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THE INTEGRAL COHOMOLOGY OF THE HILBERT SCHEME OF TWO POINTS

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Abstract

The Hilbert scheme $X^{[a]}$ of points on a complex manifold X is a compactification of the configuration space of a-element subsets of X. The integral cohomology of $X^{[a]}$ is more subtle than the rational cohomology. In this paper, we compute the mod 2 cohomology of $X^{[2]}$ for any complex manifold X, and the integral cohomology of $X^{[2]}$ when X has torsion-free cohomology.

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For a complex manifold X and a natural number a, the Hilbert scheme $X^{[a]}$ (also called the Douady space) is the space of 0-dimensional subschemes of degree a in X. It is a compactification of the configuration space $B(X, a)$ of a-element subsets of X. The Hilbert scheme is smooth if and only if X has dimension at most 2 or $a \leqslant 3$ [3, equation (0.2.1)]. The integral cohomology of the Hilbert scheme is more subtle than the rational cohomology. Markman computed the integral cohomology of the Hilbert schemes $X^{[a]}$ for X of dimension 2 with effective anticanonical divisor [10]. In this paper, we compute the mod 2 cohomology of $X^{[2]}$ for any complex manifold X, and the integral cohomology of $X^{[2]}$ when X has torsion-free cohomology.

In one way, things are unexpectedly good: the Hilbert scheme $X^{[2]}$ has torsion-free cohomology if X does (Theorem 2.2). On the other hand, the details are intricate, and it was not clear that complete answers would be possible. The behavior of the inclusion of the exceptional divisor E_X into $X^{[2]}$ is related to the Steenrod operations on the mod 2 cohomology of X (Theorem 2.1). To explain one difficulty: some cohomology classes on $X^{[2]}$ can be defined as the classes of $Y^{[2]}$ in $X^{[2]}$ for complex submanifolds Y of X, which we study in Lemma 6.1. But because the Hilbert scheme is only defined for complex manifolds, it is harder to construct 'interesting' classes on $X^{[2]}$ associated to arbitrary cohomology classes on X, for example to odd-degree cohomology classes.

Why look at two points? Configurations of two points come up naturally in geometry, but one especially relevant use of the Hilbert scheme $X^{[2]}$ is in Voisin's paper on the universal CH_0 group of cubic hypersurfaces [22]. The background is that major recent advances have been made in determining which algebraic varieties are stably rational, that is, become birational to projective space after multiplying by projective space of some dimension [5, 19, 21]. These papers are based on the observation that if a smooth projective variety is stably rational, then its Chow group of 0-cycles is universally trivial, meaning that CH_0 does not increase when the base field is increased.

The Chow group $CH_0 \otimes \mathbf{Q}$ is universally trivial for all rationally connected varieties, and so proving that varieties of interest are not stably rational requires looking at torsion in the Chow group, with the best results coming from 2-torsion. As a result, Voisin's work on cubics X uses information on the integral or mod 2 cohomology of the Hilbert scheme $X^{[2]}$, including results from this paper [22, proof of Proposition 2.6]. A typical application is that smooth cubic 3-folds in $\mathbf{CP}^4$ have CH_0 universally trivial for at least a countable union of codimension-3 subvarieties in the moduli space of cubics [22, Theorem 1.5]. (Smooth cubic 3-folds are all nonrational by Clemens and Griffiths [4], but it is wide open whether all, or some, smooth cubic 3-folds are stably rational.)

Since the Hilbert cube $X^{[3]}$ of a complex manifold X is again a complex manifold, it is natural to ask whether the results of this paper extend to that case. In particular, does $X^{[3]}$ have torsion-free cohomology if X does? The explicit geometric description of $X^{[3]}$ by Shen and Vial should help [16, Section 4].

1. Torsion-free cohomology in even degrees

Here we give a short proof that the Hilbert scheme $X^{[2]}$ of a compact complex manifold has torsion-free cohomology if the cohomology of X is torsion-free and concentrated in even degrees. We show this without the restriction to even degrees and without assuming compactness in Theorem 2.2, but that proof is considerably harder.

THEOREM 1.1. *Let X be a compact complex manifold whose integral cohomology is torsion-free and concentrated in even degrees. Then the cohomology of the Hilbert scheme $X^{[2]}$ is also torsion-free and concentrated in even degrees.*

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Proof. Nakaoka and Milgram computed the integral homology of the symmetric product $S^a X$, the quotient of X^a by the symmetric group S_a, for any finite CW-complex X and any natural number a [13]. We now state their result on $S^2 X$ when the homology of X is torsion-free; Theorem 4.1 will give their computation of the mod 2 homology of $S^2 X$ for any X. For an element u in $H_j X$, we write $|u|$ for the dimension j, and likewise for a cohomology class.

THEOREM 1.2. *Let X be a finite CW-complex such that $H_*(X, \mathbf{Z})$ is torsion-free. Let $u_0, \ldots, u_s$ be a basis for $H_*(X, \mathbf{Z})$ as a free graded abelian group. Then $H_*(S^2 X, \mathbf{Z})$ is the direct sum of one copy of $\mathbf{Z}$ in dimension $|u_i| + |u_j|$ for each $0 \leqslant i \leqslant j \leqslant s$ except when $i = j$ and $|u_i|$ is odd, together with one copy of $\mathbf{Z}/2$ in degrees*

$$|u_i| + 2, |u_i| + 4, \ldots, 2|u_i| - 2$$

for each i with $|u_i|$ even and greater than 0, and one copy of $\mathbf{Z}/2$ in degrees

$$|u_i| + 2, |u_i| + 4, \ldots, 2|u_i| - 1$$

for each i with $|u_i|$ odd.

Let X be a compact complex manifold whose integral cohomology is torsion-free and concentrated in even degrees. The universal coefficient theorem implies the same statement for the integral homology of X [9, Theorem 3.2]. Then Theorem 1.2 gives that $H_*(S^2 X, \mathbf{Z})$ is also concentrated in even degrees, although it may have torsion.

A point of the Hilbert scheme $X^{[2]}$ represents either an unordered pair of distinct points in X or a point x in X together with a complex line in the tangent space $T_x X$. As a result, the Hilbert scheme $X^{[2]}$ is related to the symmetric square $S^2 X$ by a blow-up square:

$$\begin{tikzcd} E_X \arrow[r] \arrow[d] & X^{[2]} \arrow[d] \\ X \arrow[r] & S^2 X. \end{tikzcd}$$

Here $X \to S^2 X$ is the diagonal inclusion. For a (real or complex) vector bundle V, we use Grothendieck's convention that $P(V)$ means the (real or complex) projective bundle of hyperplanes in V, and $O(1)$ means the quotient line bundle on $P(V)$. Then the exceptional divisor E_X is the complex projective bundle $P(T^* X)$ of lines in the tangent bundle TX. (To say that this is a blow-up square means that it is a Cartesian diagram with $X \to S^2 X$ a closed embedding, $X^{[2]} \to S^2 X$ a proper morphism, and $X^{[2]} - E_X \to S^2 X - X$ an isomorphism.)

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The blow-up square gives a long exact sequence of integral homology groups:

$$H_i E_X \to H_i X \oplus H_i X^{[2]} \to H_i S^2 X \to H_{i-1} E_X.$$

(This follows by showing that a blow-up square is a homotopy pushout diagram, and applying the Mayer–Vietoris sequence [11, Lemma, page 78].) We know that $H_*(X, \mathbf{Z})$ and $H_*(S^2X, \mathbf{Z})$ are concentrated in even degrees. Since E_X is the projectivization of a complex vector bundle over X, its homology is also concentrated in even degrees. So the long exact sequence gives that $H_*(X^{[2]}, \mathbf{Z})$ is concentrated in even degrees.

We also want to show that the integral homology of $X^{[2]}$ is torsion-free. Let n be the complex dimension of X. Because $X^{[2]}$ is a closed oriented real manifold of dimension $4n$, Poincaré duality gives a duality between the finite abelian groups $H_i(X^{[2]}, \mathbf{Z})_{\text{tors}}$ and $H_{4n-1-i}(X^{[2]}, \mathbf{Z})_{\text{tors}}$ [9, Theorems 3.2 and 3.30]. Since $H_{\text{odd}}(X^{[2]}, \mathbf{Z})_{\text{tors}} = 0$, it follows that $H_{\text{ev}}(X^{[2]}, \mathbf{Z})_{\text{tors}} = 0$. By the universal coefficient theorem, the integral cohomology of $X^{[2]}$ is also torsion-free and concentrated in even degrees. Theorem 1.1 is proved. $\square$

Let X be a compact complex manifold of dimension n. Let $\widetilde{X \times X}$ be the blow-up of $X \times X$ along the diagonal. The exceptional divisor E_X in $X^{[2]}$ is known to be 2 times an element e in $H^2(X^{[2]}, \mathbf{Z})$. This follows from the existence of the double covering g from $S := \widetilde{X \times X}$ to $T := X^{[2]}$, ramified along E_X. Namely, we can define e to be $-c_1$ of the holomorphic line bundle $(g_* O_S)/O_T$. (When X has torsion-free homology, the fact that the class of E_X in $X^{[2]}$ is divisible by 2 can also be seen from Theorem 1.2 and the blow-up sequence, above.) We also write e for the associated element of $H^2(X^{[2]}, \mathbf{F}_2)$.

