1

Nicolas' $\pi(\mathbf{x}) < \mathrm{li}(\theta(\mathbf{x}))$ Equivalence

1.1 Introduction

To begin this introduction, we give a summary of results for two inequalities which are closely related to the inequality of Jean-Louis Nicolas, which is the subject of this chapter. Numerical evaluation up to modest values of x gives $\pi(x) < \mathrm{li}(x)$. It was thought by many in the early part of the twentieth century that this might always be the case. Given the prime number theorem (PNT) estimate

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right),$$

Nicolas' inequality would have provided a useful simplification. However, in 1914 Littlewood showed, using a method developed by Landau, that $\mathrm{li}(x) - \pi(x)$ changed sign infinitely often as $x \to \infty$ [116, chapter V]. Littlewood's research student Skewes set about finding the first number for which $\mathrm{li}(x) < \pi(x)$. In 1933, assuming RH, Skewes showed that such a number would not be greater than

$$10^{10^{10^{34}}}.$$

He continued to work on this problem and by 1955 had shown, unconditionally, that the number would need to be no greater than the astronomical

$$10^{10^{10^{964}}}.$$

Many number theorists were fascinated by this problem and progressively reduced the proved upper bound, or found an interval in which there was at least one zero crossing for $\mathrm{li}(x) - \pi(x)$. They included Lehman (1966), te Riele (1987), Bays and Hudson (2000), Chao and Plymen (2010), Saouter and Demichel (2014), Zegowitz (2010), and Stoll (2011).

For the initial interval of positivity, J. B. Rosser and L. Schoenfeld (1962) [206] showed that $\pi(x) < \mathrm{li}(x)$ continued to hold at least up until 10^8. R. Brent (1975) [24] improved this to 8×10^{10}, T. Kotnic (2008) [129] to 10^{14}, D. J. Platt and T. S. Trudgian (2016) [188] to 1.39×10^{17}, and J. Büthe (2017) [39] to 10^{19}. We note Littlewood's theorem of 1914 reveals there is an infinite number of crossings [116, theorem 35]. It takes the form

$$\mathrm{li}(x) - \pi(x) = \Omega_{\pm}\left(\frac{\sqrt{x}\,\mathrm{logloglog}\,x}{\log x}\right).$$

Michael Rubinstein and Peter Sarnak in 1994 [208] showed that the logarithmic density of positive integers for which $\mathrm{li}(x) < \pi(x)$ exists and is about 2.6×10^{-7} of all integers.

The difference $x - \theta(x)$ has a similar set of behaviours, although not as extensively studied as $\mathrm{li}(x) - \pi(x)$. The method of Landau, when applied to $x - \psi(x)$, because

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + O(x^{1/3+\epsilon}),$$

can be used to show $x - \theta(x)$ changes sign infinitely often as $x \to \infty$. Indeed, more precisely [116, theorem 33]

$$x - \theta(x) = \Omega_{\pm}\left(x^{1/2-\epsilon}\right).$$

Regarding the initial interval, Schoenfeld (1976) showed that $\theta(x) < x$ up to 10^{11}, Dusart (2010) to 8×10^{11}, and Platt and Trudgian in Theorem B.2 (2015) that there is an

$$x \in [e^{x_0-h}, e^{x_0+h}],\ x_0 = 727.951332655,\ h = 1.3 \times 10^{-8},$$

for which $x < \theta(x)$.

It came as a surprise to the author that the "irregularities of distribution" ([116, chapter V]) exhibited by the three functions $\pi(x)$, $\mathrm{li}(x)$ and $\theta(x)$ would give rise to an RH equivalence. Indeed, that the functions might conspire together to give an inequality closely related to $\theta(x) < x$ and $\pi(x) < \mathrm{li}(x)$, which was true on an unbounded interval if RH was true, but alternated between true and false infinitely if RH was false. This result was published by Jean-Louis Nicolas in 2017 [172] and has the statement

$$RH \iff \pi(x) < \mathrm{li}(\theta(x)),\ x \geq 11.$$

The proof is set out in this chapter as Theorem 1.17. Consistent with $\pi(x) < \mathrm{li}(x)$ and $\theta(x) < x$ the proof in the RH is false case, gives not just one counterexample but an infinite set x_n of counterexamples with $x_n \to \infty$. In the RH is true case $\mathrm{li}(\theta(x)) - \pi(x)$ is not only positive but has limit value infinity. This can be derived from a different equivalence of Nicolas, stated in an end note to the chapter.

To prove his result Nicolas defines the difference $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$ and splits it into two parts using the function $\Pi(x)$. The definitions follow:

$$\Pi(x) := \sum_{p^j \le x} \frac{1}{j} = \sum_{j=1}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \frac{\pi(x^{1/j})}{j},$$
$$A_1(x) := \mathrm{li}(\psi(x)) - \Pi(x),$$
$$A_2(x) := \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x),$$
$$A(x) := \mathrm{li}(\theta(x)) - \pi(x) = A_1(x) + A_2(x).$$

The intricate detailed relationships between the lemmas required to prove the theorem are described in Figure 1.1. Note the important role played by the imported results set out in Appendix B.

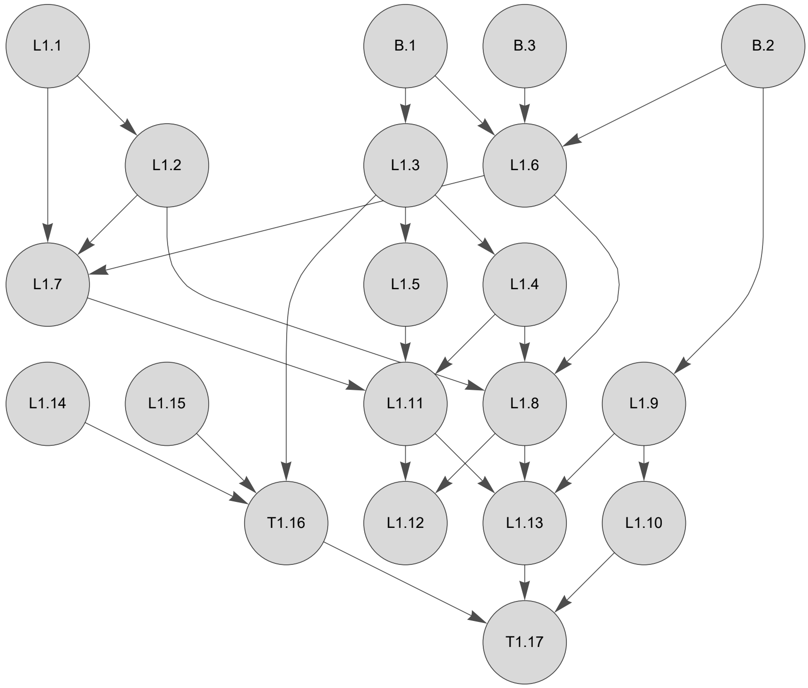

Figure 1.1 Dependencies for Theorem 1.17.

We don't develop the fascinating consequences of Nicolas' theorem, such as if we assume RH is true we get

$$\theta(x) < x \implies \pi(x) < \mathrm{li}(x).$$

Because of this, the first crossing point for x and $\theta(x)$, under RH, must come before that of $\pi(x)$ and $\mathrm{li}(x)$, and the reverse is true for the second one. Any density which exists for $\pi(x) - \mathrm{li}(x)$ must be no greater than that for $\theta(x) - x$.

In Section 1.2 we estimate $\mathrm{li}(x)$, in Section 1.3 the function $A_1(x)$, in Section 1.4 $A_2(x)$, and in Section 1.5 the function $A(x)$, all assuming RH is true. Where it is needed, we use the equivalence of Schoenfeld given in Volume One and quoted in this volume in Appendix B. Then for the case RH is false we first prove part of Guy Robin's result, Theorem 1.16 which is

$$A(x) = \Omega_-(x^\alpha),\ 0 < \alpha < \Theta,$$

where $\Theta := \sup\{\beta : \zeta(\beta + i\gamma) = 0\} > \frac{1}{2}$, which is all we need. This is then used to easily complete the proof of the equivalence, which is a little weaker than the result of Nicolas.

1.2 Estimating the Logarithmic Integral

First, we define the logarithmic integral valid for all $x > 1$ using the Cauchy principal value:

$$\mathrm{li}(x) := \lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t},$$

so

$$\mathrm{li}(x) := \mathrm{li}(2) + \int_2^x \frac{dt}{\log t},$$

with $\mathrm{li}(2) = 1.045163780117....$

For $x \to \infty$ we have the asymptotic expansions for the logarithmic integral valid for all $N \in \mathbb{N}$:

$$\begin{aligned}\mathrm{li}(x) &= \sum_{j=1}^{N} \frac{(j-1)!x}{(\log x)^j} + O\left(\frac{x}{(\log x)^{N+1}}\right)\\ &= \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)\\ &= \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).\end{aligned}$$

To see this note that by splitting the integral at $\sqrt{x}$ we get for $n \in \mathbb{N}$

$$\int_0^x \frac{1}{(\log x)^n}\, dx = O\left(\frac{x}{(\log x)^n}\right).$$

The expansion follows using integration by parts. In Figure 1.2 we show $\mathrm{li}(x)$ around its singularity, and in Figure 1.3 we give $\mathrm{li}(x)$ and its asymptotic approximation

$$\frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2x}{(\log x)^3},$$

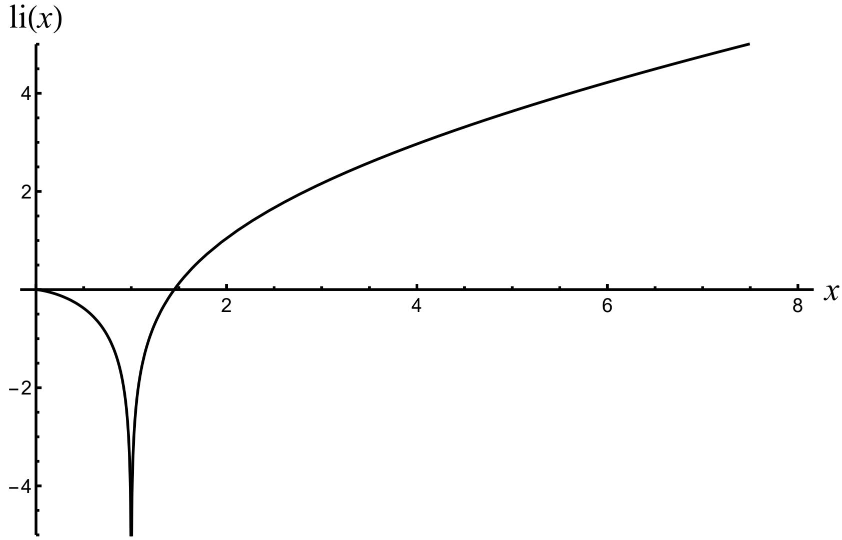

Figure 1.2 A plot of li(x) for $0 \le x \le 8$.

