SOME RELATIONSHIPS BETWEEN BERS AND BELTRAMI SYSTEMS AND LINEAR ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. Much work has been done in the investigation of the properties of solutions of linear elliptic systems of partial differential equations. Among these systems, the class of Beltrami systems has been studied for many years and has been shown to be of fundamental importance. Another class, perhaps of equal importance, is the class defined by Bers **(1)**, which the author has taken the liberty of calling Bers systems. Solutions of these systems will be called Beltrami and Bers functions respectively.

Gergen and Dressel **(5)**, as well as Bers himself, have shown that the Bers and Beltrami systems are in a sense the canonical forms of systems of linear elliptic equations. The purpose of this paper is to investigate further the topological and algebraic properties of collections of Bers and Beltrami functions and to show explicitly the connections between linear elliptic systems $\mathfrak{L}$ of type (2.1) having C^1 coefficients and uniquely determined Bers and Beltrami systems. Most of this work was accomplished by developing and exploiting a matrix representation for the Jacobian matrices of an elliptic system $\mathfrak{L}$ of type (2.1).

In 1954, Titus and McLaughlin **(9)** proved that if $\mathfrak{V}$ is a vector space of real 2×2 matrices with non-negative determinants and having the rank property, then $\mathfrak{V}$ is either one-dimensional and isomorphic to the real numbers or two-dimensional and equivalent to the complex numbers. This result suggested the possibility of a matrix representation of the Jacobian matrices of solutions to an elliptic system of type (2.1). In the same year, Golomb **(6)** showed that if $\mathfrak{W}$ is a real linear vector space of pseudo-regular functions in a domain $\mathfrak{D}$ which contains two functions $f = u + iv$ and $g = p + iq$ such that $v_x q_y - v_y q_x \neq 0$ in $\mathfrak{D}$, then $\mathfrak{W}$ contains only solutions of a uniquely determined elliptic system of first-order partial differential equations. In Section 3, it is shown that if $\mathfrak{V}$ consists of the Jacobian matrices of solutions to such a system defined in a domain $\mathfrak{D}$ in the complex plane, then there exists a uniquely determined matrix representation of $\mathfrak{V}$. Conversely, Theorem 3.1 shows that if $\mathfrak{V}$ is a vector space over the real numbers of Jacobian matrices with non-negative determinants and having the rank property, and if $\mathfrak{V}$ contains two linearly independent elements, then $\mathfrak{V}$ consists of Jacobian matrices of solutions to a uniquely determined elliptic system of type (2.1).

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Solutions of a system $\mathfrak{L}$ of type (2.1) are, of course, light and interior. While such functions do not in general have derivatives, Theorem 3.2 shows that if the coefficients of $\mathfrak{L}$ are C^1, one can associate with $\mathfrak{L}$ a uniquely determined system $\mathfrak{L}^*$ of type (3.18) such that if f is a solution of $\mathfrak{L}$, there corresponds a unique solution g of $\mathfrak{L}^*$ whose zeros are the critical points of f. Since Bers **(1)** has shown that the zeros of a solution of $\mathfrak{L}^*$ are isolated and have no interior limit point (in the domain $\mathfrak{D}$ of definition), it follows that the solutions of $\mathfrak{L}$ are pseudo-regular functions. In view of Bers' result, one would like to conclude that solutions of a system of type (3.18) are light. Unfortunately, one can find functions mapping $\mathfrak{D}$ into the real line which are solutions of a system $\mathfrak{L}^*$ of type (3.18).