The restriction of e to the exceptional divisor $E_X = P(T^*X)$ is $e = c_1 O(-1)$. The cohomology of E_X with any coefficient ring is a free module over H^*X with basis $1, e, \ldots, e^{n-1}$. Let $i\colon E_X \to X^{[2]}$ be the inclusion, and let $\pi\colon E_X \to X$ be the projection. To simplify notation, we omit the symbol π^* when considering cohomology classes on X pulled back to E_X.

2. Main results

THEOREM 2.1. *Let X be a complex manifold of complex dimension n. Suppose that X is homeomorphic to the complement of a closed subcomplex in a finite CW-complex; this is no restriction for X compact. Then the kernel of the pushforward* *homomorphism $i_*\colon H^*(E_X, \mathbf{F}_2) \to H^*(X^{[2]}, \mathbf{F}_2)$ is spanned over $\mathbf{F}_2$ by the following elements, for u in $H^*(X, \mathbf{F}_2)$:*

$$\begin{aligned}
e^j(e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots + \operatorname{Sq}^{2a} u) &\quad \textit{for } |u| = 2a, 0 \leqslant j \leqslant n-1-a;\\
e^j(e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots + \operatorname{Sq}^{2a} u) &\quad \textit{for } |u| = 2a+1,\\
&\qquad\qquad\qquad 0 \leqslant j \leqslant n-1-a;\\
e^j(e^{a-1}\operatorname{Sq}^1 u + e^{a-2}\operatorname{Sq}^3 u + \cdots + \operatorname{Sq}^{2a-1} u) &\quad \textit{for } |u| = 2a, 0 \leqslant j \leqslant n-1-a;\\
e^j(e^a \operatorname{Sq}^1 u + e^{a-1}\operatorname{Sq}^3 u + \cdots + \operatorname{Sq}^{2a+1} u) &\quad \textit{for } |u| = 2a+1,\\
&\qquad\qquad\qquad 0 \leqslant j \leqslant n-2-a.
\end{aligned}$$

We have a localization exact sequence, in particular with $\mathbf{F}_2$ coefficients:

$$\to H^{j+1}X^{[2]} \to H^{j+1}(S^2X - X) \to H^j E_X \to H^{j+2}X^{[2]} \to .$$

Moreover, the $\mathbf{F}_2$-Betti numbers of E_X and $S^2X - X$ are determined by those of X (see Theorem 4.2 for $S^2X - X$). So Theorem 2.1 determines the $\mathbf{F}_2$-Betti numbers of $X^{[2]}$ in terms of the action of Steenrod operations on $H^*(X, \mathbf{F}_2)$. The description is complicated, but this is unavoidable: Example 2.5 shows that the $\mathbf{F}_2$-Betti numbers of $X^{[2]}$ are not determined by the $\mathbf{F}_2$-Betti numbers of X, in general.

On the other hand, the following result implies that $X^{[2]}$ has several good properties when the integral cohomology of X has no 2-torsion; in particular, its $\mathbf{F}_2$-Betti numbers are determined by those of X in that case.

THEOREM 2.2. *Let X be a complex manifold of complex dimension n whose integral cohomology has no 2-torsion. Suppose that X is homeomorphic to the complement of a closed subcomplex in a finite CW-complex; this is no restriction for X compact. Then a basis over $\mathbf{F}_2$ for the kernel of the pushforward homomorphism $i_*\colon H^*(E_X, \mathbf{F}_2) \to H^*(X^{[2]}, \mathbf{F}_2)$ is given by the elements:*

$$\begin{aligned}
e^j(e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots + \operatorname{Sq}^{2a} u) &\quad \textit{for } |u| = 2a, 0 \leqslant j \leqslant n-1-a;\\
e^j(e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots + \operatorname{Sq}^{2a} u) &\quad \textit{for } |u| = 2a+1, 0 \leqslant j \leqslant n-1-a,
\end{aligned}$$

where u runs through a basis for $H^*(X, \mathbf{F}_2)$.

Moreover, if the integral cohomology of X has no 2-torsion (respectively no torsion), then the integral cohomology of $X^{[2]}$ has no 2-torsion (respectively no torsion).

The following corollary gives a basis for the cohomology of $X^{[2]}$ when the cohomology of X has no 2-torsion. We use the maps g and i from the end of Section 1. Note that any cohomology class on X can be represented by a pseudomanifold (as discussed in Section 3).

COROLLARY 2.3. *Let X be as in Theorem 2.2. Let $z_0, \dots, z_s$ be $\mathbf{Z}$-cohomology classes that form a basis for $H^*(X, \mathbf{Z}_{(2)})$. For each j, let Z_j be a closed pseudomanifold in X that represents the class z_j.*

For each j from 0 to s with $|z_j| \leqslant 2n-2$, if $|z_j|$ is even, $|z_j| = 2a$, then there is an element x_j of $H^{4a}(X^{[2]}, \mathbf{Z})$ that restricts to the class $[S^2 Z_j - Z_j]$ in the $\mathbf{Z}$-cohomology of $S^2X - X$; and if $|z_j| = 2a+1$, there is an element y_j of $H^{4a+3}(X^{[2]}, \mathbf{Z})$ that restricts to the Bockstein $\beta[S^2 Z_j - Z_j]$ in the $\mathbf{Z}$-cohomology of $S^2X - X$. Choose such elements. Then $H^*(X^{[2]}, \mathbf{Z}_{(2)})$ is a free $\mathbf{Z}_{(2)}$-module, with a basis given by the elements:

$$\begin{aligned} g_*(z_j \otimes z_k) \quad & \text{for } j < k, \\ i_*(e^m z_j) \quad & \text{for each } j,\ 0 \leqslant m \leqslant \lfloor |z_j|/2 \rfloor - 1, \\ e^m x_j \quad & \text{for each } j \text{ with } |z_j| = 2a,\ 0 \leqslant m \leqslant n-1-a, \\ e^m y_j \quad & \text{for each } j \text{ with } |z_j| = 2a+1,\ 0 \leqslant m \leqslant n-2-a. \end{aligned}$$

If z_j is the cohomology class of a complex submanifold Z_j of X, then the element x_j in Corollary 2.3 can be taken to be the class of the sub-Hilbert scheme $Z_j^{[2]}$ in $X^{[2]}$. Beyond that case, it is not clear how to describe the classes x_j and y_j in geometric terms.

The following statement is used in Voisin's paper on cubic hypersurfaces. It is proved there in the case of odd-degree complete intersections in projective space [22, Lemma 2.8].

COROLLARY 2.4. *Let X be a compact complex manifold whose integral cohomology has no 2-torsion. Let $k \geqslant l$ be integers, and let α be an element of $H^{2k}(E_X, \mathbf{Z})$ of the form*

$$\alpha = e^{k-l}\beta_l + e^{k-l-1}\beta_{l+1} + \cdots$$

with β in $H^{2j}(X, \mathbf{Z})$. If $i_*\alpha$ is divisible by 2 in $H^{2k+2}(X^{[2]}, \mathbf{Z})$ and $2l > k$, then β_l is divisible by 2 in $H^{2l}(X, \mathbf{Z})$.