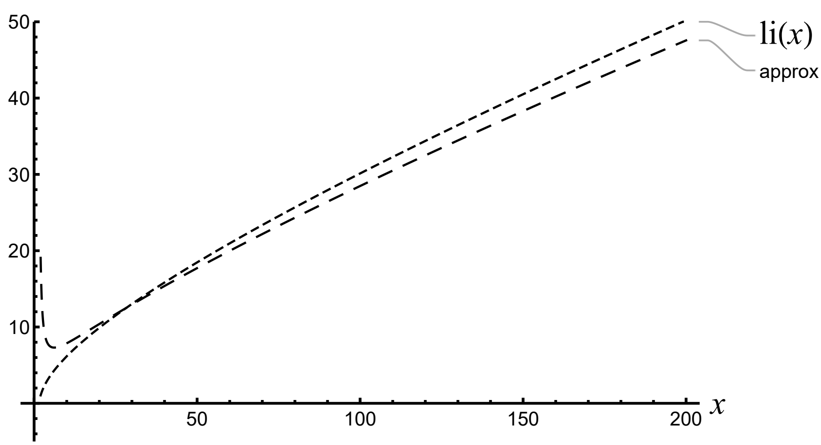

Figure 1.3 A plot of li(x) and an approximation for $2 \le x \le 200$.

which for at least $x \ge 20$ is less than li(x).

We note that the finite sum approximations are increasing with the number of terms and all terms, even the error for x sufficiently large, are positive for $x \ge 2$.

We use in the sequel the following functions relating to the difference between li(x) and its asymptotic expansions. We need only go to the second order:

$$L_1(x) := \mathrm{li}(x) - \frac{x}{\log x},$$

$$L_2(x) := \mathrm{li}(x) - \frac{x}{\log x} - \frac{x}{(\log x)^2},$$
$$F_1(x) := \frac{(\log x)^2\,\mathrm{li}(x) - x(\log x)}{x} = L_1(x)\frac{(\log x)^2}{x},$$
$$F_2(x) := \frac{(\log x)^3\,\mathrm{li}(x) - x(\log x)^2 - x(\log x)}{x} = L_2(x)\frac{(\log x)^3}{x}.$$

With these definitions we will see that $F_1(x)$ and $F_2(x)$ are bounded and have well-defined asymptotic limits.

Lemma 1.1 *The function $F_1(x)$ has the following and no other zeros or critical points on $[1,\infty)$:*
(i) $\lim_{x\to 1+} F_1(x) = 0$.
(ii) An absolute minimum at $x_3 = 1.85...$ with value -0.488.
(iii) A positive zero at $x_0 = 3.8464...$.
(iv) An absolute maximum at $x_4 = 94.6...$ with value $1.784...$.
(v) $\lim_{x\to\infty} F_1(x) = 1$.
In addition
(vi) For all $x > 1$ we have $\mathrm{li}(x) < 3x/4$.

Proof **(1)** First, note that for $x > 1$ we have the Taylor expansion

$$\mathrm{li}(x) = \log\log x + \gamma_0 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n \cdot n!}.$$

Since the sum is $O((x-1)e^{x-1})$, we can write as $x \to 1+$, $\mathrm{li}(x) = \log\log x + \gamma_0 + o(1)$. Thus, using l'Hôspital's rule to derive

$$\lim_{x\to 1+} (\log x)\log\log x = \lim_{y\to 0+} y\log y = \lim_{y\to 0+} \frac{\log y}{1/y} = -\lim_{y\to 0+} y = 0,$$

we get

$$\lim_{x\to 1+} F_1(x) = \frac{1}{x}\Big((\log x)^2(\log\log x + \gamma_0 + o(1)) - x\log x\Big)$$
$$= \lim_{x\to 1+} \frac{(\log x)}{x}(\log x)\log\log x = 0.$$

This proves (i).

(2) We now define three related functions which will enable the properties of $F_1(x)$ to be deduced:

$$f_1(x) := \frac{x^2}{\log x} F_1'(x),$$

$$= 2\,\mathrm{li}(x) + x - \frac{x}{\log x} - \log(x)\,\mathrm{li}(x),$$

$$f_2(x) := x f_1'(x),$$

$$= -\left(\mathrm{li}(x) - \frac{x}{\log x} - \frac{x}{(\log x)^2}\right) = -L_2(x) = -F_2(x)\frac{x}{(\log x)^3},$$

$$f_3(x) := f_2'(x) = -\frac{2}{(\log x)^3}.$$

Figures 1.4 and 1.5 indicate how the first two functions behave.

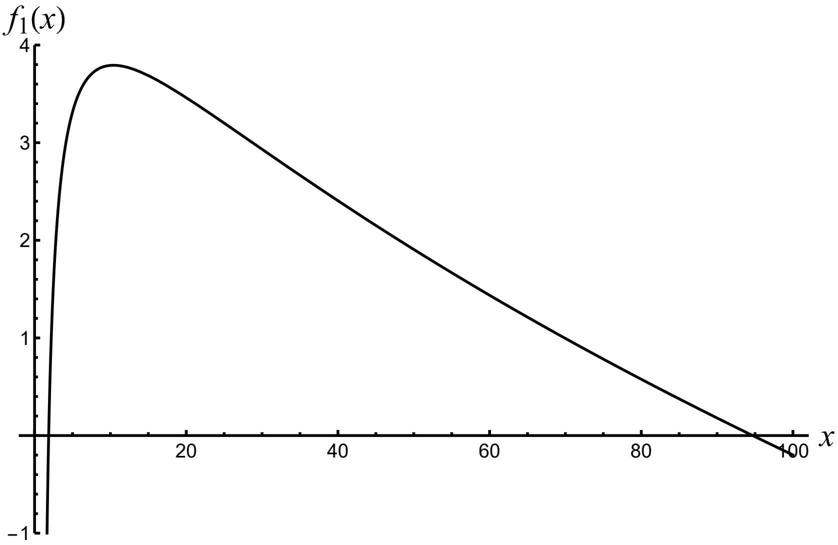

Figure 1.4 A plot of $f_1(x)$ for $2 \le x \le 100$.

Note that since $x > 1$, $f_2(x)$ and $f_1'(x)$ have the same sign, and that $f_3(x)$, hence $f_2'(x)$, is strictly negative. Thus, $f_2(x)$ is decreasing. Also the limit of $f_2(x)$ at $1+$ is $+\infty$ and at ∞ is $-\infty$. Therefore $f_2(x)$ has a unique zero in $(1, \infty)$ which we compute as $x_2 = 10.3973\ldots$. See Figure 1.5.

(3) We also derive

$$\lim_{x\to\infty} F_1(x) = \lim_{x\to\infty} \frac{(\log x)^2\left(\frac{x}{(\log x)} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right)\right) - x(\log x)}{x}$$

$$= \lim_{x\to\infty} \frac{x + O(x/(\log x))}{x} = 1.$$

This proves (v).

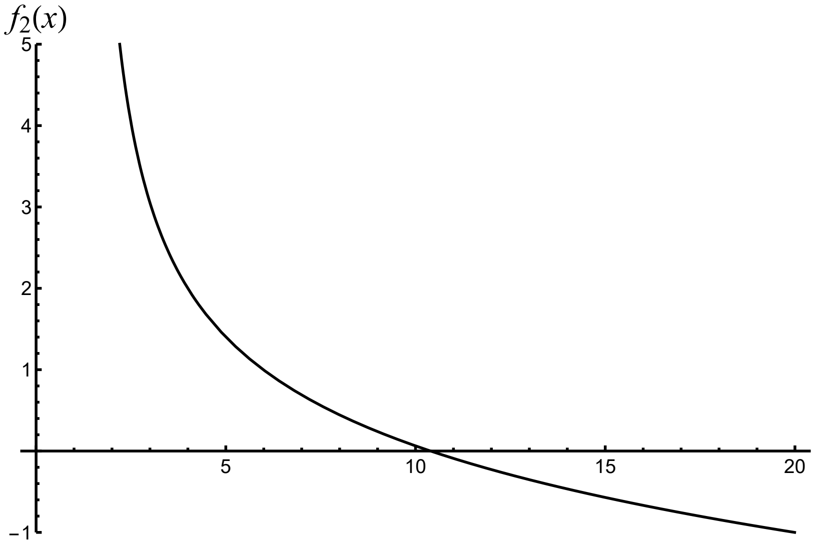

Figure 1.5 A plot of $f_2(x)$ for $1 \le x \le 20$.

(4) A computation shows $f_1(x)$ has precisely two zeros on $(1, \infty)$, at $x_3 = 1.85...$ and $x_4 = 94.6....$ Hence $F_1(x)$ has two corresponding critical points. Thus, we can say, moving from left to right, $F_1(1) = 0$, then $F_1(x)$ decreases to its minimum $F_1(x_3)$, then increases to its maximum $F_1(x_4)$, passing through a zero which we compute as $x_0 = 3.846467717...$, and then descends to its asymptotic limit 1 at ∞. Thus, we have (ii) and (iv). See Figures 1.6 and 1.7.

(5) Because

$$\frac{d}{dx}\left(\frac{\mathrm{li}(x)}{x}\right) = -\frac{F_1(x)}{x(\log x)^2}$$

is positive for $1 < x < x_0$ and negative for $x_0 < x$, $\mathrm{li}(x)/x$ has a maximum at x_0, and so we can write for all $x > 1$

$$\frac{\mathrm{li}(x)}{x} \le \frac{\mathrm{li}(x_0)}{x_0} \le 0.743 < \frac{3}{4},$$

so $\mathrm{li}(x) < 3x/4$. This proves (vi).

(6) In addition note that in the range $x > x_3$ we have $F_1(x) > 1$ so

$$\mathrm{li}(x) - \frac{x}{\log x} = L_1(x) = F_1(x)\frac{x}{(\log x)^2} > \frac{x}{(\log x)^2}$$

and so

$$\mathrm{li}(x) > \frac{x}{\log x} + \frac{x}{(\log x)^2}, x > x_3.$$

□

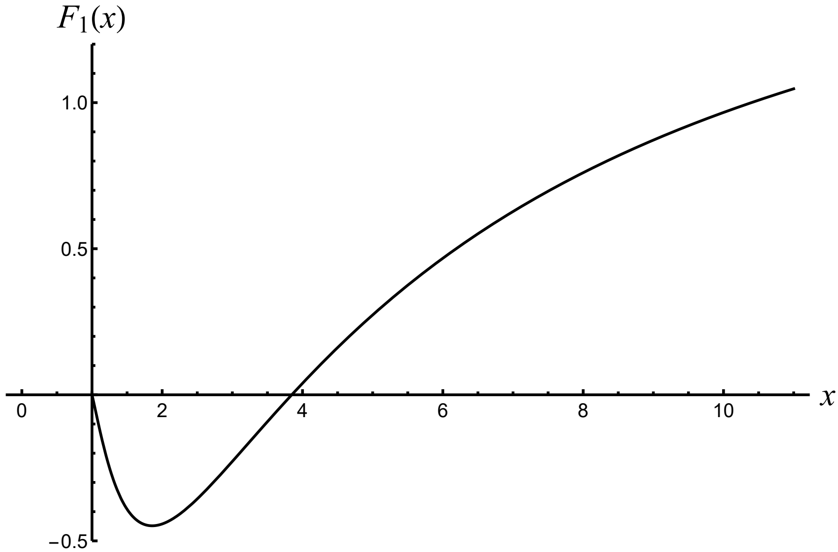

Figure 1.6 A plot of $F_1(x)$ for $1 \le x \le 11$.