In one of the classic theorems in topological analysis, Stöilow **(8)** proved that if f is light and interior in $\mathfrak{D}$, there exists a homeomorphism h defined in $\mathfrak{D}$ and a function g analytic in $h(\mathfrak{D})$ such that $f = g \circ h$. In general, h depends on f and one is led to wonder what conditions must be placed on two linearly independent functions f_1 and f_2 defined in $\mathfrak{D}$ to ensure that there exist two functions g_1 and g_2 analytic in $h(\mathfrak{D})$, such that $f_i = g_i \circ h$ for $i = 1, 2$. A partial answer to this question was given in 1938 by Kakutani **(7)**, who showed that a necessary and sufficient condition for a collection of pseudo-regular functions to form a ring is that they all be analytic functions of a fixed pseudo-regular function. In an earlier paper **(3)**, the author showed that such collections are algebras of solutions of a uniquely determined Beltrami system. One is led to suspect that if these conditions are relaxed somewhat, further results might be obtained. In Theorem 4.3, it is shown that if $\mathfrak{W}$ is the set of solutions of an elliptic system $\mathfrak{L}$ of type (2.1) defined in $\mathfrak{D}$, one can find a homeomorphic Beltrami function h defined in $\mathfrak{D}$ and a Bers system $\mathfrak{L}_1$ defined in $h(\mathfrak{D})$ such that if f is an element of $\mathfrak{W}$, there exists a Bers function g which is a solution of $\mathfrak{L}_1$ such that $f = g \circ h$. Conversely, Theorem 4.4 shows that if h is a homeomorphic Bers function defined in $\mathfrak{D}$, there exists a uniquely determined Beltrami system $\mathfrak{L}_1$ defined in $h(\mathfrak{D})$ such that if f is analytic in $\mathfrak{D}$, there exists a Beltrami function g, a solution of $\mathfrak{L}_1$, such that $f = g \circ h$. Furthermore, every such composition mapping is analytic in $\mathfrak{D}$. This latter theorem yields an easy method of extending many theorems about analytic functions to theorems about Beltrami functions. That this is possible is, of course, no surprise, since it is well known that solutions of a Beltrami system $\mathfrak{L}$ are analytic with respect to a Riemannian metric determined by the coefficients of $\mathfrak{L}$.

In an earlier paper **(3)**, it was shown that if $\mathfrak{L}$ is a Bers system with C^1 coefficients which has a harmonic mapping as a solution, then all solutions of $\mathfrak{L}$ are harmonic mappings and the coefficients of $\mathfrak{L}$ are harmonic conjugates. In Theorem 4.5, it is shown that if $\mathfrak{L}$ is an elliptic system of type (2.1) with C^1 coefficients which has only harmonic mappings as solutions, then $\mathfrak{L}$ is a Bers system.

The author would like to express his gratitude to C. J. Titus, to whom he is

indebted for the original idea of the matrix representation and for the statement and method of proof of Theorem 4.2.

2. Preliminary definitions. All matrices considered will be 2×2 matrices whose entries are Hölder-continuous real-valued functions defined in a domain $\mathfrak{D}$ in the plane. If J is a matrix, we shall denote the determinant of J by $|J|$. If f is a C^1 function defined in $\mathfrak{D}$, we shall denote the Jacobian matrix of f by $J(f)$.

DEFINITION 2.1. *A matrix J will be said to have the rank property if $|J| = 0$ implies that the rank of J is zero.*

DEFINITION 2.2. *A function f will be said to be pseudo-regular in $\mathfrak{D}$ if* (i) $f \in C^1$, (ii) $|J(f)| \geqslant 0$, (iii) $J(f)$ *has the rank property, and* (iv) *the set of critical points in $\mathfrak{D}$ has no interior limit point.*

Now let $\mathfrak{A}$ be the set of all matrices with non-negative determinants, $\mathfrak{B}^{\sim}$ the set of all elements of $\mathfrak{A}$ that have the rank property, $\mathfrak{B}$ the set of all Jacobian matrices in $\mathfrak{B}^{\sim}$, $\mathfrak{C}$ the set of all elements of $\mathfrak{A}$ of the form

$$\begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix},$$

and $\mathfrak{S}$ the set of all elements of $\mathfrak{A}$ of the form

$$\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \quad \text{where } \alpha > 0 \text{ in } \mathfrak{D}.$$