Proof. Consider α as a class in $H^{2k}(E_X, \mathbf{F}_2)$. We are assuming that α is in the kernel of $i_* \colon H^{2k}(E_X, \mathbf{F}_2) \to H^{2k+2}(X^{[2]}, \mathbf{F}_2)$. The kernel of i_* on $H^*(E_X, \mathbf{F}_2)$ is computed in Theorem 2.2, which implies the conclusion here. □

EXAMPLE 2.5. We give an example of compact complex manifolds X and Y with the same $\mathbf{F}_2$-Betti numbers such that $X^{[2]}$ and $Y^{[2]}$ do not have the same

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$\mathbf{F}_2$-Betti numbers. First, let $W \to \mathbf{P}^1$ be a minimal rational elliptic surface with section, for example defined by blowing up the intersection of two cubic curves in $\mathbf{P}^2$. Then W has second Betti number equal to 10. By Ogg and Shafarevich, for any finite sequence of integers $m_1, \ldots, m_r \geqslant 2$, there is a smooth projective elliptic surface over $\mathbf{P}^1$ which is a principal homogeneous space for $W \to \mathbf{P}^1$ outside r points in $\mathbf{P}^1$ and which has multiple fibers with multiplicity $m_1, \ldots, m_r$ at those points [7, Theorem III.6.12]. Such a surface automatically has $b_2 = 10$, since $b_2(W) = 10$ [7, Lemma I.3.18, Proposition I.3.21, Theorem I.6.7]. Let X and Y be such elliptic surfaces with multiple fibers of multiplicities 2, 2 and 4, 4, respectively. Then $\pi_1(X) \cong \mathbf{Z}/2$ and $\pi_1(Y) \cong \mathbf{Z}/4$ [7, Theorem I.2.3]. Here X is an Enriques surface and Y has Kodaira dimension 1.

By Poincaré duality and the universal coefficient theorem, the integral cohomology groups of X and Y are:

	0	1	2	3	4
X	$\mathbf{Z}$	0	$\mathbf{Z}^{10} \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}$
Y	$\mathbf{Z}$	0	$\mathbf{Z}^{10} \oplus \mathbf{Z}/4$	$\mathbf{Z}/4$	$\mathbf{Z}$

It follows that the Enriques surface X and the surface Y have the same $\mathbf{F}_2$-Betti numbers: 1, 1, 12, 1, 1. Because the Bockstein Sq^1 is zero on $H^*(Y, \mathbf{F}_2)$ but not on $H^*(X, \mathbf{F}_2)$, $Y^{[2]}$ has smaller $\mathbf{F}_2$-Betti numbers than $X^{[2]}$. Explicitly, by Theorem 2.1, the $\mathbf{F}_2$-Betti numbers are:

	0	1	2	3	4	5	6	7	8
$X^{[2]}$	1	1	13	15	94	15	13	1	1
$Y^{[2]}$	1	1	13	14	92	14	13	1	1

3. The boundary map

Recall the localization exact sequence with $\mathbf{F}_2$ coefficients:

$$H^{j+1}X^{[2]} \to H^{j+1}(S^2X - X) \to H^j E_X \to H^{j+2}X^{[2]}.$$

The key step in determining the kernel of the pushforward $i_*\colon H^j E_X \to H^{j+2}X^{[2]}$ is to compute the boundary homomorphism on interesting elements of $H^{j+1}(S^2X - X)$, as we now do.

LEMMA 3.1. *Let Z be a closed C^∞ submanifold of real codimension r in a complex manifold X. Let u be the cohomology class of Z in $H^r(X, \mathbf{F}_2)$. Then the boundary in $H^{2r-1}(E_X, \mathbf{F}_2)$ of the class $[S^2Z - Z]$ in $H^{2r}(S^2X - X, \mathbf{F}_2)$ is*

$$\begin{cases} e^{a-1}\,\mathrm{Sq}^1 u + e^{a-2}\,\mathrm{Sq}^3 u + \cdots + \mathrm{Sq}^{2a-1} u & \text{if } r = 2a, \\ e^a u + e^{a-1}\,\mathrm{Sq}^2 u + \cdots + \mathrm{Sq}^{2a} u & \text{if } r = 2a+1. \end{cases}$$

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Proof. We can view $S^2Z - Z$ as the interior of a manifold with boundary, where the boundary is the real projective bundle $P_{\mathbf{R}}(T^*Z)$ over Z. So the boundary of $[S^2Z - Z]$ is the class t_*1 in $H^{2r-1}(E_X, \mathbf{F}_2)$, where t is the proper map $P_{\mathbf{R}}(T^*Z) \to E_X = P_{\mathbf{C}}(T^*X)$, taking a real line in T_zZ for a point z in Z to the complex line that it spans in T_zX.

We can factor t as $P_{\mathbf{R}}(T^*Z) \hookrightarrow P_{\mathbf{R}}(T^*X)|_Z \hookrightarrow P_{\mathbf{R}}(T^*X) \twoheadrightarrow P_{\mathbf{C}}(T^*X)$. Write $\rho\colon P_{\mathbf{R}}(T^*X)|_Z \to Z$ for the projection. Then $P_{\mathbf{R}}(T^*Z)$ is the zero set of a transverse section of the real vector bundle $\mathrm{Hom}(O(-1), \rho^*N_{Z/X})$ over $P_{\mathbf{R}}(T^*X)|_Z$; that section is the one associated to the subbundle $O(-1) \subset \rho^*TX|_Z$. So the cohomology class of $P_{\mathbf{R}}(T^*Z)$ on $P_{\mathbf{R}}(T^*X)|_Z$ is the top Stiefel–Whitney class $w_r(O(1) \otimes \rho^*N_{Z/X})$. (This follows from the description of the top Stiefel–Whitney class in Milnor and Stasheff [15, page 145].) The top Stiefel–Whitney class of the tensor product of a line bundle L with a vector bundle W of rank r is

$$w_r(L \otimes W) = (w_1L)^r + (w_1L)^{r-1}w_1W + \cdots + w_rW$$

by the splitting principle, as in Bott and Tu [2, page 279]. Write b for the class $w_1O(1)$ in $H^1(P_{\mathbf{R}}(TX), \mathbf{F}_2)$. We deduce that the $\mathbf{F}_2$-cohomology class of $P_{\mathbf{R}}(T^*Z)$ on $P_{\mathbf{R}}(T^*X)|_Z$ is $b^r + b^{r-1}w_1N_{Z/X} + \cdots + w_rN_{Z/X}$, where we omit the symbol ρ^* for cohomology classes on Z pulled back to $P_{\mathbf{R}}(T^*X)|_Z$.

Write s for the inclusion $Z \hookrightarrow X$, and u for the cohomology class $s_*1 = [Z]$ in $H^r(X, \mathbf{F}_2)$. Then the pushforward of the class of $P_{\mathbf{R}}(T^*Z)$ from $P_{\mathbf{R}}(T^*X)|_Z$ to $P_{\mathbf{R}}(T^*X)$ is $b^ru + b^{r-1}s_*w_1N_{Z/X} + \cdots + s_*w_rN_{Z/X}$. The Steenrod squares of the class $u = [Z]$ in $H^*(X, \mathbf{F}_2)$ are the pushforward to X of the Stiefel–Whitney classes of the normal bundle $N_{Z/X}$ by the inclusion $s\colon Z \to X$,

$$\mathrm{Sq}^j u = s_*w_j(N_{Z/X}),$$

by Thom [17]. (For an introduction to Steenrod squares, see Hatcher [9, Section 4.L].) So the class of $P_{\mathbf{R}}(T^*Z)$ on $P_{\mathbf{R}}(T^*X)$ is $b^ru + b^{r-1}\,\mathrm{Sq}^1 u + \cdots + \mathrm{Sq}^r u$.

Finally, we have to push this class forward via the S^1-bundle $h\colon P_{\mathbf{R}}(T^*X) \to P_{\mathbf{C}}(T^*X)$. (We sometimes write $O(1)_{\mathbf{R}}$ instead of $O(1)$ for the natural real line bundle on $P_{\mathbf{R}}(T^*X)$, and likewise $O(1)_{\mathbf{C}}$ instead for $O(1)$ for the natural complex line bundle on $P_{\mathbf{C}}(T^*X)$, to avoid confusion.) The class $e = c_1O(-1)$ in $H^2(P_{\mathbf{C}}(T^*X), \mathbf{F}_2)$ pulls back to b^2, since the complex line bundle $O(1)_{\mathbf{C}}$ on $P_{\mathbf{C}}(T^*X)$ pulls back to $O(1)_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ on $P_{\mathbf{R}}(T^*X)$, and $c_1(O(1)_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}) = w_1(O(1)_{\mathbf{R}})^2 = b^2$. All classes on $P_{\mathbf{R}}(T^*X)$ pulled back from $H^*(X, \mathbf{F}_2)$ (such as the classes $\mathrm{Sq}^j u$) are also pulled back from $P_{\mathbf{C}}(T^*X)$. Here $H^*(P_{\mathbf{R}}(T^*X), \mathbf{F}_2)$ is a free module over $H^*(X, \mathbf{F}_2)$ with basis $1, b, \ldots, b^{2n-1}$, where n is the complex dimension of X. (This follows from the Leray–Hirsch theorem [9, Theorem 4D.1].) So to compute the pushforward h_* on $\mathbf{F}_2$-cohomology, it suffices to compute h_*1 and h_*b. Here h_*1 is in $H^{-1}(P_{\mathbf{C}}(T^*X), \mathbf{F}_2) = 0$, and so $h_*1 = 0$. Also, h_*b is in $H^0(P_{\mathbf{C}}(T^*X), \mathbf{F}_2)$, and so it is either 0 or 1. In fact, $h_*b = 1$. This can be proved by restricting over a point in X, and noting that the inclusion of a hyperplane $b = [\mathbf{RP}^{2n-2}]$ in $\mathbf{RP}^{2n-1}$ composed with the surjection to $\mathbf{CP}^{n-1}$ has degree $1 \pmod 2$, as it restricts to a diffeomorphism from $\mathbf{R}^{2n-2}$ to $\mathbf{C}^{n-1}$.

