J. Austral. Math. Soc. (*Series A*) **29** (1980), 310–315

NOTE ON SUPPORT-CONCENTRATED BOREL MEASURES

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(Received 17 June; revised 30 October 1979)

Communicated by E. Strzelecki

Abstract

Every τ-smooth Borel measure is support-concentrated. We shall prove in this note that the converse of this statement is not true, in general. Furthermore, we shall give some conditions assuring that a support-concentrated Borel measure be τ-smooth.

1980 Mathematics subject classification (*Amer. Math. Soc.*): 28 C 15.

In this note X will always be a topological space. We denote by $\mathscr{G}(X)$, $\mathscr{F}(X)$, $\mathscr{B}(X)$ the collection of all open, closed, Borel subsets of X, respectively. By a *Borel measure* (on X) we understand a nonnegative, finite, countably additive set-function defined on $\mathscr{B}(X)$. Let $\mathscr{M}_+(X)$ denote the family of all Borel measures on X.

If $\mu \in \mathscr{M}_+(X)$ then the set

$$\begin{aligned}\operatorname{supp}\mu := X - \bigcup\{G \in \mathscr{G}(X) : \mu(G) = 0\} \\ = \{x \in X : \mu(G) > 0 \text{ for every open neighbourhood } G \text{ of } x\}\end{aligned}$$

is called the *support* of μ.

$\mu \in \mathscr{M}_+(X)$ is said to be

(i) *support-concentrated* if $\mu(\operatorname{supp}\mu) = \mu(X)$;

(ii) [*weakly*] τ-*smooth* if $\mu(\bigcup_\alpha G_\alpha) = \sup_\alpha \mu(G_\alpha)$ for every increasing net (G_α) in $\mathscr{G}(X)$ [with $\bigcup_\alpha G_\alpha = X$];

(iii) *regular* if $\mu(B) = \sup\{\mu(F) : F \in \mathscr{F}(X), F \subset B\}$ for all $B \in \mathscr{B}(X)$.

We remark that Okada (1979) uses the terminology 'the strong support of μ exists' for expressing that μ is support-concentrated. It is an immediate consequence of the definitions that every τ-smooth Borel measure is support-concentrated. However, the converse of this statement is, in general, not true, as the following example shows.

EXAMPLE 1. On the one hand, consider the compact Hausdorff space $X = \{0, 1\}^{\aleph_1}$. X is separable (Willard (1970), Theorem 16.4), but not Borel-complete (Hager and others (1972), Corollary 2.10). Thus, by Gardner (1975), Theorem 5.7, there exists a non-τ-smooth $\nu \in \mathscr{M}_+(X)$.

On the other hand, let X be the Novak space (Steen and Seebach (1978), Counterexample 112). X is a completely regular Hausdorff space which is separable and countably compact, but not realcompact. Thus, by Dykes (1970), Corollary 1.10, X is not α-realcompact and hence, by Gardner (1975), Theorem 3.5, there exists a regular, non-τ-smooth $\nu \in \mathscr{M}_+(X)$.

In either case, let $\{x_n\}$ be a countable dense subset of X and put

$$\mu := \nu + \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n},$$

where δ_{x_n} denotes the Dirac measure at x_n. Then μ is a non-τ-smooth Borel measure on X with $\operatorname{supp} \mu = X$. In the case of the Novak space μ is even regular.

Let ν be the Dieudonné measure on $[0, \Omega)$, Ω denoting the first uncountable ordinal. Then ν is a regular Borel measure which is not support-concentrated (cf. Okada (1979), Example 2.3). On the other hand, the following example shows that there also exist support-concentrated (even τ-smooth) Borel measures that are not regular. Thus support-concentration is a property of Borel measures being incomparable with regularity.

EXAMPLE 2. Let $X := [0, 1]$ and let Q be a subset of X such that $\lambda_*(Q) = 0$ and $\lambda^*(Q) = 1$ where λ denotes the Lebesgue measure (Halmos (1950), Theorem E, p. 70). Let X be equipped with the topology generated by Q and the usual topology τ_0 on X, that is $\mathscr{G}(X) = \{G_1 \cup (Q \cap G_2) : G_1, G_2 \in \tau_0\}$. Then X is a Hausdorff space being second countable but not regular. Furthermore we have

$$\mathscr{B}(X) = \{(B_1 \cap Q) \cup (B_2 - Q) : B_1, B_2\ \tau_0\text{-Borel sets}\}.$$

For any two τ_0-Borel sets B_1, B_2 put

$$\mu((B_1 \cap Q) \cup (B_2 - Q)) := \lambda(B_1).$$

By this definition an element $\mu \in \mathscr{M}_+(X)$ is unambiguously defined. Now $\mu(X - Q) = 0$ and

$$\begin{aligned} \inf\{\mu(G) : X - Q \subset G \in \mathscr{G}(X)\} &\geqslant \inf\{\mu(G_1) : X - Q \subset G_1 \in \tau_0\} \\ &= \inf\{\lambda(G_1) : X - Q \subset G_1 \in \tau_0\} = \lambda^*(X - Q) = 1, \end{aligned}$$

hence μ is not regular. However, μ is τ-smooth, since X is second countable.