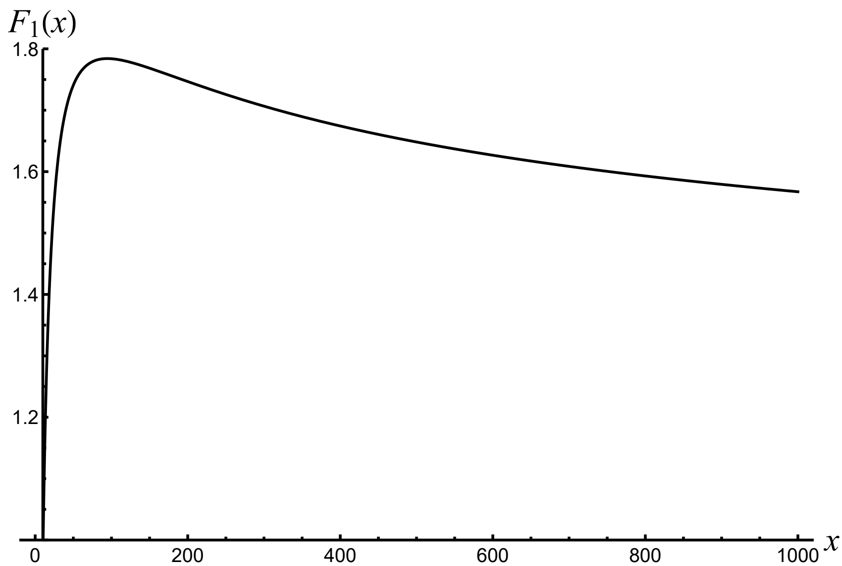

Figure 1.7 A plot of $F_1(x)$ for $11 \le x \le 1000$.

The function $F_2(x)$ behaves, qualitatively, in the same manner as $F_1(x)$. This gives rise to the possible use of higher-order approximations, $F_n(x)$, if needed.

Lemma 1.2 *The function $F_2(x)$ has the following and no other zeros or critical points:*

(i) $\lim_{x\to 1+} F_2(x) = 0$.
(ii) An absolute minimum at $x_3 = 3.38...$ *with value* $-1.369496....$
(iii) A positive zero at $x_0 = 10.39....$
(iv) An absolute maximum at $x_4 = 380.15...$ *with value* $4.040415....$
(v) $\lim_{x\to\infty} F_2(x) = 2$.

Proof The proof is similar to that of Lemma 1.1. In this case we define

$$f_1(x) := \frac{x^2 F_2'(x)}{(\log x)^2},$$
$$f_2(x) := x f_1'(x),$$
$$f_3(x) := f_2'(x) = -\frac{6}{(\log x)^4} < 0,$$

and proceed using the same steps as in that lemma. The function $F_2(x)$ is plotted in Figures 1.8 and 1.9. □

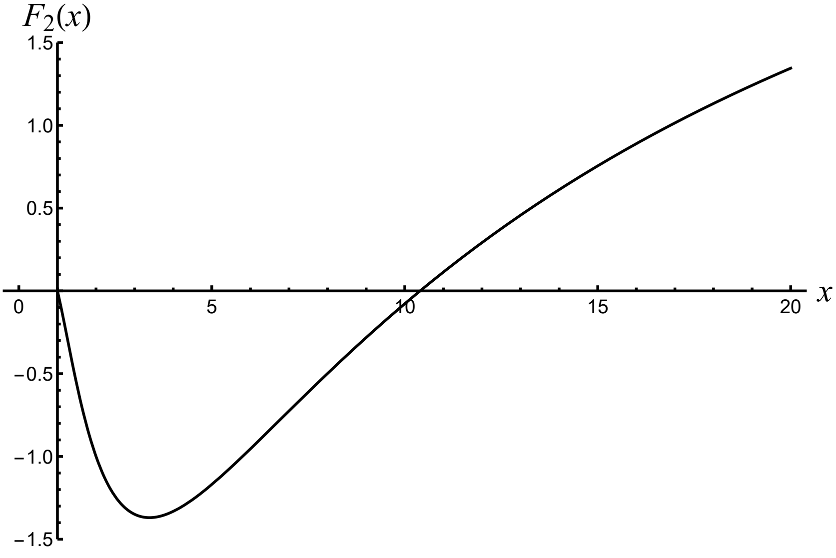

Figure 1.8 A plot of $F_2(x)$ for $1 \le x \le 20$.

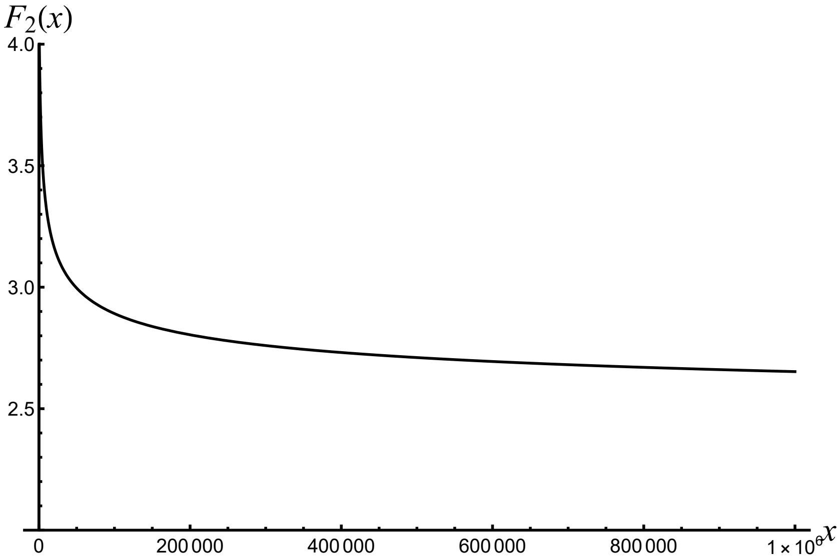

Figure 1.9 A plot of $F_2(x)$ for $20 \le x \le 10^6$.

1.3 The Function $A_1(x)$

Sums over $\rho = \beta + i\gamma$ are assumed to be over all of the non-trivial zeros of $\zeta(s)$. We define $A_1(x) := \mathrm{li}(\psi(x)) - \Pi(x)$ where $\Pi(x) := \sum_{j=1}^{\kappa} \frac{\pi(x^{1/j})}{j}$ with $\kappa := \lfloor \log x / \log 2 \rfloor$. The symbol $\Lambda(x)$ is the von Mangoldt function, set to zero if x is not a prime power.

We also define

$$\widetilde{\psi}(x) := \psi(x) - \tfrac{1}{2}\Lambda(x),$$
$$\widetilde{\Pi}(x) := \frac{\Pi(x)}{2(\log x)^2} - \tfrac{1}{2}\Lambda(x),$$

and use the explicit formulas (which can be derived for example from [30, theorem 9.5]) valid for $x > 1$

$$\widetilde{\psi}(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \tfrac{1}{2}\log\left(1 - \tfrac{1}{x^2}\right), \tag{1.1}$$

$$\widetilde{\Pi}(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{1}{t(t^2-1)(\log t)}\, dt. \tag{1.2}$$

Lemma 1.3 *Assume RH is true. For $x \geq 599$ we have*

$$\frac{\theta(x) - x}{\log x} - \frac{9(\log x)^2}{10^4} \leq \mathrm{li}(\theta(x)) - \mathrm{li}(x) \leq \frac{\theta(x) - x}{\log x},$$

and the same inequalities are valid if we replace $\theta(x)$ by $\psi(x)$.

Proof Let $x \geq 599$. By Theorem B.1 we have $|\theta(x) - x| \leq \sqrt{x}(\log x)^2/(8\pi)$. In addition $(\log x)^2/\sqrt{x}$ is decreasing for $x \geq 599$, so

$$\frac{\theta(x)}{x} \geq \frac{1}{x}\left(x - \frac{\sqrt{x}(\log x)^2}{8\pi}\right) \geq 1 - \frac{(\log 599)^2}{8\pi\sqrt{599}} > 0.93350 =: b \implies \theta(x) > bx. \tag{1.3}$$

Let $h > 1 - x$ and note that $\mathrm{li}'(x) = 1/\log(x)$ and $\mathrm{li}''(x) = -1/(x(\log x)^2)$. By Taylor's theorem, for some ξ with $\xi > \min(x, x+h)$, we can write

$$\mathrm{li}(x+h) = \mathrm{li}(x) + \frac{h}{\log x} - \frac{h^2}{2\xi(\log \xi)^2}. \tag{1.4}$$

Now let $h = \theta(x) - x$ so $h + x = \theta(x) \geq \theta(599) > 1$ and $\xi > bx$. Thus, we can write, noting that $\log b < 0$,

$$\begin{aligned}
\xi(\log \xi)^2 &\geq bx(\log(bx))^2 \\
&= bx(\log x)^2\left(1 + \frac{\log b}{\log x}\right)^2 \\
&\geq bx(\log x)^2\left(1 + \frac{\log b}{\log 599}\right)^2 \\
&\geq 0.91353x(\log x)^2.
\end{aligned}$$

Using this bound and Theorem B.1 again we get

$$0 \leq \frac{h^2}{2\xi(\log \xi)^2} \leq \frac{x(\log x)^4}{128\pi^2\xi(\log \xi)^2} \leq \frac{(\log x)^2}{0.91353 \times 128\pi^2} < \frac{9(\log x)^2}{10^4}.$$

Using these bounds in (1.4), neglecting the final term to get the upper bound, gives

$$\frac{\theta(x) - x}{\log x} - \frac{9(\log x)^2}{10^4} \leq \mathrm{li}(\theta(x)) - \mathrm{li}(x) \leq \frac{\theta(x) - x}{\log x},$$

which completes the proof. □

We have the definitions: γ_0 is Euler's constant and

$$A_1(x) := \mathrm{li}(\psi(x)) - \Pi(x) \text{ where } \Pi(x) := \sum_{j=1}^{\kappa} \frac{\pi(x^{1/j})}{j}.$$

Sums over $\rho = \beta + i\gamma$ are assumed to be over all of the non-trivial zeros of $\zeta(s)$. We use the explicit so-called Landau formulas

$$\widetilde{\psi}(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \tfrac{1}{2}\log\left(1 - \tfrac{1}{x^2}\right) = \psi(x) - \tfrac{1}{2}\Lambda(x),$$

$$\widetilde{\Pi}(x) = \mathrm{li}(x) - \sum_\rho \mathrm{li}(x^\rho) - \log 2 + \int_x^\infty \frac{1}{t(t^2-1)(\log t)}\, dt = \Pi(x) - \frac{\Lambda(x)}{2\log x},$$

where Nicolas replaces the summand $\mathrm{li}(x^\rho)$, using [137, section 88], with

$$\mathrm{li}(x^\rho) = \int_0^\infty \frac{x^{\rho-t}}{\rho - t}\, dt.$$

(Note the sum is over all zeros so Landau's $\pm i\pi$ cancel.) In [137, section 5] we see the definition for $w = u + iv$

$$\mathrm{li}(e^w) := \int_{-\infty+vi}^{w} \frac{e^s}{s}\, ds \pm i\pi,\ v \neq 0.$$

Hence using the substitutions $\rho = \beta + i\gamma$ then $y = \rho - z$ and finally $s = y + \log x$ we get

$$\int_0^\infty \frac{x^{\rho-t}}{\rho-t}\, dt = \int_{-\infty+\gamma i}^{\rho} \frac{x^y}{y}\, dy = \int_{-\infty+\gamma i \log x}^{\rho\log x} \frac{e^s}{s}\, ds = \mathrm{li}(x^\rho) \mp i.$$

Note the sum is over all zeros so Landau's $\pm i\pi$ cancel.