Therefore, for Z of codimension $r = 2a$, the boundary of $[S^2Z - Z]$ in $H^{4a-1}(E_X, \mathbf{F}_2)$ is

$$h_*(b^{2a}u + b^{2a-1}\,\mathrm{Sq}^1\,u + \cdots + \mathrm{Sq}^{2a}\,u) = e^{a-1}\,\mathrm{Sq}^1\,u + e^{a-2}\,\mathrm{Sq}^3\,u + \cdots + \mathrm{Sq}^{2a-1}\,u.$$

For Z of codimension $r = 2a + 1$, the boundary of $[S^2Z - Z]$ in $H^{4a+1}(E_X, \mathbf{F}_2)$ is

$$h_*(b^{2a+1}u + b^{2a}\,\mathrm{Sq}^1\,u + \cdots + \mathrm{Sq}^{2a+1}\,u) = e^a u + e^{a-1}\,\mathrm{Sq}^2\,u + \cdots + \mathrm{Sq}^{2a}\,u.$$

□

Let b be the element of $H^1(S^2X - X, \mathbf{F}_2)$ associated to the double cover $X \times X - X \to S^2X - X$.

LEMMA 3.2. *Let Z be a closed C^∞ submanifold of real codimension r in a complex manifold X. Let u be the cohomology class of Z in $H^r(X, \mathbf{F}_2)$. Then the boundary in $H^{2r}(E_X, \mathbf{F}_2)$ of the product $b[S^2Z - Z]$ in $H^{2r+1}(S^2X - X, \mathbf{F}_2)$ is*

$$\begin{cases} e^a u + e^{a-1}\,\mathrm{Sq}^2\,u + \cdots + \mathrm{Sq}^{2a}\,u & \text{if } r = 2a, \\ e^a\,\mathrm{Sq}^1\,u + e^{a-1}\,\mathrm{Sq}^3\,u + \cdots + \mathrm{Sq}^{2a+1}\,u & \text{if } r = 2a+1. \end{cases}$$

Proof. We can think of $S^2X - X$ as the interior of a real manifold with boundary, where the boundary is the real projective bundle $P_{\mathbf{R}}(T^*X)$. Then the element b in $H^1(S^2X - X, \mathbf{F}_2)$ restricts to the class $b = w_1(O(1)_{\mathbf{R}})$ on $P_{\mathbf{R}}(T^*X)$.

As in the proof of Lemma 3.1, the boundary in $H^{2r}(E_X, \mathbf{F}_2)$ of the product $b[S^2Z - Z]$ is the pushforward of the cohomology class b on $P_{\mathbf{R}}(T^*Z)$ via the proper map $P_{\mathbf{R}}(T^*Z) \to P_{\mathbf{C}}(T^*X)$. We can factor that map as $P_{\mathbf{R}}(T^*Z) \hookrightarrow P_{\mathbf{R}}(T^*X) \twoheadrightarrow P_{\mathbf{C}}(T^*X)$, and the class b is pulled back from $P_{\mathbf{R}}(T^*X)$. So the pushforward of b to $P_{\mathbf{R}}(T^*X)$ is b times the class of $P_{\mathbf{R}}(T^*Z)$ on $P_{\mathbf{R}}(T^*X)$, as computed in the proof of Lemma 3.1. Thus, the pushforward of b to $P_{\mathbf{R}}(T^*X)$ is $b(b^r u + b^{r-1}\,\mathrm{Sq}^1\,u + \cdots + \mathrm{Sq}^r\,u) = b^{r+1}u + b^r\,\mathrm{Sq}^1\,u + \cdots + b\,\mathrm{Sq}^r\,u$.

It remains to push this class forward via the S^1-bundle $h\colon P_{\mathbf{R}}(T^*X) \to P_{\mathbf{C}}(T^*X)$. We recall from the proof of Lemma 3.1 that $h^*e = b^2$, $h_*1 = 0$, and $h_*b = 1$. Thus, for Z of even codimension $r = 2a$, the boundary of $b[S^2Z - Z]$ in $H^{4a}(E_X, \mathbf{F}_2)$ is

$$h_*(b^{2a+1}u + b^{2a}\,\mathrm{Sq}^1\,u + \cdots + b\,\mathrm{Sq}^{2a}\,u) = e^a u + e^{a-1}\,\mathrm{Sq}^2\,u + \cdots + \mathrm{Sq}^{2a}\,u.$$

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For Z of codimension $r = 2a+1$, the boundary of $b[S^2 Z - Z]$ in $H^{4a+2}(E_X, \mathbf{F}_2)$ is

$$h_*(b^{2a+2}u + b^{2a+1}\operatorname{Sq}^1 u + \cdots + b\operatorname{Sq}^{2a+1} u) = e^a \operatorname{Sq}^1 u + e^{a-1}\operatorname{Sq}^3 u + \cdots + \operatorname{Sq}^{2a+1} u.$$

$\square$

Next, we prove the same formulas for any $\mathbf{F}_2$-cohomology class on X, not necessarily the class of a submanifold. We can view any cohomology class on a manifold as the class of a pseudomanifold, that is, a closed piecewise linear subspace that is a manifold outside a closed subset of real codimension at least 2.

LEMMA 3.3. *Let X be a complex manifold, and let u be an element of $H^r(X, \mathbf{F}_2)$ for some r. Consider u as the class of a closed pseudomanifold Z in X. Then the boundary in $H^{2r-1}(E_X, \mathbf{F}_2)$ of the class $[S^2 Z - Z]$ in $H^{2r}(S^2 X - X, \mathbf{F}_2)$ is*

$$\begin{cases} e^{a-1}\operatorname{Sq}^1 u + e^{a-2}\operatorname{Sq}^3 u + \cdots + \operatorname{Sq}^{2a-1} u & \text{if } r = 2a, \\ e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots + \operatorname{Sq}^{2a} u & \text{if } r = 2a+1. \end{cases}$$

Also, the boundary in $H^{2r}(E_X, \mathbf{F}_2)$ of the product $b[S^2 Z - Z]$ in $H^{2r+1}(S^2 X - X, \mathbf{F}_2)$ is

$$\begin{cases} e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots + \operatorname{Sq}^{2a} u & \text{if } r = 2a, \\ e^a \operatorname{Sq}^1 u + e^{a-1}\operatorname{Sq}^3 u + \cdots + \operatorname{Sq}^{2a+1} u & \text{if } r = 2a+1. \end{cases}$$

Proof. By Thom, the $\mathbf{F}_2$-homology of any space X is generated by classes of closed (unoriented) C^∞ manifolds Z with continuous maps $Z \to X$ [18, Théorème III.2]. When X is a manifold, Thom also showed that $H_*(X, \mathbf{F}_2)$ is not always generated by submanifolds; that is, we cannot always take the maps $Z \to X$ to be embeddings [18, page 46]. For a locally compact space X, Thom's argument shows that the Borel–Moore homology of X with $\mathbf{F}_2$ coefficients is generated by C^∞ manifolds Z with proper maps $Z \to X$. (The Borel–Moore homology of a 'reasonable' locally compact space such as a manifold is isomorphic to the singular homology with locally finite chains. For a survey, see Fulton [8, Section 19.1].)

