij} are linear operators (B_{ij} being bounded) on Hilbert spaces $\mathcal{H}_i$, $1 \leq i \leq n$. Under "regularity" (see below), which implies, in particular, that $m = n$, some elegant connections are available between the singularity (in the sense of 0 belonging to the spectra) of the $L_i(\lambda)$ and the joint spectrum of certain commuting operators Γ_j defined on the tensor product $\mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$. For finite-dimensional $\mathcal{H}_i$, where only point spectra are involved, the theory is detailed in the treatise of Atkinson [2]. In the case of infinite-dimensional spaces, Sleeman's monograph [23] develops the theory for selfadjoint operators A_i and B_{ij} and it includes completeness theorems involving the joint spectral measure of the Γ_j. In the nonselfadjoint case, Rynne [22] investigates the relation between the Taylor spectra of $L_i(\lambda)^\dagger$ and Γ_j.

Here † denotes induction from $\mathcal{H}_i$ to $\mathcal{H}$, so that $B_{ij}^\dagger$ is the operator $I \otimes I \otimes \cdots \otimes B_{ij} \otimes \cdots \otimes I$, where B_{ij} appears in the i-th position of the tensor product. Because the operators $B_{ij}^\dagger$ commute for different i, the formal determinant Δ_0 of the square matrix $B = [B_{ij}^\dagger]_{i,j=1}^m$ is a well-defined linear operator on $\mathcal{H}$. The theory is advanced under the "regularity" assumption that Δ_0 is nonsingular. In finite dimensions, all operators are bounded and so one can define the operators $\Gamma_j = \Delta_0^{-1}\Delta_j$, where Δ_j is the operator determinant obtained by replacing the j-th column of the matrix B with $[A_1^\dagger \cdots A_m^\dagger]^t$. In infinite dimensions, the ideas are similar if Δ_0^{-1} is bounded, but significant complications arise if Δ_0 is only one-to-one.

Multiparameter equations of the form $L_i(\lambda)x = 0$ arise in separation of variables for partial differential equations, in linearised bifurcation models, and in certain inverse problems. Equations of the form $\Gamma_j x = 0$ are connected not only with multiparameter equations, but also with simultaneous diagonability of more general (not necessarily determinantal) operators Δ_j [3]. Here we shall discuss how the dilation of Theorem 1.1 relates to a basic idea in multiparameter theory, namely the solubility of the equation

$$(*) \qquad Bx = y$$

for $x = [x_1 \cdots x_m]^t$ and $y = [y_1 \cdots y_m]^t$, where each y_j is in the range of $A_j^\dagger$.

This appears implicitly in several works (see [2; Chapter 6],[23; Chapter 3]) and is explicitly studied in [14], [15]. In 1976, Isaev [14; Theorem 4] stated that the commutativity of the B_{ij} for each fixed i suffices, but this fails for $m = 1$, $B_{11} = 0 \neq A_1$, so some extra condition is obviously missing. One version of Isaev's theorem suitable for dilation is as follows.

THEOREM 4.1. *If the operators B_{ij} commute for each fixed i, and if Δ_0^{-1} exists and is bounded, then* $(*)$ *is soluble for all y.*

PROOF. Because all of the entries of B commute, the operator $\Delta_0^{-1}C$ is the inverse of B, where C_{ij} is the (j,i)-cofactor of Δ_0. ∎

In infinite dimensions, the solubility of $(*)$ is nontrivial, so the following result is of some interest.

COROLLARY 4.2. *Every bounded selfadjoint multiparameter system has a dilation such that (the dilated equivalent of)* $(*)$ *is soluble.*

PROOF. For each i, suppose that the joint range $W(A_i, B_{i1}, \ldots, B_{im})$ is contained in $\mathcal{V}_i$, which we may assume (without loss of generality) to be a simplex. Using Theorem 1.1, dilate $B_{i1}, \ldots, B_{im}$ to commuting hermitian operators $D_{i1}, \ldots, D_{im}$. The (bounded) determinantal operator $D_0 = \det[D_{ij}^{\dagger}]$, which is the dilated equivalent of Δ_0, is diagonal with entries that are determinants whose columns are the (final m components of) vertices of the $\mathcal{V}_i$. By a perturbation of the vertices, if necessary, we can assume that D_0^{-1} exists. ∎

Theorem 1.1 is not the only dilation relevant for multiparameter spectral theory. The dilation given by T. Kosir [16] shows that an arbitrary set of m commuting matrices can be dilated to the Γ_j for some regular multiparameter system. This is not obvious, since the Γ_j are constrained by the relations $A_i^{\dagger} = \sum_j B_{ij}^{\dagger}\Gamma_j$.

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