Lemma 1.4 *Assume RH is true. Then for $x \geq 599$ we have*

$$A_1(x) = \sum_\rho \frac{x^\rho}{\rho^2(\log x)^2} + J(x),$$

where the error term $J(x)$ satisfies

$$-\frac{9}{10^4}(\log x)^2 - \frac{1}{150}\frac{\sqrt{x}}{(\log x)^3} \leq J(x) \leq \frac{1}{150}\frac{\sqrt{x}}{(\log x)^3} + \log(2).$$

Proof **(1)** First, note that by RH for all $\rho = 1/2 + i\gamma$ we have $|\rho|^2 = \rho(1-\rho) = \frac{1}{4} + \gamma^2$, and the imaginary part of the first zero of $\zeta(s)$ has absolute value greater than 14.134. Thus,

$$\sum_\rho \frac{1}{\gamma^2} = \sum_\rho \frac{1 + 1/(4\gamma^2)}{\frac{1}{4}+\gamma^2} \leq \sum_\rho \frac{1 + 1/(4\times 14.134^2)}{\frac{1}{4}+\gamma^2} \leq \frac{800}{799}\sum_\rho \frac{1}{\rho(1-\rho)}.$$

In addition we have (see for example [29, lemma 2.10(b)])

$$\sum_\rho \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log(4\pi) = 0.0461914... \tag{1.5}$$

Combining these we get

$$\sum_\rho \frac{1}{|\gamma|^3} \le \frac{1}{14.134} \sum_\rho \frac{1}{|\gamma|^2} \le 0.0032722 < \frac{1}{300}.$$

(2) Next, integrating the left-hand side of $x^{\rho-t}/(\rho - t)$ by parts twice we get

$$\int_0^\infty \frac{x^{\rho-t}}{\rho - t}\,dt = \frac{x^\rho}{\rho(\log x)} + \frac{x^\rho}{\rho^2(\log x)^2} + \frac{2}{(\log x)^2}\int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3}\,dt.$$

We also have the bound

$$\left|\int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3}\,dt\right| \le \frac{1}{|\Im\rho|^3}\int_0^\infty x^{1/2-t}\,dt = \frac{1}{|\Im\rho|^3}\frac{\sqrt{x}}{(\log x)}.$$

Therefore, if the error term is

$$K(x) := \sum_\rho \frac{2}{(\log x)^2}\int_0^\infty \frac{x^{\rho-t}}{(\rho-t)^3}\,dt,$$

then using the bound derived in Step (1) we get

$$|K(x)| \le \frac{2\sqrt{x}}{(\log x)^3}\sum_\rho \frac{1}{|\Im\rho|^3} \le \frac{\sqrt{x}}{150(\log x)^3}.$$

Thus,

$$\sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho-t}\,dt = \sum_\rho \frac{x^\rho}{\rho\log x} + \sum_\rho \frac{x^\rho}{\rho^2(\log x)^2} + K(x),\ |K(x)| \le \frac{\sqrt{x}}{150(\log x)^3}.$$

(3) By the derivation of Lemma 1.3, replacing $h = \theta(x) - x$ with $h = \psi(x) - x$, and noting for the equation corresponding to (1.3) that $\psi(x)/x \ge \theta(x)/x$, we get

$$\frac{\psi(x)-x}{\log x} - \frac{9(\log x)^2}{10^4} \le \mathrm{li}(\psi(x)) - \mathrm{li}(x) \le \frac{\psi(x)-x}{\log x}.$$

(4) In this step, maybe the most intricate, we rearrange the expression for $A_1(x)$. First, note we have

$$\begin{aligned}\mathrm{li}(\psi(x)) &= \mathrm{li}(x) + \frac{\psi(x)-x}{\log x} + J_1(x),\\ &= \mathrm{li}(x) + \frac{\widetilde{\psi}(x) - x + \frac{1}{2}\Lambda(x)}{\log x} + J_1(x).\end{aligned}$$

By Step (3) we have for $x \ge 599$, $-(9/10^4)(\log x)^2 \le J_1(x) \le 0$. Thus, if we define

$$J_2(x) := \log 2 - \frac{\log(2\pi)}{\log x} \text{ and}$$
$$J_3(x) := -\frac{\log(1 - 1/x^2)}{2\log x} - \int_x^\infty \frac{1}{t(t^2-1)(\log t)}\, dt,$$

and use the explicit formulas for $\widetilde{\psi}(x)$ and Landau's form for $\widetilde{\Pi}(x)$, we can write

$$\begin{aligned}
A_1(x) &= \mathrm{li}(\psi(x)) - \Pi(x)\\
&= \mathrm{li}(x) + \frac{1}{\log x}\left(-\sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \tfrac{1}{2}\log\left(1 - \tfrac{1}{x^2}\right) + \tfrac{\Lambda(x)}{2}\right)\\
&+ J_1(x) - \mathrm{li}(x) + \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho - t}\, dt - \int_x^\infty \frac{1}{t(t^2-1)(\log t)}\, dt + \log 2 - \frac{\Lambda(x)}{2(\log x)}\\
&= \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho-t}\, dt - \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + J_1(x) + J_2(x) + J_3(x)
\end{aligned}$$

Hence we can write, recalling the definition of $K(x)$ from Step (2),

$$A_1(x) = \sum_\rho \frac{x^\rho}{\rho^2(\log x)^2} + J(x) \text{ where } J(x) := K(x) + J_1(x) + J_2(x) + J_3(x).$$

(5) In this penultimate step we will bound $J_1(x) + J_2(x)$. From the definition in Step (4) we can write, with $x \geq 599$,

$$J_3(x) = \int_x^\infty \frac{1}{t(t^2-1)}\left(\frac{1}{\log x} - \frac{1}{\log t}\right) dt \geq 0,$$

so

$$\begin{aligned}
J_3(x) &\leq \frac{1}{\log x}\int_x^\infty \frac{1}{t(t^2-1)}\, dt\\
&= \frac{\log(1 + 1/(x^2-1))}{2\log x} \leq \frac{1}{2(x^2-1)\log x} < \frac{\log(2\pi)}{\log x}.
\end{aligned}$$

Therefore $0 < J_2(x) + J_3(x) < \log 2$.

(6) Combining the result of Step (5) with the results from Steps (1)–(4), we get

$$-\frac{9}{10^4}(\log x)^2 - \frac{1}{150}\frac{\sqrt{x}}{(\log x)^3} \leq J(x) \leq \frac{1}{150}\frac{\sqrt{x}}{(\log x)^3} + \log(2),$$

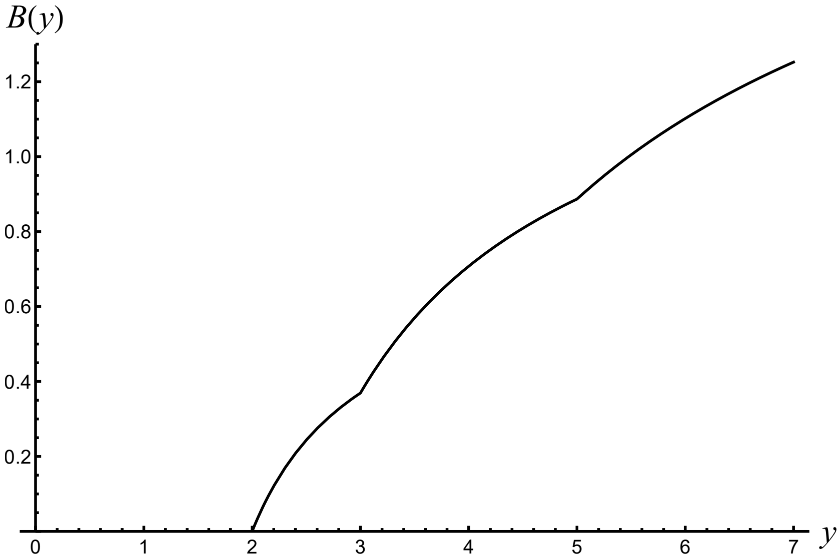

Figure 1.10 A plot of $B(y)$ for $2 \le y \le 80$.

which completes the proof. □

1.4 The Functions $B(x)$ and $A_2(x)$

Now for $y \ge 2$ let

$$B(y) := \pi(y) - \frac{\theta(y)}{\log y} = \sum_{p \le y} \left(1 - \frac{\log p}{\log y}\right).$$

Figure 1.10 is a plot of $B(y)$ for $2 \le y \le 7$ showing its continuity and increasing nature. Figure 1.11 is a plot of $B(x)$ for $2 \le x \le 100$ showing its square root order of increase. Also recall the definitions

$$A_2(x) := \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x),$$
$$\Pi(x) := \sum_{j=1}^{\kappa} \frac{\pi(x^{1/j})}{j}.$$

Lemma 1.5 *Assume RH is true. For $x \ge 599$, if we define $\kappa := \lfloor \log x / \log 2 \rfloor$, then*

$$A_2(x) = \sum_{j=2}^{\kappa} \frac{B(x^{1/j})}{j} + U(x),$$

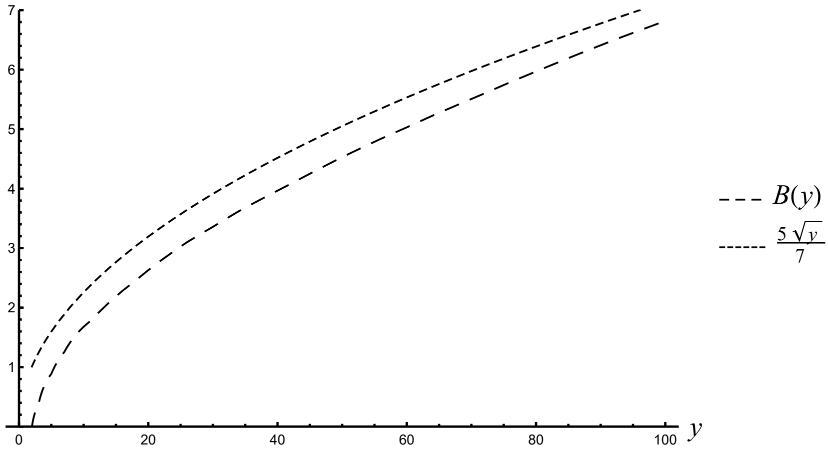

Figure 1.11 A plot of $B(y)$ and $5\sqrt{y}/7$ for $2 \le y \le 100$.

where $|U(x)| \le 9(\log x)^2/10^4$.