Let X be a complex manifold of complex dimension n. By Thom's theorem, it suffices to prove the lemma for the class u in $H^r(X, \mathbf{F}_2)$ of a C^∞ manifold Z of real dimension $2n - r$ with a proper map $Z \to X$. The idea is that for N large enough, the composition $Z \to X \hookrightarrow X \times \mathbf{P}^N$ associated to a point in complex projective space $\mathbf{P}^N$ can be approximated by a proper C^∞ embedding, by Whitney. (Namely, it suffices that $\dim_{\mathbf{R}}(X \times \mathbf{P}^N) \geqslant 2\dim_{\mathbf{R}}(Z) + 1$.) Perturbing Z in this way does not change the class of $S^2 Z - Z$ in $H^*(S^2(X \times \mathbf{P}^N) - X \times \mathbf{P}^N, \mathbf{F}_2)$.

Let v be the generator of $H^2(\mathbf{P}^N, \mathbf{F}_2)$; then v^N is the class of a point on $\mathbf{P}^N$, and so the class of Z on $X \times \mathbf{P}^N$ is uv^N. Lemmas 3.1 and 3.2 compute the boundary of the classes $[S^2 Z - Z]$ and $b[S^2 Z - Z]$ in $H^*(E_{X\times \mathbf{P}^N}, \mathbf{F}_2)$, whether r is even or odd. For example, suppose $r = 2a$ and look at the boundary of $b[S^2 Z - Z]$; the argument is completely analogous in the other three cases. The boundary of $b[S^2 Z - Z]$ in $H^{4a+4N}(E_{X\times \mathbf{P}^N}, \mathbf{F}_2)$ is $e^{a+N} uv^N + e^{a+N-1}\operatorname{Sq}^2(uv^N) + \cdots$. Since v^N is in the top-degree cohomology group $H^{2N}(\mathbf{P}^N, \mathbf{F}_2)$, we have $\operatorname{Sq}^j(v^N) = 0$ for all $j > 0$. By the Cartan formula $\operatorname{Sq}^i(xy) = \sum_{j=0}^{i} \operatorname{Sq}^j(x)\operatorname{Sq}^{i-j}(y)$ [9, Section 4.L], the boundary of $b[S^2 Z - Z]$ in $H^{4a+4N}(E_{X\times \mathbf{P}^N}, \mathbf{F}_2)$ can be rewritten as $e^{a+N} uv^N + e^{a+N-1}\operatorname{Sq}^2(u)v^N + \cdots$.

We want to compute the boundary of $[S^2 Z - Z]$ in $H^{4a}(E_X, \mathbf{F}_2)$. Clearly this element pushes forward to the boundary of $[S^2 Z - Z]$ in $H^{4a+4N}(E_{X\times \mathbf{P}^N}, \mathbf{F}_2)$. Since E_X is the complex projective bundle $P(T^*X)$ and $E_{X\times \mathbf{P}^N}$ is $P(T^*(X \times \mathbf{P}^N))$, it is straightforward to check that this pushforward homomorphism is injective. So to show that the boundary of $b[S^2 Z - Z]$ in $H^{4a}(E_X, \mathbf{F}_2)$ is $e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots$ as we want, it suffices to show that the latter element pushes forward to $e^{a+N} uv^N + e^{a+N-1}\operatorname{Sq}^2(u)v^N + \cdots$.

We can factor the inclusion we are considering as $P(T^*X) \hookrightarrow P(T^*(X \times \mathbf{P}^N))|_X \hookrightarrow P(T^*(X \times \mathbf{P}^N))$. Here $p\colon P(T^*X) \to P(T^*(X \times \mathbf{P}^N))|_X$ is the zero set of a transverse section of the complex vector bundle $O(1) \otimes N_{X/X\times \mathbf{P}^N} = O(1)^{\oplus N}$ over $P(T^*(X \times \mathbf{P}^N))|_X$. So $p_* 1$ is the top Chern class $c_N(O(1)^{\oplus N}) = e^N$. So

$$\begin{aligned} p_*(e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots) &= (p_*1)(e^a u + e^{a-1}\operatorname{Sq}^2 u + \cdots) \\ &= e^{a+N} u + e^{a+N-1}\operatorname{Sq}^2 u + \cdots . \end{aligned}$$

Next, we push this class forward by $q\colon P(T^*(X \times \mathbf{P}^N))|_X \hookrightarrow P(T^*(X \times \mathbf{P}^N))$. Here $q_*1 = v^N$. So

$$\begin{aligned} q_*(e^{a+N} u + e^{a+N-1}\operatorname{Sq}^2 u + \cdots) &= (q_*1)(e^{a+N} u + e^{a+N-1}\operatorname{Sq}^2 u + \cdots) \\ &= e^{a+N} uv^N + e^{a+N-1}\operatorname{Sq}^2(u)v^N + \cdots . \end{aligned}$$

By the previous paragraph, this proves that the formulas we want hold in $H^*(E_X, \mathbf{F}_2)$. □

4. Cohomology of the configuration space, and proof of Theorem 2.1

Milgram, Löffler, Bödigheimer, Cohen, and Taylor computed the $\mathbf{F}_2$-homology of the configuration space $B(X, a)$ of subsets of X of order a in terms of the $\mathbf{F}_2$-homology of X and the dimension of X, for any compact manifold X (possibly with boundary) and any natural number a [1, 14]. Since we need explicit generators for the cohomology of $B(X, 2) = S^2X - X$, we compute this cohomology directly for X a closed manifold in Theorem 4.2, not relying on their work. It would be interesting to compute the ring structure on the $\mathbf{F}_2$-cohomology of the configuration spaces $B(X, a)$ for manifolds X.

As a tool, we use the calculation of the homology of symmetric products by Nakaoka and Milgram [13], as follows. We need a statement (unlike Theorem 1.2) that does not require X to have torsion-free integral cohomology. Let $f : X \times X \to S^2X$ be the obvious map.

THEOREM 4.1. *Let X be the complement of a closed subcomplex in a finite CW-complex. Let $u_0, \ldots, u_s$ be a basis for $H_*^{BM}(X, \mathbf{F}_2)$ over $\mathbf{F}_2$. Then $H_*^{BM}(S^2X, \mathbf{F}_2)$ has a basis consisting of the element $f_*(u_i \otimes u_j)$ in degree $|u_i| + |u_j|$ for each $i < j$, one element in each degree*

$$|u_i| + 2, |u_i| + 3, \ldots, 2|u_i|$$

for each i with $|u_i| > 0$, and one element in degree 0 for each i with $|u_i| = 0$.

Proof. First suppose that X is a finite CW-complex, so that Borel–Moore homology coincides with homology in the usual sense. Dold showed that the $\mathbf{F}_p$-homology of a symmetric product S^aX (as a graded vector space) is determined by the $\mathbf{F}_p$-homology of X [6, Theorem 7.2]. So to compute the $\mathbf{F}_2$-homology of S^2X, it suffices to compute this when X is a wedge (one-point union) of spheres. That easily reduces to the case of a single sphere X, where the calculation of $H_*(S^2X, \mathbf{F}_2)$ was made by Nakaoka. For any finite CW-complex X, the identification of some of the generators as pushforwards $f_*(u_i \otimes u_j)$ is part of Milgram's calculation of the 'Pontrjagin product' on symmetric products, the action on homology of the natural maps $S^aX \times S^bX \to S^{a+b}X$ [13, Theorem 5.2].

More generally, let $X = Y - Z$ for some finite CW-complex Y and closed subcomplex Z. Then X is the complement of a point p_0 in the quotient space Y/Z, which is a finite CW-complex. So $S^2X = S^2(Y/Z) - (Y/Z)$, where the inclusion $Y/Z \to S^2(Y/Z)$ is the map $x \mapsto x + p_0$. Steenrod showed that this inclusion induces an injection on $\mathbf{F}_2$-homology [6, Theorem 2]. So the exact sequence

$$H_j(Y/Z) \to H_jS^2(Y/Z) \to H_j^{BM}S^2X \to H_{j-1}(Y/Z)$$

with $\mathbf{F}_2$ coefficients determines the Borel–Moore homology of S^2X with $\mathbf{F}_2$ coefficients from the results of Nakaoka and Milgram. □

Using that, we give an explicit basis for the cohomology of $S^2X - X$.