REMARK. The preceding two examples answer two questions that have been raised in the Introduction of Okada (1979).

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Let $\mu \in \mathscr{M}_+(X)$ and $B_0 \in \mathscr{B}(X)$. Then the measure $\mu_{B_0} \in \mathscr{M}_+(X)$ is defined by $\mu_{B_0}(B) := \mu(B_0 \cap B)$ for $B \in \mathscr{B}(X)$. The following proposition shows that τ-smoothness can be characterized by means of support-concentration.

PROPOSITION 1. *For a measure $\mu \in \mathscr{M}_+(X)$ the following three conditions are equivalent:*

(1) *μ is τ-smooth.*

(2) *Every measure $\nu \in \mathscr{M}_+(X)$ being absolutely continuous with respect to μ is support-concentrated.*

(3) *$\mu_{G_2 - G_1}$ is support-concentrated for all $G_1, G_2 \in \mathscr{G}(X)$ with $G_1 \subset G_2$.*

PROOF. (1) $\rightarrow$ (2) As μ is τ-smooth, so is every $\nu \in \mathscr{M}_+(X)$ being absolutely continuous with respect to μ. Thus (2) is obvious.

(2) $\rightarrow$ (3) Trivial.

(3) $\rightarrow$ (1) Let (G_α) be an increasing net in $\mathscr{G}(X)$. Put $G := \bigcup_\alpha G_\alpha$ and $a := \sup_\alpha \mu(G_\alpha)$. Choose a sequence (α_n) such that $\lim_n \mu(G_{\alpha_n}) = a$. Then $G^* := \bigcup_n G_{\alpha_n} \subset G$ and $\mu(G^*) = a$. It is easy to see that $\mu_{G - G^*}(G_\alpha) = 0$ for all α. This implies $G \subset X - \operatorname{supp} \mu_{G - G^*}$, hence, by (3), $\mu_{G - G^*}(G) = 0$ and thus $\mu(G) = \mu(G^*) = a$.

X is said to be a *τ-space* (Adamski (1977), p. 99) if every $\mu \in \mathscr{M}_+(X)$ is τ-smooth. According to Gardner (1975), Theorem 5.1, the τ-spaces are identical with the HB-spaces introduced by Gardner.

The following result is a direct consequence of Proposition 1.

PROPOSITION 2. *X is a τ-space if and only if every Borel measure on X is support-concentrated.*

In the remaining part of this note we shall give some sufficient conditions that a support-concentrated Borel measure be τ-smooth. At first we consider regular Borel measures with a Lindelöf support.

PROPOSITION 3. *Let $\mu \in \mathscr{M}_+(X)$ be regular and assume that* supp μ *be Lindelöf. Then the following three conditions are equivalent:*

(1) *μ is τ-smooth.*

(2) *μ is support-concentrated.*

(3) *There exists a Lindelöf set $S \in \mathscr{B}(X)$ such that $\mu(S) = \mu(X)$.*

PROOF. (1) $\rightarrow$ (2) Obvious.

(2) $\rightarrow$ (3) Put $S := \operatorname{supp} \mu$.

(3) $\rightarrow$ (1) Let (G_α) be an increasing net in $\mathscr{G}(X)$ such that $\bigcup_\alpha G_\alpha = X$. As S is Lindelöf, we can find a sequence (α_n) such that $S \subset \bigcup_n G_{\alpha_n}$ This implies $\mu(X) = \mu(S)$

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$\leqslant \mu(\bigcup_n G_{\alpha_n}) \leqslant \sup_\alpha \mu(G_\alpha)$. Thus μ is weakly τ-smooth. Now (1) follows from Gardner (1975), Theorem 4.3.