Proof Applying the method of Lemma 1.3, but replacing $\theta(x)$ by $\psi(x)$, and then subtracting the results of the $\theta(x)$ and $\psi(x)$ forms, gives the estimates

$$\frac{\psi(x)-\theta(x)}{\log x} - \frac{9(\log x)^2}{10^4} \le \operatorname{li}(\psi(x)) - \operatorname{li}(\theta(x)) \le \frac{\psi(x)-\theta(x)}{\log x} + \frac{9(\log x)^2}{10^4}.$$

Thus,

$$\operatorname{li}(\psi(x)) - \operatorname{li}(\theta(x)) = \frac{\psi(x)-\theta(x)}{\log x} + U(x),\ |U(x)| \le \frac{9(\log x)^2}{10^4}.$$

Inserting this in the definition of $A_2(x)$, and then using the definitions of $\psi(x)$, $\Pi(x)$ and $B(y)$ gives

$$A_2(x) = \sum_{j=2}^{\kappa} \left(\frac{\pi(x^{1/j})}{j} - \frac{\theta(x^{1/j})}{\log x} \right) + U(x) = \sum_{j=2}^{\kappa} \frac{B(x^{1/j})}{j} + U(x).$$

This completes the proof. □

Recall the definitions:

$$L_1(t) := \operatorname{li}(t) - \frac{t}{\log t},$$

$$B(x) := \pi(x) - \frac{\theta(x)}{\log x}.$$

Lemma 1.6 *If* $8.3 = y_0 \le y \le 1.39 \times 10^{17}$ *we have* $B(y) \le L_1(y)$. *If* $y \ge 599$ *and RH is true then*

$$B(y) \le L_1(y) + \frac{\sqrt{y}}{4\pi}.$$

Proof **(1)** By Abel's theorem [3, theorem 4.2] we have

$$\pi(y) = \frac{\theta(y)}{\log y} + \int_2^y \frac{\theta(t)}{t(\log t)^2}\, dt.$$

Thus,

$$\begin{aligned} B(y) &= \int_2^y \frac{\theta(t)}{t(\log t)^2}\, dt \\ &= \int_2^{y_0} \frac{\theta(t)}{t(\log t)^2}\, dt + \int_{y_0}^y \frac{\theta(t)}{t(\log t)^2}\, dt \\ &= B(y_0) + \int_{y_0}^y \frac{\theta(t)}{t(\log t)^2}\, dt. \end{aligned} \tag{1.6}$$

By Theorem B.2 we have $\theta(x) < x$ for $0 < x \le 1.39 \times 10^{17}$, and for $n \in \mathbb{N}$, $x > 1$, using induction and integration by parts, we have

$$\int \frac{dx}{(\log x)^n} = \frac{1}{(n-1)!}\left(\mathrm{li}(x) - \sum_{j=1}^{n-1} \frac{(j-1)!x}{(\log x)^j}\right).$$

Thus, for $y \le 1.39 \times 10^{17}$ we get

$$\int_{y_0}^y \frac{\theta(t)}{t(\log t)^2}\, dt \le \int_{y_0}^y \frac{dt}{(\log t)^2} = \mathrm{li}(y) - \mathrm{li}(y_0) + \frac{y_0}{\log y_0} - \frac{y}{\log y},$$

so

$$B(y) \le L_1(y) + B(y_0) - L_1(y_0) \le L_1(y) - 0.0012 < L_1(y).$$

(2) Now use $y_1 = 599$ instead of y_0 in Equation (1.6) and define

$$T(y, y_1) := \int_{y_1}^y \frac{\theta(t) - t}{t(\log t)^2}\, dt,$$

to get

$$B(y) = B(y_1) - L_1(y_1) + L_1(y) + T(y, y_1).$$

By Theorem B.1 we get

$$|T(y, y_1)| \le \int_{y_1}^y \frac{\sqrt{t}(\log t)^2}{8\pi t(\log t)^2}\, dt = \frac{\sqrt{y} - \sqrt{y_1}}{4\pi}.$$

Therefore, since $B(y_1) - L_1(y_1) - \sqrt{y_1}/(4\pi) < 0$, we get

$$B(y) \leq L_1(y) + \frac{\sqrt{y}}{4\pi} + B(y_1) - L_1(y_1) - \frac{\sqrt{y_1}}{4\pi} < L_1(y) + \frac{\sqrt{y}}{4\pi}.$$

This completes the proof. □

Recall the definitions:

$$\widetilde{F}_1(t) = \begin{cases} 1.785, & t \leq 95, \\ F_1(t) = \frac{(\log x)^2 \operatorname{li}(t) - t(\log x)}{t}, & t > 95, \end{cases}$$

$$\widetilde{F}_2(t) = \begin{cases} 4.05, & t \leq 381, \\ F_2(t) = \frac{(\log x)^3 \operatorname{li}(t) - t(\log x)^2 - t(\log x)}{t}, & t > 381, \end{cases}$$

$$L_1(t) := \operatorname{li}(t) - \frac{t}{\log t},$$

$$F_1(x) := \frac{L_1(x)(\log x)^2}{t},$$

$$B(x) := \pi(x) - \frac{\theta(x)}{\log x},$$

$$\kappa := \left\lfloor \frac{\log x}{\log 2} \right\rfloor,$$

$$\kappa_2 := \left\lfloor \frac{\log x}{\log a} \right\rfloor,$$

$$\epsilon(y) := \chi_{(1.39\times 10^{15},\infty)}(y), \text{ the characteristic function.}$$

In addition we set $a = 10.4$ and let κ_1 be any integer in the range $3 \leq \kappa_1 < \kappa_2$.

Lemma 1.7 *Let RH be true and $x \geq 10^8$. Then with the given definitions and parameter settings we have*

$$\sum_{j=2}^{\kappa} \frac{B(x^{1/j})}{j} \leq \frac{2\sqrt{x}}{(\log x)^2} + \frac{4\sqrt{x}}{(\log x)^3}\widetilde{F}_2(\sqrt{x})$$
$$+ \sum_{j=3}^{\kappa_1} \frac{jx^{1/j}}{(\log x)^2}\widetilde{F}_1(x^{1/j}) + \frac{7.23\kappa_1^3 x^{1/\kappa_1}}{(\log x)^3} + 2.35 + 0.94\frac{\sqrt{x}}{(\log x)^5}.$$

Proof **(1)** To begin the proof we split an upper bound for the sum on the left into five separate sums. First, we note that by Lemma 1.6 for $y \geq 8.3$ we have $B(y) \leq L_1(y) + 0.25\epsilon(y)\sqrt{y}/\pi$, and for $2 \leq j \leq \kappa_2$ we get

$$a = x^{\log a/\log x} \leq x^{1/\kappa_2} \leq x^{1/j}.$$

Thus, for these values of j we have

$$B(x^{1/j}) \le L_1(x^{1/j}) + \epsilon(x^{1/j})\frac{x^{1/(2j)}}{4\pi}.$$

Thus, if we define

$$\begin{aligned}
T_1 &:= \tfrac{1}{2}L_1(\sqrt{x}),\\
T_2 &:= \sum_{j=3}^{\kappa_1}\frac{L_1(x^{1/j})}{j},\\
T_3 &:= \sum_{j=\kappa_1+1}^{\kappa_2}\frac{L_1(x^{1/j})}{j},\\
T_4 &:= \sum_{j=\kappa_2+1}^{\kappa}\frac{B(x^{1/j})}{j},\\
T_5 &:= \sum_{j=2}^{\kappa_2}\epsilon(x^{1/j})\frac{x^{1/(2j)}}{4\pi j},
\end{aligned}$$

we get

$$\sum_{j=2}^{\kappa}\frac{B(x^{1/j})}{j} \le \sum_{i=1}^{5} T_i.$$

In Step (2) we bound T_1 and T_2. In (3) we bound T_3, in (4) T_4 and in (5) T_5.

(2) Since for $t > 1$

$$L_1(t) = \frac{tF_1(t)}{(\log t)^2} \le \frac{t\widetilde{F}_1(t)}{(\log t)^2} \text{ and } L_2(t) = \frac{tF_2(t)}{(\log t)^3} \le \frac{t\widetilde{F}_2(t)}{(\log t)^3},$$

we get

$$T_1 = \tfrac{1}{2}L_2(\sqrt{x}) + \frac{\sqrt{x}}{2(\log\sqrt{x})^2} = \frac{2\sqrt{x}}{(\log x)^2} + \frac{4\sqrt{x}F_2(\sqrt{x})}{(\log x)^3} \le \frac{2\sqrt{x}}{(\log x)^2} + \frac{4\sqrt{x}\widetilde{F}_2(\sqrt{x})}{(\log x)^3}$$

and

$$T_2 = \sum_{j=3}^{\kappa_1}\frac{jx^{1/j}}{(\log x)^2}F_1(x^{1/j}) \le \sum_{j=3}^{\kappa_1}\frac{jx^{1/j}}{(\log x)^2}\widetilde{F}_1(x^{1/j}).$$

(3) For $t \ge 1$, using Lemma 1.1, the maximum value of $F_1(t)$ is $F_1(x_4)$, and we get

$$L_1(t) \le 1.785\frac{t}{(\log t)^2} \implies T_3 \le \frac{1.785}{(\log x)^2}\sum_{j=\kappa_2+1}^{\kappa} jx^{1/j}.$$

Since $x > 1$, in the range $0 < t \le \log x$ the function $t \to tx^{1/t}$ is strictly positive and decreasing. Hence we can write using the change of variables $u^t = x$,

$$T_3 \le 1.785(L_2(x^{1/\kappa_1}) - L_2(a)) \le 1.785\left(4.05\frac{x^{1/\kappa_1}}{(\log x^{1/\kappa_1})^3} - L_2(a)\right).$$

Next, by Lemma 1.2 we have $L_2(10.4) > 0$, so setting $a = 10.4$ by the result of Step (3) we get

$$T_3 \le 1.785\left(4.05\frac{\kappa_1^3 x^{1/\kappa_1}}{(\log x)^3} - L_2(10.4)\right) \le 7.23\frac{7.23\kappa_1^3 x^{1/\kappa_1}}{(\log x)^3}.$$