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THEOREM 4.2. *Let X be a C^∞ manifold of real dimension m. Assume that X is homeomorphic to the complement of a closed subcomplex in a finite CW-complex; this is automatic for X compact. Let $z_0, \ldots, z_s$ be a basis for $H^*(X, \mathbf{F}_2)$, and let Z_i be a closed pseudomanifold in X that represents the class z_i. Let b in $H^1(S^2X - X, \mathbf{F}_2)$ be the class of the double cover $g\colon X \times X - X \to S^2X - X$. Then a basis for $H^*(S^2X - X, \mathbf{F}_2)$ is given by the elements $g_*(z_i \otimes z_j)$ in degree $|z_i| + |z_j|$ for $i < j$, together with the elements $b^j[S^2Z_i - Z_i]$ in degree $2|z_i| + j$ for all i and all $0 \leqslant j \leqslant m - 1 - |z_i|$.*

Proof. We compute the $\mathbf{F}_2$-Betti numbers of $S^2X - X$ by reducing to the better-understood homology of symmetric products. Namely, we have an exact sequence of Borel–Moore homology groups with $\mathbf{F}_2$ coefficients:

$$\to H_i^{BM}X \to H_i^{BM}S^2X \to H_i^{BM}(S^2X - X) \to H_{i-1}^{BM}X \to .$$

By Poincaré duality, $H_i^{BM}(S^2X - X, \mathbf{F}_2) \cong H^{2m-i}(S^2X - X, \mathbf{F}_2)$; so the $\mathbf{F}_2$-Betti numbers of $S^2X - X$ are determined by the pushforward homomorphism associated to the diagonal inclusion $\Delta\colon X \to S^2X$.

In fact, this homomorphism is zero in positive degrees. To see this, note that it suffices to prove this for X a finite CW-complex, by the proof of Theorem 4.1. We can also assume that X is connected. Fix a base point p_0 in X. This determines a sequence of inclusions

$$X \to S^2X \to S^3X \to \cdots$$

given by adding the point p_0. (Do not confuse this map $X \to S^2X$, $x \mapsto x + p_0$, with the diagonal inclusion, $x \mapsto 2x$.) The colimit of this sequence is called the *infinite symmetric product* $S^\infty X$. It can be viewed as a topological commutative monoid, with the homotopy type of the product of Eilenberg–MacLane spaces $\prod_{j>0} K(H_j(X, \mathbf{Z}), j)$, by Dold and Thom [9, Section 4.K]. This product decomposition is compatible with the addition on $S^\infty X$, up to homotopy. Moreover, by Steenrod, all the maps $X \to S^2X \to \cdots \to S^\infty X$ given by adding p_0 give injections on $\mathbf{F}_2$-homology [6, Theorem 2].

The Dold–Thom theorem implies that $R := H^*(S^\infty X, \mathbf{F}_2)$ is a primitively generated Hopf algebra, with generators given by applying Steenrod operations to generators of $H^j(K(H_j(X, \mathbf{Z}), j), \mathbf{F}_2)$ [12, Section 4.4, Theorem 6.19]. The multiplication by 2 map on $S^\infty X$ is the composition of the diagonal $S^\infty X \to S^\infty X \times S^\infty X$ with the addition $S^\infty X \times S^\infty X \to S^\infty X$, and so the corresponding pullback homomorphism on R is the composition of the coproduct $R \to R \otimes R$ with the product $R \otimes R \to R$. Pulling back the multiplication by 2 map sends a primitive class x in R to zero (as $x \mapsto 1 \otimes x + x \otimes 1 \mapsto 2x = 0$).

Since R is primitively generated, the multiplication by 2 map induces zero on R in positive degrees. Equivalently, the multiplication by 2 map induces zero on the $\mathbf{F}_2$-homology of $S^\infty X$ in positive degrees. By the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d] & S^2X \arrow[d] \\ S^\infty X \arrow[r, "2"] & S^\infty X, \end{tikzcd}$$

the composition of the diagonal map $\Delta\colon X \to S^2X$ with the inclusion $S^2X \to S^\infty X$ induces zero on $\mathbf{F}_2$-homology in positive degrees. By Steenrod's theorem (above), it follows that Δ induces zero on $\mathbf{F}_2$-homology in positive degrees, as we want.

We return to considering a C^∞ manifold X, possibly noncompact. We know the Borel–Moore homology of S^2X with $\mathbf{F}_2$ coefficients from Theorem 4.1. Together with the previous paragraph, this determines the cohomology of $S^2X - X$ with $\mathbf{F}_2$ coefficients, by the exact sequence

$$H_i^{BM}X \to H_i^{BM}S^2X \to H^{2m-i}(S^2X - X) \to H_{i-1}^{BM}X.$$

Namely, given a basis $z_0, \ldots, z_s$ for $H^*(X, \mathbf{F}_2)$, $H^*(S^2X - X, \mathbf{F}_2)$ has a basis with one element in degree $|z_i| + |z_j|$ for all $i < j$ and one element in each degree

$$2|z_i|, 2|z_i| + 1, \ldots, |z_i| + m - 1$$

for each i.

We want to show that a basis for $H^*(S^2X - X, \mathbf{F}_2)$ is given by the classes $g_*(z_i \otimes z_j)$ for $i < j$ together with the elements $b^j[S^2Z_i - Z_i]$ for all i and all $0 \leqslant j \leqslant m - 1 - |z_i|$. Since we know the dimension of $H^*(S^2X - X, \mathbf{F}_2)$, it suffices to show that these elements are linearly independent over $\mathbf{F}_2$.

To see this, think of $S^2X - X$ as the interior of a manifold with boundary, where the boundary is the real projective bundle $P_{\mathbf{R}}(T^*X)$. This gives a restriction homomorphism

$$H^*(S^2X - X, \mathbf{F}_2) \to H^*(P_{\mathbf{R}}(T^*X), \mathbf{F}_2).$$

I claim that the elements $g_*(z_i \otimes z_j)$ restrict to zero on $P_{\mathbf{R}}(T^*X)$. For this, think of $X \times X - X$ as the interior of a manifold with boundary, where the boundary is the unit sphere bundle $S_{\mathbf{R}}(TX)$ inside TX. Because the cohomology class $z_i \otimes z_j$ on $X \times X - X$ extends to $X \times X$, the restriction of $z_i \otimes z_j$ to $S_{\mathbf{R}}(TX)$ extends to the unit disc bundle $D_{\mathbf{R}}(TX)$, which is homotopy equivalent to X. Clearly this restriction of $z_i \otimes z_j$ to $H^*D_{\mathbf{R}}(TX) \cong H^*X$ is $z_iz_j \in H^*X$; so the restriction of

$z_i \otimes z_j$ to $H^* S_{\mathbf{R}}(TX)$ is the pullback of $z_i z_j$. So the pushforward by the double cover $g \colon S_{\mathbf{R}}(TX) \to P_{\mathbf{R}}(T^*X)$ is $g_* g^*(z_i z_j) = (g_* 1) z_i z_j = 0$, where $g_* 1 = 0 \in H^0(P_{\mathbf{R}}(T^*X), \mathbf{F}_2)$ since g has degree $0 \pmod 2$. That is, the classes $g_*(z_i \otimes z_j)$ restrict to zero on $P_{\mathbf{R}}(T^*X)$.

Next, let u be the class in $H^r(X, \mathbf{F}_2)$ of a closed pseudomanifold Z in X. The restriction of b in $H^1(S^2X - X, \mathbf{F}_2)$ to $P_{\mathbf{R}}(T^*X)$ is the Stiefel–Whitney class $b = w_1(O(1)_{\mathbf{R}})$. By the proof of Lemma 3.1, the restriction of $[S^2Z - Z]$ from $H^{2r}(S^2X - X, \mathbf{F}_2)$ to $H^{2r}(P_{\mathbf{R}}(T^*X), \mathbf{F}_2)$ is $b^r u + b^{r-1} \operatorname{Sq}^1 u + \cdots$. (To be precise, Lemma 3.1 proves this when Z is a closed C^∞ submanifold of X, but the proof of Lemma 3.3, replacing X by a product $X \times \mathbf{P}^N_{\mathbf{C}}$ (or $X \times S^N$) for N large, extends this to arbitrary cohomology classes u.) So the element $b^j[S^2Z - Z]$ for $0 \leqslant j \leqslant m - 1 - r$ restricts to $b^j(b^r u + b^{r-1} \operatorname{Sq}^1 u + \cdots)$ on $P_{\mathbf{R}}(T^*X)$. Since $H^*(P_{\mathbf{R}}(T^*X), \mathbf{F}_2)$ is a free module over $H^*(X, \mathbf{F}_2)$ with basis $1, b, \ldots, b^{m-1}$, we read off that the elements $b^j(S^2Z_i - Z_i)$ for $z_i = [Z_i]$ running through a basis for $H^*(X, \mathbf{F}_2)$ and $0 \leqslant j \leqslant m - 1 - |z_i|$ have linearly independent restrictions to $P_{\mathbf{R}}(T^*X)$.