Example 1 shows that neither the regularity of μ nor the assumption that $\operatorname{supp}\mu$ be Lindelöf can be omitted from Proposition 3. X is said to be an *SL-space* (Okada (1979), Definition 4.1) if $\operatorname{supp}\mu$ is Lindelöf for every $\mu \in \mathscr{M}_+(X)$. Furthermore, X is called a *Borel-regular space* (Okada and Okazaki (1978), p. 184) if every $\mu \in \mathscr{M}_+(X)$ is regular.

COROLLARY 1. *Let X be a Borel-regular SL-space. Then every support-concentrated Borel measure on X is τ-smooth.*

It follows from Choquet's capacity theorem (Meyer (1966), III, T 19) that X is a Borel-regular space if every $G \in \mathscr{G}(X)$ is an $\mathscr{F}(X)$-Souslin set (in particular, if every $G \in \mathscr{G}(X)$ is an F_σ-set). Furthermore, by Okada (1979), Theorem 4.2, every metacompact space is an SL-space. Thus, in particular, every metrizable space is a Borel-regular SL-space, and we obtain from Proposition 3:

PROPOSITION 4. *Let X be a metrizable space. For $\mu \in \mathscr{M}_+(X)$ the following three conditions are equivalent:*

(1) *μ is τ-smooth.*

(2) *μ is support-concentrated.*

(3) *There exists a separable set $S \in \mathscr{B}(X)$ such that $\mu(S) = \mu(X)$.*

We remark that condition (3) of Proposition 4 is Billingsley's definition of a τ-smooth (Billingsley uses the term '*separable*') Borel measure on a metrizable space (compare Billingsley (1968), p. 234).

For 0, 1-valued measures we have the following result.

PROPOSITION 5. *Every* 0, 1-*valued support-concentrated Borel measure is τ-smooth.*

PROOF. Let $\mu \in \mathscr{M}_+(X)$ be 0, 1-valued and assume that there is an increasing net (G_α) in $\mathscr{G}(X)$ such that $\sup_\alpha \mu(G_\alpha) < \mu(\bigcup_\alpha G_\alpha)$. This implies $\mu(\bigcup_\alpha G_\alpha) = 1$ and $\mu(G_\alpha) = 0$ for all α, hence $\bigcup_\alpha G_\alpha \subset X\text{-}\operatorname{supp}\mu$ and therefore $\mu(X\text{-}\operatorname{supp}\mu) = 1$. Thus μ is not support-concentrated.

Finally, we shall consider residual measures. A Borel measure is called a *residual measure*, if every nowhere dense Borel set has measure zero (or equivalently, if every Borel set of first category has measure zero). On the other hand, a Borel measure is called a *category measure* if the Borel sets of measure zero are exactly the Borel sets of first category.

PROPOSITION 6. *Every support-concentrated residual Borel measure is τ-smooth.*

PROOF. Let μ be a support-concentrated residual Borel measure on X and let $G_1, G_2 \in \mathscr{G}(X)$ with $G_1 \subset G_2$. According to Proposition 1 it suffices to show that $\nu := \mu_{G_2 - G_1}$ be support-concentrated.

Put $H_0 := X - \operatorname{supp} \nu = \bigcup \{G \in \mathscr{G}(X) : \mu(G \cap (G_2 - G_1)) = 0\}$, $H_1 := X - \operatorname{supp} \mu$ and $B := H_0 \cap (G_2 - G_1)$. It is easy to see that the following two inclusions are valid :

(*) $\operatorname{int} B \subset H_1$;

(**) $B - \operatorname{int} B \subset \partial(H_0 \cap G_1)$.

From (*) we obtain $\mu(\operatorname{int} B) = 0$, since μ is support-concentrated. From (**) we conclude $\mu(B - \operatorname{int} B) = 0$, since $\partial(H_0 \cap G_1)$ is nowhere dense and μ is residual. Thus we have $\mu(B) = 0$, that is $\nu(H_0) = 0$.

COROLLARY 2. *Every category Borel measure is τ-smooth.*

PROOF. Let $\mu \in \mathscr{M}_+(X)$ be a category measure. In view of Proposition 6 it suffices to prove that μ is support-concentrated. Since every open μ-null set is an open set of first category, Banach's category theorem (Oxtoby (1971), Satz 16.1) implies that $X - \operatorname{supp} \mu$ is a set of first category, too, and hence a μ-null set.

It follows from Proposition 6 respectively Corollary 2 that in the paper of Armstrong and Prikry (1978) both the equivalence of the assertions (a) and (b) within Proposition 1 and the statement of Corollary 1 are valid for arbitrary topological spaces.

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https://doi.org/10.1017/S1446788700021303 Published online by Cambridge University Press