(4) We next derive a bound for T_4. First, note that

$$j \ge \kappa_2 + 1 > (\log x)/\log a \implies x^{1/j} < a.$$

Thus, because the function $y \to B(y)$ is increasing, for the given values of j we get

$$B(x^{1/j}) \le B(a) = B(10.4) < 1.72.$$

Therefore, from the definition of T_4 in Step (1)

$$\begin{aligned}
T_4 &\le 1.72\sum_{\kappa_2+1}^{\kappa}\frac{1}{j}\\
&\le 1.72\int_{\kappa_1}^{\kappa_2-1}\frac{dt}{t}\\
&= 1.72\left(\log\left(\frac{\log x}{\log 2}\right) - \log\left(\frac{\log(x/a)}{\log a}\right)\right)\\
&= 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \log\left(\frac{\log(x)}{\log(x/a)}\right)\right)\\
&\le 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \left(\frac{\log(x)}{\log(x/a)} - 1\right)\right)\\
&= 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \frac{\log a}{\log(x/a)}\right)\\
&\le 1.72\left(\log\left(\frac{\log a}{\log 2}\right) + \frac{\log a}{\log(10^8/a)}\right)\\
&\le 2.3445.
\end{aligned}$$

(5) Finally, we bound T_5 as defined in Step (1). If we set

$$S := \sum_{j=2}^{\kappa_2}\frac{x^{1/(2j)}}{j},$$

then we **claim** for $a \geq 2.11$ and $x \geq a^3$ we get $S \leq 1.25x^{1/4}$. To see this note that with the given constraint on x the function $t \to x^{1/(2t)}$ is positive and decreasing for $t > 0$. Thus, using the change of variables $u^{2t} = x$, and using Lemma 1.1(vii) that for all $t > 1$ we have $\mathrm{li}(t) < 3t/4$, and that $\mathrm{li}(\sqrt{a}) > 0$ to get the final inequality, we get

$$\begin{aligned} S &= \tfrac{1}{2}x^{1/4} + \textstyle\sum_{j=3}^{\kappa_2} \frac{x^{1/(2j)}}{j} \\ &\leq \tfrac{1}{2}x^{1/4} + \int_2^{\frac{\log x}{\log a}} \frac{x^{1/(2t)}}{t}\, dt \\ &= \tfrac{1}{2}x^{1/4} + \int_{\sqrt{a}}^{x^{1/4}} \frac{du}{\log u} \\ &\leq \tfrac{1}{2}x^{1/4} + \mathrm{li}(x^{1/4}) - \mathrm{li}(\sqrt{a}) \\ &\leq \frac{5}{4}x^{1/4} - \mathrm{li}(\sqrt{a}) < \frac{5}{4}x^{1/4}. \end{aligned}$$

This completes the proof of the claim. Next, because $\epsilon(t)$ is increasing and vanishes when $t \leq 10^{17}$ we can write

$$\begin{aligned} T_5 &\leq \epsilon(\sqrt{x}) \sum_{j=2}^{\kappa_2} \frac{x^{1/(2j)}}{4\pi j} \\ &\leq \frac{5\epsilon(\sqrt{x})}{16\pi} x^{1/4} \\ &= \frac{5\epsilon(\sqrt{x})}{16\pi} \frac{\sqrt{x}}{(\log x)^5} \frac{(\log x)^5}{x^{1/4}} \\ &< \frac{5\epsilon(\sqrt{x})}{16\pi} \frac{\sqrt{x}}{(\log x)^5} \frac{(\log 10^{34})^5}{10^{34/4}} \\ &< 0.94 \frac{\sqrt{x}}{(\log x)^5}. \end{aligned}$$

Combining the bounds from each of the steps completes the proof. □

1.5 Asymptotic and Explicit Bounds for the Function $A(x)$

We next derive a lower bound for $A(x)$. Recall the definitions,

$$\Pi(x) := \sum_{p^j \leq x} \frac{1}{j} = \sum_{j=1}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \pi(x^{1/j}),$$

$$\begin{aligned}
A_1(x) &:= \mathrm{li}(\psi(x)) - \Pi(x),\\
A_2(x) &:= \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x),\\
A(x) &:= \mathrm{li}(\theta(x)) - \pi(x) = A_1(x) + A_2(x),\\
L_1(x) &:= \mathrm{li}(x) - \frac{t}{\log t},\\
L_2(x) &:= L_1(x) - \frac{t}{(\log t)^2},\\
\Delta &:= \sum_{\rho} \frac{1}{|\rho|^2},\\
B(x) &:= \pi(x) - \frac{\theta(x)}{\log x} = \sum_{p\le x}\left(1 - \frac{\log p}{\log x}\right).
\end{aligned}$$

We use the bounds we have derived for $A_1(x)$ and $A_2(x)$ to derive a lower bound for $A(x)$.

Lemma 1.8 *Assume RH is true. For all $x \ge 9 \times 10^6$ we have*

$$A(x) \ge \frac{\sqrt{x}}{(\log x)^2}\left(2 - \Delta + \frac{1}{\log x}\left(7.993 - \frac{(\log x)^3}{8\pi x^{1/4}} - \frac{18(\log x)^5}{10^4\sqrt{x}}\right)\right).$$

Proof **(1)** First, we bound the function $A_2(x)$. Using Lemma 1.5, for $x \ge 599$ we get

$$A_2(x) \ge \tfrac{1}{2}B(\sqrt{x}) - \tfrac{9(\log x)^2}{10^4}.$$

We have $x \ge 2903^2$. Thus, by Lemma 1.6, which gives for $y \ge 2903$, because each $B(x^{1/j}) \ge 0$, the bound

$$B(y) \ge L_1(y) - \frac{\sqrt{y}}{4\pi},$$

and, using Lemma 1.2(v) to get $L_2(t) > 2t/(\log t)^3$ for $t > 29^2$, to derive the third line, we have

$$\begin{aligned}
A_2(x) &\ge \tfrac{1}{2}\left(L_1(\sqrt{x}) - \tfrac{x^{1/4}}{4\pi}\right) - \tfrac{9(\log x)^2}{10^4}\\
&= \tfrac{1}{2}\left(\tfrac{\sqrt{x}}{(\log\sqrt{x})^2} + L_2(\sqrt{x}) - \tfrac{x^{1/4}}{4\pi}\right) - \tfrac{9(\log x)^2}{10^4}\\
&\ge \tfrac{1}{2}\left(\tfrac{\sqrt{x}}{(\log\sqrt{x})^2} + \tfrac{2\sqrt{x}}{(\log\sqrt{x})^3} - \tfrac{x^{1/4}}{4\pi}\right) - \tfrac{9(\log x)^2}{10^4}\\
&= \frac{\sqrt{x}}{(\log x)^2}\left(2 + \frac{8}{\log x} - \frac{(\log x)^2}{8\pi x^{1/4}} - \frac{9(\log x)^4}{10^4\sqrt{x}}\right).
\end{aligned}$$

(2) Next, we bound $A_1(x)$ using Lemma 1.4 to get

$$A_1(x) \geq -\left|\sum_\rho \frac{x^\rho}{\rho^2(\log x)^2}\right| - \frac{9}{10^4}(\log x)^2 - \frac{1}{150}\frac{\sqrt{x}}{(\log x)^3}.$$

Therefore, using Step (1), RH, and $A(x) = A_1(x) + A_2(x)$, we get

$$A(x) \geq \frac{\sqrt{x}}{(\log x)^2}\left(2 - \sum_\rho \frac{1}{|\rho|^2} + \frac{8 - 1/150}{\log x} - \frac{(\log x)^2}{8\pi x^{1/4}} - \frac{18(\log x)^4}{10^4\sqrt{x}}\right).$$

Substituting Δ for the sum completes the proof. □

The following result, Lemma 1.9, shows that even though $A(x)$ appears initially to be nondecreasing, this reasonable assumption is false unconditionally for an infinite number of integer values. Let $Q(p)$ be the largest prime strictly smaller than the prime $p \geq 3$, namely we have three consecutive primes $Q(p) < p < P(p)$.

Lemma 1.9 *$A(x)$ is nondecreasing in the range $1 \leq x \leq 1.39 \times 10^{17}$. However there are infinitely many primes p for which $A(p) < A(Q(p))$ is true.*

Proof **(1)** Let $p \in (3, 1.3 \times 10^{17})$. Then, using Theorem B.2 to get the final bound,

$$\begin{aligned} A(p) - A(Q(p)) &= \mathrm{li}(\theta(p)) - \mathrm{li}(\theta(Q(p)) - 1 \\ &= \int_{Q(p)}^{\theta(p)} \frac{dt}{\log t} - 1 \\ &> \frac{\theta(p) - \theta(Q(p))}{\log \theta(p)} - 1 \\ &= \frac{\log p}{\log \theta(p)} - 1 > 0. \end{aligned}$$

(2) By Littlewood's theorem (see for example Volume One [29, theorem 4.13]) there is a constant $C > 0$ and an infinite increasing sequence x_n with limit value infinity such that for all $n \in \mathbb{N}$ we have

$$\theta(x_n) \geq x_n + C\sqrt{x_n}\,\mathrm{logloglog}(x_n).$$

Note that we can replace $\psi(x)$ by $\theta(x)$ in Littlewood's theorem because, by the prime number theorem, we have

$$\psi(x) = \theta(x) + \sqrt{x} + o(\sqrt{x}).$$

Let $p \leq x_n$ be the largest such prime and assume $x_n,\ p$ are sufficiently large so that

$$\theta(p) = \theta(x_n) \geq x_n + C\sqrt{x_n}\,\mathrm{logloglog}\,x_n > p + \log p \implies \theta(p) - \log p > p.$$

Then, using the first part of the derivation in Step (1), we get

$$A(p) - A(Q(p)) < \frac{\log p}{\log \theta(Q(p))} - 1 = \frac{\log p}{\log(\theta(p) - \log p)} - 1 < 0.$$

This completes the proof. □

Lemma 1.10 gives some simple indicative bounds for $A(x)$ in finite ranges. These are a prelude to Theorem 1.12 which gives asymptotic upper and lower bounds, and then Lemma 1.13 which gives absolute bounds, all depending on RH. If RH is false all of the infinite range bounds fail – this is the subject of Theorem 1.17, which depends on the result of Guy Robin, Theorem 1.16.

Lemma 1.10 *(1) If* $x \in [11, 1.39 \times 10^{17}]$ *we have* $A(x) > 0$.

(2) Let $x \in [2, 10^4]$. *Then*

$$A(x) \leq 5.0644 \frac{\sqrt{x}}{(\log x)^2}.$$

(3) For $x \in [37, 89]$ *we have*

$$A(x) \geq \frac{\sqrt{x}}{(\log x)^2}(2 - \Delta).$$

Proof **(1)** This follows from $A(11) = 0.1301...$ and Lemma 1.9.