By the previous two paragraphs, to show that the given elements are linearly independent in $H^*(S^2X - X, \mathbf{F}_2)$ and hence form a basis, it suffices to show that the elements $g_*(z_i \otimes z_j)$ for $i < j$ are linearly independent in $H^*(S^2X - X, \mathbf{F}_2)$. But this is clear from the exact sequence with $\mathbf{F}_2$ coefficients:

$$H_a^{BM} X \to H_a^{BM} S^2X \to H^{2m-a}(S^2X - X) \to H_{a-1}^{BM} X.$$

Indeed, the elements $g_*(z_i \otimes z_j)$ in the cohomology of $S^2X - X$ are the restrictions of the Borel–Moore homology classes $f_*(z_i \otimes z_j)$ on S^2X, where $f \colon X \times X \to S^2X$ is the obvious map. These classes in $H_*^{BM} S^2X$ are linearly independent for $i < j$ by Theorem 4.1. Since the diagonal homomorphism $H_a^{BM} X \to H_a^{BM} S^2X$ is zero in positive degrees, the elements $g_*(z_i \otimes z_j)$ for $i < j$ are linearly independent in $H^*(S^2X - X, \mathbf{F}_2)$. Theorem 4.2 is proved. □

Proof of Theorem 2.1. By the exact sequence of $\mathbf{F}_2$-cohomology groups

$$H^{j+1} X^{[2]} \to H^{j+1}(S^2X - X) \to H^j E_X \to H^{j+2} X^{[2]},$$

the kernel of the pushforward $i_* \colon H^* E_X \to H^* X^{[2]}$ is equal to the image of the boundary homomorphism from $H^*(S^2X - X, \mathbf{F}_2)$. Theorem 4.2 gives a basis for $H^*(S^2X - X, \mathbf{F}_2)$, and Lemma 3.3 computes the boundary of the classes $[S^2Z - Z]$ and $b[S^2Z - Z]$, for a pseudomanifold Z in X. That determines the boundary of all classes $b^j[S^2Z - Z]$, since b^2 in $H^2(S^2X - X, \mathbf{F}_2)$ is the pullback of the class e in $H^2(X^{[2]}, \mathbf{F}_2)$. This gives the elements of $\ker(i_*)$ listed in Theorem 2.1.

It remains to show that the boundary of each remaining basis element $g_*(v_i \otimes v_j)$ for $H^*(S^2X - X, \mathbf{F}_2)$ (where $i < j$) is zero. By the exact sequence, it suffices to show that these classes are restrictions of cohomology classes on $X^{[2]}$. To see this, let $g\colon \widetilde{X \times X} \to X^{[2]}$ be the obvious degree-2 map. Since g is a proper map of manifolds, it induces a pushforward homomorphism on cohomology. Consider each cohomology class $v_i \otimes v_j$ in $H^*(X \times X)$ as a class on $\widetilde{X \times X}$ by pulling back. Then the class $g_*(v_i \otimes v_j)$ on $S^2X - X$ is the restriction of the cohomology class $g_*(v_i \otimes v_j)$ on $X^{[2]}$. Theorem 2.1 is proved. $\square$

5. Torsion-free cohomology

Proof of Theorem 2.2. The Adem relations among Steenrod operations imply that $\mathrm{Sq}^1\,\mathrm{Sq}^{2j} = \mathrm{Sq}^{2j+1}$ on the $\mathbf{F}_2$-cohomology of any space [9, Section 4.L]. Here Sq^1 is the Bockstein on $\mathbf{F}_2$-cohomology. Since we assume that $H^*(X, \mathbf{Z})$ has no 2-torsion, we have $\mathrm{Sq}^1 = 0$ on $H^*(X, \mathbf{F}_2)$, and hence all odd Steenrod operations are zero.

As a result, Theorem 2.1 gives that the kernel of the pushforward homomorphism $i_*\colon H^*(E_X, \mathbf{F}_2) \to H^*(X^{[2]}, \mathbf{F}_2)$ is spanned by the elements

$$\begin{aligned} &e^j(e^a u + e^{a-1}\,\mathrm{Sq}^2\,u + \cdots + \mathrm{Sq}^{2a}\,u) \quad \text{for } |u| = 2a, 0 \leqslant j \leqslant n-1-a, \quad \text{and}\\ &e^j(e^a u + e^{a-1}\,\mathrm{Sq}^2\,u + \cdots + \mathrm{Sq}^{2a}\,u) \quad \text{for } |u| = 2a+1, 0 \leqslant j \leqslant n-1-a, \end{aligned}$$

where u runs through a basis for $H^*(X, \mathbf{F}_2)$. Since $H^*(E_X, \mathbf{F}_2)$ is a free module over $H^*(X, \mathbf{F}_2)$ with basis $1, e, \ldots, e^{n-1}$, the elements listed are linearly independent over $\mathbf{F}_2$.

Thus, we have a basis for $\ker(i_*)$. By the exact sequence with $\mathbf{F}_2$ coefficients

$$H^{j+1}X^{[2]} \to H^{j+1}(S^2X - X) \to H^j E_X \to H^{j+2}X^{[2]},$$

we now know the $\mathbf{F}_2$-Betti numbers of $X^{[2]}$. Namely, let $v_0, \ldots, v_s$ be a basis for $H^*(X, \mathbf{F}_2)$. Then $H^*(X^{[2]}, \mathbf{F}_2)$ has a basis with one element in degree $|v_i| + |v_j|$ for each $i \leqslant j$ except when $i = j$ and $|v_i|$ is odd, together with one element in each degree

$$|v_i| + 2, |v_i| + 4, \ldots, |v_i| + 2n - 2$$

for each i.

Since $H^*(X, \mathbf{Z})$ has no 2-torsion, the rational cohomology of X has a basis $v_0, \ldots, v_s$ in the same degrees. To show that $H^*(X^{[2]}, \mathbf{Z})$ has no 2-torsion, it suffices to show that the rational cohomology of $X^{[2]}$ has a basis in the same degrees as the basis above for $H^*(X^{[2]}, \mathbf{F}_2)$. Since the Hilbert scheme $X^{[2]}$ is the quotient by the symmetric group S_2 of the blow-up $\widetilde{X \times X}$ along the diagonal, $H^*(X^{[2]}, \mathbf{Q})$ is the subspace of S_2-invariants in $H^*(\widetilde{X \times X}, \mathbf{Q})$. Since $\widetilde{X \times X}$ is the blow-up of the complex manifold $X \times X$ along the closed complex submanifold X of codimension n, we have

$$H^*(\widetilde{X \times X}) = H^*(X \times X) \oplus E \cdot H^*X \oplus \cdots \oplus E^{n-1} \cdot H^*X,$$

where E denotes the class of the exceptional divisor E_X in $H^2(\widetilde{X \times X})$ [20, Theorem 7.31]. Since the nontrivial element of S_2 acts on $H^*(X \times X, \mathbf{Q})$ by $v_i \otimes v_j \mapsto (-1)^{|v_i||v_j|} v_j \otimes v_i$, the S_2-invariants in $H^*(X \times X, \mathbf{Q})$ have a basis with one element in each degree $|v_i| + |v_j|$ for each $i \leqslant j$ except when $i = j$ and $|v_i|$ is odd. The other summands $E^j \cdot H^*(X, \mathbf{Q})$ of $H^*(\widetilde{X \times X}, \mathbf{Q})$ are fixed by S_2. We conclude that $H^*(X^{[2]}, \mathbf{Q}) = H^*(\widetilde{X \times X}, \mathbf{Q})^{S_2}$ has a basis in the same degrees as the basis above for $H^*(X^{[2]}, \mathbf{F}_2)$. So the integral cohomology of $X^{[2]}$ has no 2-torsion.

Finally, suppose that $H^*(X, \mathbf{Z})$ has no torsion. The easy computation of the rational cohomology of $X^{[2]}$ above works with $\mathbf{Z}[1/2]$-coefficients. In particular, the integral cohomology of $X^{[2]}$ has no odd torsion. By the previous paragraph, it follows that the integral cohomology of $X^{[2]}$ is torsion-free. Theorem 2.2 is proved. □

Proof of Corollary 2.3. We first construct the classes x_j and y_j in the $\mathbf{Z}$-cohomology of the Hilbert scheme $X^{[2]}$.

Suppose that $|z_j|$ is odd, $|z_j| = 2a + 1$. Then $S^2 Z_j - Z_j$ is typically not orientable, because the action of $\mathbf{Z}/2$ on $Z_j \times Z_j$ does not preserve orientation. In this case, the Bockstein $\beta[S^2 Z_j - Z_j]$ is an integral cohomology class on $S^2 X - X$ killed by 2. Consider the exact sequence of $\mathbf{Z}$-cohomology groups

$$H^{l+1} X^{[2]} \to H^{l+1}(S^2 X - X) \to H^l E_X.$$

Because the integral cohomology of E_X has no 2-torsion, the boundary of $\beta[S^2 Z_j - Z_j]$ must be zero in the integral cohomology of E_X. So there is an element y_j in $H^{4a+3}(X^{[2]}, \mathbf{Z})$ that restricts to $\beta[S^2 Z_j - Z_j]$ on $S^2 X - X$, as we want. In the $\mathbf{F}_2$-cohomology of $S^2 X - X$, y_j restricts to $b[S^2 Z_j - Z_j]$, by Thom's formula (see Section 3) applied to the smooth locus of $S^2 Z_j - Z_j$. This uses that the orientation class w_1 on $S^2 Z_j - Z_j$ (corresponding to the double cover $Z_j \times Z_j - Z_j \to S^2 Z_j - Z_j$) is the restriction of $b \in H^1(S^2 X - X, \mathbf{F}_2)$.

Suppose that $|z_j|$ is even, $|z_j| = 2a$. Then $[S^2 Z_j - Z_j]$ is a $\mathbf{Z}$-cohomology class on $S^2 X - X$, and $2[S^2 Z_j - Z_j]$ is the restriction of the class $g_*(z_j \otimes z_j)$ on $X^{[2]}$. It follows that the boundary of $2[S^2 Z_j - Z_j]$ is zero in the integral cohomology of E_X. Because the cohomology of E_X has no 2-torsion, it follows that the boundary of $[S^2 Z_j - Z_j]$ is also zero. So there is an element x_j in $H^{4a}(X^{[2]}, \mathbf{Z})$ that restricts to $[S^2 Z_j - Z_j]$ on $S^2 X - X$, as we want.

Since $H^*(X, \mathbf{Z}_{(2)})$ is torsion-free, we know by Theorem 2.2 that $H^*(X^{[2]}, \mathbf{Z}_{(2)})$ is torsion-free. Being finitely generated, it is in fact a free $\mathbf{Z}_{(2)}$-module. Therefore, in order to show that the elements listed in Corollary 2.3 form a basis for $H^*(X^{[2]}, \mathbf{Z}_{(2)})$, it suffices to show that they form a basis for $H^*(X^{[2]}, \mathbf{F}_2)$ as an $\mathbf{F}_2$-vector space.

This follows by going through the proof of Theorem 2.2. Namely, we have computed the boundary map in the exact sequence of $\mathbf{F}_2$-cohomology groups

$$H^{l+1}X^{[2]} \to H^{l+1}(S^2X - X) \to H^l E_X \to H^{l+2}X^{[2]},$$

as well as a basis for the kernel of the pushforward $i_*\colon H^*E_X \to H^*X^{[2]}$. As a result, we can write down a basis for the image of i_*, as listed in Corollary 2.3. We also know, using Lemma 3.3 and Theorem 4.2, a basis for the kernel of the boundary map on $H^*(S^2X - X)$: the elements $g_*(z_j \otimes z_k)$ for $j < k$, and the elements $b^m[S^2Z_j - Z_j]$ for $0 \leqslant m \leqslant 2n - 2 - |z_j|$ with $m \equiv |z_j| \pmod 2$. Equivalently, this is a basis for $H^*X^{[2]}$ modulo the image of i_*.

Using that the element e in $H^2X^{[2]}$ restricts to b^2 in $H^2(S^2X - X)$, we read off that the elements listed in Corollary 2.3 form a basis for the $\mathbf{F}_2$-cohomology of $X^{[2]}$. As mentioned above, it follows that the corresponding classes in $\mathbf{Z}_{(2)}$-cohomology also form a basis. Corollary 2.3 is proved. □

6. Complex submanifolds

In a special case, the formulas in Theorems 2.1 and 2.2 have a simple geometric explanation, and that is what led to guessing those formulas in general. Namely, let Y be a closed complex submanifold of codimension a in a complex manifold X, and let u be the cohomology class of Y in $H^{2a}(X, \mathbf{F}_2)$. The Hilbert scheme $Y^{[2]}$ is a closed complex submanifold of codimension $2a$ in $X^{[2]}$. As throughout the paper, we omit the symbol π^* for cohomology classes on X pulled back to E_X.

LEMMA 6.1. *The restriction of the cohomology class of $Y^{[2]}$ in $H^{4a}(X^{[2]}, \mathbf{F}_2)$ to E_X is*

$$e^a u + e^{a-1}\,\mathrm{Sq}^2 u + \cdots + \mathrm{Sq}^{2a} u.$$

Proof. We have an exact sequence of holomorphic vector bundles on Y, $0 \to TY \to TX|_Y \to N_{Y/X} \to 0$. The exceptional divisor E_X is the complex projective bundle $\pi\colon P(T^*X) \to X$ of lines in TX. (Following our conventions for projective bundles from Section 1, the natural line subbundle in the vector bundle π^*TX is called $O(-1)$.) We also write π for the projection $P(T^*X)|_Y \to Y$. The intersection $Y^{[2]} \cap E_X$, which is transverse, is $W := P(T^*Y) \subset P(T^*X)|_Y \subset P(T^*X) = E_X$. The submanifold W is the zero set of a transverse section of the vector bundle $\mathrm{Hom}(O(-1), \pi^* N_{Y/X})$ over $P(T^*X)|_Y$; that section is the one associated to the subbundle $O(-1) \subset \pi^*(TX|_Y)$.

So the cohomology class of W on $P(T^*X)|_Y$ is the top Chern class $c_a(O(1) \otimes N_{Y/X})$. The top Chern class of the tensor product of a line bundle L with a vector bundle F of rank a is

$$c_a(L \otimes F) = (c_1 L)^a + (c_1 L)^{a-1} c_1 F + \cdots + c_a F.$$

The class e on $X^{[2]}$ restricted to E_X is $e = c_1 O(-1)$. So the cohomology class of W on $P(T^*X)|_Y$ in $\mathbf{F}_2$-cohomology is $e^a + e^{a-1} c_1 N_{Y/X} + \cdots + c_a N_{Y/X}$.

The Steenrod squares of the class $u = [Y]$ in $H^*(X, \mathbf{F}_2)$ are the pushforward to X of the Stiefel–Whitney classes of the normal bundle $N_{Y/X}$ by the inclusion $s : Y \to X$,

$$\mathrm{Sq}^j u = s_* w_j(N_{Y/X}),$$

by Thom [17]. Since $N_{Y/X}$ is a complex vector bundle, the odd Stiefel–Whitney classes are zero and the even Stiefel–Whitney classes are the Chern classes in $\mathbf{F}_2$-cohomology:

$$w_{2i} N_{Y/X} = c_i N_{Y/X} \pmod 2$$

[15, Problem 14-B]. We conclude that the class of $W = Y^{[2]} \cap E_X$ in $H^*(E_X, \mathbf{F}_2)$ is $e^a u + e^{a-1} \mathrm{Sq}^2 u + \cdots + \mathrm{Sq}^{2a} u$. $\square$

To relate this to Theorems 2.1 and 2.2, note that the element e in $H^2(E_X, \mathbf{F}_2)$ is in the image of restriction i^* from $X^{[2]}$. So Lemma 6.1 implies that for the class u in $H^{2a}(X, \mathbf{F}_2)$ of a complex submanifold, the image of restriction i^* contains $e^j(e^a u + e^{a-1} \mathrm{Sq}^2 u + \cdots + \mathrm{Sq}^{2a} u)$ for all $j \geqslant 0$, in particular for all $0 \leqslant j \leqslant n - 1 - a$.

Moreover, the class $[E_X]$ in $H^2(X^{[2]}, \mathbf{Z})$ is equal to $2e$, and so $i_* 1 = [E_X] = 0$ in $H^2(X^{[2]}, \mathbf{F}_2)$. So $i_* i^* y = (i_* 1) y = 0$ for all y in $H^*(E_X, \mathbf{F}_2)$. Thus, Lemma 6.1 shows that the kernel of i_* contains $e^j(e^a u + e^{a-1} \mathrm{Sq}^2 u + \cdots + \mathrm{Sq}^{2a} u)$ for all classes u in $H^{2a}(X, \mathbf{F}_2)$ of complex submanifolds and all $0 \leqslant j \leqslant n-1-a$. This calculation suggested the complete description of the kernel of i_* in Theorems 2.1 and 2.2.

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