(2) On the domain $[1, \infty)$ the function $\varphi(x) = (\log x)^2/\sqrt{x}$ has a maximum at $x = e^4$ with value $16/e^2$. Because $A(x)$ is nondecreasing when $x < 59$ we get

$$\frac{A(x)(\log x)^2}{\sqrt{x}} \leq \frac{16A(53)}{e^2} \leq 2.502.$$

If $p \geq 59$ and $p \leq x < P(p)$ then, since the maximum of $A(p)(\log p)^2/\sqrt{(p)}$ for $p \in [59, 10^4]$ is at $p = 3643$, we get

$$\frac{A(x)(\log x)^2}{\sqrt{x}} = \frac{A(p)(\log x)^2}{\sqrt{x}} \leq \frac{A(p)(\log p)^2}{\sqrt{(p)}} \leq 5.0644,$$

which gives (2).

(3) Using the function $\varphi(x)$ again, for $1 < a < b$, a lower bound for φ on $[a, b]$ is $\min(\varphi(a), \varphi(b))$. If the prime $p \in [11, 83]$ then by Step (1) we have $A(p) > 0$ and for $p \leq x < P(p)$ we can write

$$\frac{A(x)(\log x)^2}{\sqrt{x}} = \frac{A(p)(\log x)^2}{\sqrt{x}} \geq A(p)\min(\varphi(p), \varphi(P(p))).$$

A computation shows that for $37 \le p \le 83$ we have

$$A(p)\min(\varphi(p), \varphi(P(p))) > 2 - \Delta.$$

This completes the proof. □

Recall that we have $\Delta = \sum_\rho 1/|\rho|^2$, and that we have defined $\kappa = \lfloor \log(x)/\log(2) \rfloor$.

Lemma 1.11 *Assume RH is true. Then for $x \ge 10^8$ we have the upper bound*

$$A(x) \le \frac{\sqrt{x}}{(\log x)^2}\left(2 + \Delta + \frac{25.212}{\log x}\right).$$

Proof By Lemma 1.4 for $x \ge 599$ we have

$$A_1(x) \le \Delta\frac{\sqrt{x}}{(\log x)^2} + \frac{1}{150}\frac{\sqrt{x}}{(\log x)^3} + 0.7.$$

By Lemma 1.5 we also have

$$A_2(x) \le \sum_{j=2}^{\kappa} \frac{B(x^{1/j})}{j} + \frac{9(\log x)^2}{10^4}.$$

Lemma 1.7 gives an upper bound for the sum in this bound. Combining these estimates we get an upper bound for $A(x) = A_1(x) + A_2(x)$. We write the bound in the form

$$A(x) \le \frac{\sqrt{x}}{(\log x)^2}\left(2 + \Delta + \frac{R(\kappa_1, x)}{\log x}\right),$$

where

$$R(\kappa_1, x) := 4\widetilde{F}_2(\sqrt{x}) + \frac{1}{150} + 3.05\frac{(\log x)^3}{\sqrt{x}} + \sum_{j=3}^{\kappa_1} \frac{j\widetilde{F}_1(x^{1/j})(\log x)}{x^{1/2-1/j}} + 7.23\frac{\kappa_1^3}{x^{1/2-1/\kappa_1}} + \frac{0.94}{(\log x)^2} + \frac{9}{10^4}\frac{(\log x)^5}{\sqrt{x}}. \tag{1.7}$$

Note that as a function of x, $R(\kappa_1, x)$, for $x \ge 10^8$, has all terms nonincreasing and positive. Evaluating with $\kappa_1 = 5$, we get

$$R(5, x) \le R(5, 10^8) \le 25.2119... \le 25.212,$$

which completes the proof. □

Theorem 1.12 *Assume RH is true. Then for $x \to \infty$ we have*

$$\frac{\sqrt{x}}{(\log x)^2}\left(2 - \Delta + \frac{7.993 + o(1)}{\log x}\right) \le A(x) \le \frac{\sqrt{x}}{(\log x)^2}\left(2 + \Delta + \frac{8.007 + o(1)}{\log x}\right).$$

Proof Lemma 1.8 gives the lower bound. To get the upper bound note that

$$\lim_{x\to\infty} \widetilde{F}_1(x) = 1 \text{ and } \lim_{x\to\infty} \widetilde{F}_2(x) = 2,$$

so considering the expression for $R(\kappa_1, x)$ from Lemma 1.11, namely (1.7), we get

$$\lim_{x\to\infty} R(3, x) = 8 + \frac{1}{150},$$

which gives the upper bound, completing the proof. □

Lemma 1.13 *Assume RH is true. Then*
(1) For all $x \ge 2$ we have

$$A(x) \le \frac{\sqrt{x}}{(\log x)^2}\left(2 + \Delta + \frac{27.727}{\log x}\right).$$

(2) For all $x \ge 84.11$ we have

$$A(x) \ge \frac{\sqrt{x}}{(\log x)^2}\left(2 - \Delta + \frac{5.12}{\log x}\right).$$

Proof **(1)** If $x \ge 10^8$, the result follows from Lemma 1.11. If $409 \le x < 10^8$, since $e^6 < 409$, then if we define

$$f(x) := \log(x)\left(A(x)\frac{(\log x)^2}{\sqrt{x}} - 2 - \Delta\right) \text{ and}$$
$$f_p(x) := \log(x)\left(A(p)\frac{(\log x)^2}{\sqrt{x}} - 2 - \Delta\right),$$

for $x \in [p, P(p))$ the function f_p is decreasing, so $f(x) \le f(p)$. In addition, evaluating

$$\max\left\{f(x)\colon 409 \le x \le 10^8\right\} = \max\left\{f(p)\colon 409 \le p \le 10^8\right\}$$
$$= f(33647) \le 27.727.$$

If $2 \le x < 409$, by Lemma 1.9 $A(x)$ is non-decreasing, so

$$A(x) \le A(Q(409)) = A(401) \le 2.52,$$

and as before

$$\frac{(\log x)^2}{\sqrt{x}} \leq \frac{16}{e^2}.$$

Thus,

$$f(x) \leq \log(409)\left(2.52\frac{16}{e^2} - 2 - \Delta\right) < 20.51,$$

completing the proof of (1).

(2) Let

$$h(x) := \frac{\sqrt{x}}{(\log x)^2}\left(2 - \Delta + \frac{5.12}{\log x}\right).$$

For $x \geq 10^8$ we have $A(x) \geq h(x)$ by Lemma 1.8. Also define for any prime p with $e^6 < 409 \leq p < 10^8$, a function

$$k_p(x) := \log(x)\left(A(p)\frac{(\log x)^2}{\sqrt{x}} - (2 - \Delta)\right).$$

If $p \leq x < P(p)$ then $A(x) = A(p)$ and as the sum of two decreasing functions in this range $k_p(x)$ is also decreasing and so

$$k_p(x) \geq \widetilde{k_p}(p) = \lim_{x \to P(p),\ x < P(p)} k_p(x) = \log P(p)\left(A(p)\frac{(\log P(p))^2}{\sqrt{P(p)}} - 2 + \Delta\right).$$

Evaluating numerically

$$\min_{409 \leq p < 10^8} \widetilde{k_p}(p) = \widetilde{k_{409}}(409) \geq 15.3734,$$

so for $409 \leq x < 10^8$ we have $k_p(x) \geq 15.3734$. Therefore in that range also $A(x) > h(x)$.

Finally, for $89 \leq p \leq P(401) = 409$ we check numerically that $A(p) > \max(h(p), h(P(p)))$ so $A(x) > h(x)$ in $[89, 409]$ also. This completes the proof. □

1.6 A Big Omega Theorem of Robin

First, we recall some definitions for $x > 0$:

$$\psi(x) := \sum_{j \in \mathbb{N},\ p^j \leq x} \log p,$$

$$\Pi(x) := \sum_{j\in\mathbb{N}} \frac{\pi(x^{1/j})}{j} = \sum_{p^j\le x} \frac{1}{j},$$
$$\mathrm{li}(x) := \int_0^x \frac{dt}{\log t}.$$

We derive a set of Mellin transforms of functions which we use.

Lemma 1.14 *Let* $\Re s > 1$. *Then*

$$(1)\ s\int_2^\infty \frac{\psi(x)}{x^{s+1}}\,dx = -\frac{\zeta'(s)}{\zeta(s)},$$
$$(2)\ s\int_2^\infty \frac{\pi(x)}{x^{s+1}}\,dx = \sum_{p\in\mathbb{P}} \frac{1}{p^s},$$
$$(3)\ s\int_2^\infty \frac{\Pi(x)}{x^{s+1}}\,dx = \log\zeta(s),$$
$$(4)\ s\int_2^\infty \frac{\mathrm{li}(x)}{x^{s+1}}\,dx = -\log(s-1) + g(s),$$

where $g(s)$ *is an entire function.*

Proof **(1)** Since the Dirichlet series for $\zeta(s)$ converges absolutely when $\Re s > 1$ and also

$$\frac{1}{\zeta(s)} = \sum_{n=1}^\infty \frac{\mu(n)}{n^s} \text{ and } \sum_{d|n} \Lambda(d) = \log n,$$

using the Dirichlet product we can derive

$$-\zeta'(s) = \sum_{n=1}^\infty \frac{\log n}{n^s} = \sum_{n=1}^\infty \frac{\sum_{d|n}\Lambda(d)}{n^s}$$
$$= \left(\sum_{n=1}^\infty \frac{1}{n^s}\right)\left(\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}\right) = \zeta(s)\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}.$$

Because $\psi(x) = \sum_{n\le x}\Lambda(n)$, $\psi(x) \ll x$, and $\psi(x)/x^s \to 0$ as $x\to\infty$, using Abel's theorem [3, theorem 4.2], we get

$$\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx.$$

Therefore, because $\zeta(s) \neq 0$ for $\sigma > 1$, dividing we have

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx.$$

(2) Let p_n be the nth prime with $p_1 = 2$. Then, bounding the difference between the left-hand side and the partial sum of terms on the right and then letting their number tend to infinity to get the first line, and using partial summation to get the last line, we have

$$\begin{aligned} s\int_1^\infty \frac{\pi(x)}{x^{s+1}}\, dx &= \sum_{n\in\mathbb{N}} \int_{p_n}^{p_{n+1}} \frac{sn}{x^{s+1}}\, dx \\ &= \sum_{n\in\mathbb{N}} n\left(-x^{-s}\Big|_{p_n}^{p_{n+1}}\right) \\ &= \sum_{n\in\mathbb{N}} n\left(\frac{1}{p_n^s} - \frac{1}{p_{n+1}^s}\right) = \sum_p \frac{1}{p^s}. \end{aligned}$$

(3) Taking logarithms of the Euler product representation for $\zeta(s)$, using absolute convergence of the inner sum to get the fourth equality and (2) to get the third, with $\kappa := \lfloor \log x/\log 2 \rfloor$ we have

$$\begin{aligned} \log\zeta(s) &= -\sum_p \log\left(1 - \frac{1}{p^s}\right) \\ &= \sum_{\substack{p\in\mathbb{P}\\ j\in\mathbb{N}}} \frac{1}{jp^{js}} \\ &= \sum_{j\in\mathbb{N}} \frac{1}{j}\left(\sum_p \frac{1}{p^{js}}\right) \\ &= \sum_{j\in\mathbb{N}} s\int_2^\infty \frac{\pi(y)}{y^{js+1}}\, dy \\ &= s\sum_{j=1}^{\kappa} \frac{1}{j}\int_1^\infty \frac{\pi(x^{1/j})}{x^{s+j}}\, dx \\ &= s\sum_1^\infty \frac{\sum_{j=1}^{\kappa} \frac{\pi(x^{1/j})}{j}}{x^{s+1}}\, dx \\ &= s\int_1^\infty \frac{\Pi(x)}{x^{s+1}}\, dx. \end{aligned}$$

(4) Because $|\operatorname{li}(x)| \le x$, for $x \ge 2$ the integral

$$\int_2^e \frac{\operatorname{li}(x)}{x^{s+1}}\,dx$$

is an entire function of s. Thus, we are able to simplify the working by shifting the lower limit of the integral of identity (4) to e, and then use integration by parts to get

$$s\int_e^\infty \frac{\operatorname{li}(x)}{x^{s+1}}\,dx = -\frac{\operatorname{li}(x)}{x^s}\Big|_e^\infty + \int_e^\infty \frac{1}{x^s \log x}\,dx.$$

Next, make the substitution $x^{s-1} = e^u$. Because the second integral in the second line of what follows is constant and the third entire, we get

$$\begin{aligned}\int_e^\infty \frac{1}{x^s \log x}\,dx &= \int_{s-1}^\infty \frac{e^{-u}}{u}\,du\\ &= \int_{s-1}^1 \frac{du}{u} + \int_1^\infty \frac{e^{-u}}{u}\,du + \int_{s-1}^1 \frac{e^{-u}-1}{u}\,du\\ &= -\log(s-1) + g_1(s),\end{aligned}$$

where $g_1(s)$ is entire. Therefore, there is an entire function $g(s)$ such that

$$s\int_2^\infty \frac{\operatorname{li}(x)}{x^{s+1}}\,dx = -\log(s-1) + g(s).$$

This completes the proof. □

Big omega is used to describe irregularities exhibited by a given function as x becomes unbounded. We say for $g(x) > 0$ that $f(x) = \Omega_+(g(x))$ if there is a sequence $x_n \to \infty$ and constant $c > 0$ such that $f(x_n) > cg(x_n)$ for all $n \in \mathbb{N}$. We say for $g(x) > 0$ that $f(x) = \Omega_-(g(x))$ if there is a sequence $y_n \to \infty$ and constant $c > 0$ such that $f(y_n) < -cg(x_n)$ for all $n \in \mathbb{N}$. Finally, we say $f(x) = \Omega(g(x))$ if there is a sequence $x_n \to \infty$ and constant $c > 0$ such that $|f(x_n)| > cg(x_n)$ for all $n \in \mathbb{N}$.

Lemma 1.15 **(Landau)***[29, theorem 4.12] Let $s \in \mathbb{C}$ and let $f(x) : [1,\infty) \to \mathbb{R}$ be measurable and bounded on all bounded intervals. Suppose that*

$$F(s) := \int_1^\infty f(x)\frac{dx}{x^s}$$

has a finite abscissa of convergence σ_c so $F(s) \in \mathbb{C}$ if $\Re s > \sigma_c$.

(a) If there exists an $a \in \mathbb{R}$ such that $f(x)$ is non-negative or non-positive for $x \ge a$, the integral $F(s)$ for $\sigma = \Re s > \sigma_c$ has a singularity at $s = \sigma_c$ and $F(s)$ converges in a half plane such that it is holomorphic for $\sigma > \sigma_c$ but not in any half plane $\sigma > \sigma_c - \epsilon$ for any $\epsilon > 0$.

(b) If $F(s)$ is holomorphic at $s = \sigma_c$, then $f(x)$ changes sign at all points in an infinite set x_n with $x_n \to \infty$. We also have for every $\epsilon > 0$

$$f(x) = \Omega_{\pm}\left(x^{\sigma_c - \epsilon}\right).$$

Recall $\Theta = \sup\{\Re\rho\colon \zeta(\rho) = 0\}$, and note that if *RH* is false there is an α with $\frac{1}{2} < \alpha \le \Theta \le 1$.

Theorem 1.16 **(Robin)***[203, lemma 2]*
If RH is false, then or all α with $0 < \alpha < \Theta$, we have as $x \to \infty$

$$A(x) = \mathrm{li}(\theta(x)) - \pi(x) = \Omega_{-}(x^{\alpha}).$$

Proof **(1)** In what follows, $h(s)$ denotes a function holomorphic on $\Re s > 0$ which is not always the same in every instance. First, we define

$$D(x) := \mathrm{li}(x) - \Pi(x) + \frac{\psi(x) - x}{\log x} \text{ and } J(s) := \int_2^{\infty} D(x) \frac{\log x}{x^{s+1}}\, dx.$$

Using Lemma 1.14, for $\Re s > 1$, since $(x^s)' = x^s \log x$, we have

$$\begin{aligned} J(s) &= \frac{d}{ds}\left(\frac{\log(s-1)}{s} + \frac{\log\zeta(s)}{s}\right) - \frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} + h(s) \\ &= -\frac{1}{s^2}\log\left((s-1)\zeta(s)\right) - \frac{1}{s} + h(s). \end{aligned}$$

The numerator of the first term on the right-hand side is holomorphic in a neighbourhood of $s = 1$. Define

$$K(s) = \int_2^{\infty} \frac{x^{\alpha}\log x}{x^{s+1}}\, dx = \frac{1}{(s-\alpha)^2} + h(s),$$

where $\frac{1}{2} < \alpha < \Theta$ and consider the difference

$$f_{\alpha}(s) := J(s) - K(s) = \int_2^{\infty} \frac{D(x)\log x - x^{\alpha}\log x}{x^{s+1}}\, dx.$$

Let σ_c be the abscissa of convergence of the Dirichlet integral defining $f_{\alpha}(s)$. This integral defines a single-valued branch of $f_{\alpha}(s)$ which is holomorphic in the right half plane $\Re s > \sigma_c$. Therefore this half plane does not contain a zero of $\zeta(s)$, and so all zeros must satisfy $\Re\rho \le \sigma_c$, giving $\Theta \le \sigma_c$.

In addition, because $(s-1)\zeta(s)$ is entire and strictly positive on $(0, 1]$, it has no singularities on $(\alpha, 1] \subset \mathbb{R}$, and thus $f_{\alpha}(s)$ has no singularities on $(\alpha, 1]$ either. Therefore

$$\alpha < \Theta \le \sigma_c,$$

so $s = \sigma_c$ is a regular point of $f_\alpha(s)$. Thus, by Lemma 1.15,

$$D(x)\log x - x^\alpha \log x$$

changes sign on a sequence $x_n \to \infty$. In other words $D(x) = \Omega_\pm(x^\alpha)$.

(2) Recall the definition $A_1(x) := \mathrm{li}(\psi(x)) - \Pi(x)$ and let $S(x) := x - \psi(x)$. Then using Equation (1.4) from the proof of Lemma 1.3 (which does not assume RH), we get

$$A_1(x) = D(x) + O\left(\frac{S(x)^2}{x(\log x)^2}\right).$$

If $\Theta < 1$ then, since $|x - \psi(x)| \ll x^\Theta \log x$ this estimate gives $A_1(x) = D(x) + O\left(x^{2\Theta-1}(\log x)^2\right)$, and we can choose α so $2\Theta - 1 < \alpha < 1$. In this case the lemma follows from the result of Step (1). If however $\Theta = 1$, from Equation (1.4) again, we can derive the inequality $A_1(x) < D(x)$, which by the result of Step (1) implies $A_1(x) = \Omega_-(x^\alpha)$.

(3) In this final step we show that $A_2(x)$ is suitably small. We have

$$|A_2(x)| \le |\,\mathrm{li}(\psi(x)) - \mathrm{li}(\theta(x))| + |\Pi(x) - \pi(x)|.$$

Using Chebyschev's estimate we have $\pi(x) \ll x/\log x$. Thus,

$$\Pi(x) - \pi(x) = \sum_{j=2}^{\kappa} \frac{\pi(x^{1/j})}{j} \ll \sqrt{x}\log x.$$

Also, using Equation (1.4) again with

$$h = \psi(x) - \theta(x) = \sum_{j=2}^{\kappa} \theta(x^{1/j}) \ll \sqrt{x}\log x,$$

for x sufficiently large we get

$$|\,\mathrm{li}(\psi(x)) - \mathrm{li}(\theta(x))| \ll \frac{h}{\log x} \ll \sqrt{x}.$$

Therefore $|A_2(x)| \ll \sqrt{x}\log x$ so, by the result of Step (2), we have

$$A(x) = A_1(x) + A_2(x) = \Omega_-(x^\alpha),$$

which completes the proof. □

Theorem 1.17 (Nicolas)
The Riemann hypothesis is equivalent to the relation $A(x) > 0$ for all $x \geq 11$.

Proof If RH is true, then by Lemmas 1.10(1) and 1.13(2) we get $A(x) > 0$ for all $x \geq 11$.

If RH is false, by Robin's result, Theorem 1.16, there exists $\alpha > \frac{1}{2}$ such that

$$\liminf_{x\to\infty} \frac{A(x)}{x^\alpha} < 0,$$

so $A(x_n) < 0$ for an infinite number of x_n with limit-value infinity. Thus, $A(x) > 0$ for $x \geq 11$ is false. Therefore RH is equivalent to the statement $A(x) > 0$ for all $x \geq 11$, and the proof is complete. □

1.7 End Note

Nicolas [172, theorem 1.1] also demonstrated a number of alternative equivalents to RH based on the function $A(x)$, which are relatively straightforward to demonstrate. Let

$$\Delta := \sum_\rho \frac{1}{\rho(1-\rho)}$$

(see for example [29, lemma 2.10(b)]). Then each of the following properties regarding $A(x)$ is equivalent to RH:

(1)

$$\limsup_{x\to\infty} \frac{A(x)(\log x)^2}{\sqrt{x}} \leq 2 + \Delta,$$

(2)

$$\liminf_{x\to\infty} \frac{A(x)(\log x)^2}{\sqrt{x}} \leq 2 - \Delta,$$

(3)

$$\frac{A(x)(\log x)^2}{\sqrt{x}} \geq 2 - \Delta, \ x \geq 37,$$

(4)

$$\frac{A(x)(\log x)^2}{\sqrt{x}} \leq \frac{A(x_0)(\log x_0)^2}{\sqrt{x_0}}, \ x \geq 2, \ x_0 = 3643.$$