Let $a(x, y)$, $b(x, y)$, $c(x, y)$, and $d(x, y)$ be Hölder-continuous real-valued functions defined in $\mathfrak{D}$. A system $\mathfrak{L}$ of first-order partial differential equations

$$U_x = aV_x + bV_y, \qquad -U_y = cV_x + dV_y \tag{2.1}$$

is said to be elliptic if $4bc - (a + d)^2 > 0$, and uniformly elliptic if a, b, c, and d are uniformly bounded and there exists a positive number ϵ such that $4bc - (a + d)^2 \geqslant \epsilon$. We shall always assume that $\mathfrak{L}$ is normalized so that $b > 0$. Two special cases of elliptic systems which are of particular interest are *Bers systems*

$$U_x = aV_x + bV_y, \qquad -U_y = bV_x - aV_y \tag{2.2}$$

and *Beltrami systems*

$$U_x = aV_x + bV_y, \qquad -U_y = cV_x + aV_y, \tag{2.3}$$

where $bc - a^2 = 1$.

A function $f = u + iv$ will be said to be a solution of (2.1) if $f \in C^1$ and if the pair (u, v) satisfies (2.1). Solutions of (2.2) will be called *Bers functions* (Bers calls them "pseudo-analytic functions of the second kind") and solutions of (2.3) will be called *Beltrami functions.* If $\mathfrak{B}$ is the set of Jacobian matrices of solutions to an elliptic system $\mathfrak{L}$ of type (2.1), it is clear that $\mathfrak{V}$ forms a real linear vector space. Using Golomb's results **(6)**, it is easy to show that $\mathfrak{V}$ is a maximal real linear vector space in $\mathfrak{B}$.

3. Matrix representation. In this section, we establish the matrix representation of the set $\mathfrak{V}$ of Jacobian matrices of solutions to an elliptic system $\mathfrak{L}$ of type (2.1). This is accomplished by showing that one can find matrices S and T in $\mathfrak{S}$ depending only on the coefficients of $\mathfrak{L}$, such that $\mathfrak{V}^{\sim} = S\mathfrak{C}T$ is a maximal linear vector space in $\mathfrak{B}^{\sim}$ and $\mathfrak{V} \subset \mathfrak{V}^{\sim}$. Further, if $\mathfrak{V}^{\sim}$ is a real linear vector space in $\mathfrak{B}^{\sim}$ (containing two linearly independent elements), then $\mathfrak{V} = \mathfrak{V}^{\sim} \cap \mathfrak{B}$ consists of solutions to a uniquely determined elliptic system $\mathfrak{L}$. We shall need several lemmas. Lemma 3.1 is classical but the details of factorization are needed here.

LEMMA 3.1. *Let P be a matrix with $|P| > 0$. Then there exist unique matrices S and T in $\mathfrak{S}$ and C_1 and C_2 in $\mathfrak{C}$ such that $P = SC_1 = C_2T$.*

Proof. It will suffice to prove the existence and uniqueness of S and C_1. It will be evident that the same kind of argument would prove the existence and uniqueness of T and C_2. Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \qquad S = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \qquad \text{and } C_1 = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix},$$

where S and C_1 are to be determined. Setting $S^{-1}P = C_1$, we obtain

$$p_{11}\alpha^{-1} - p_{21}\beta = \lambda = p_{22}\alpha \tag{3.1}$$

and

$$p_{12}\alpha^{-1} - p_{22}\beta = -\mu = -p_{21}\alpha, \tag{3.2}$$

or, multiplying by α and rearranging terms,

$$p_{22}\alpha^2 + p_{21}\alpha\beta = p_{11}, \tag{3.3}$$

and

$$p_{21}\alpha^2 - p_{22}\alpha\beta = -p_{12}. \tag{3.4}$$

Solving for α^2, we obtain

$$\alpha^2 = \frac{p_{11}p_{22} - p_{12}p_{21}}{p_{22}^2 + p_{21}^2} = \frac{|P|}{p_{22}^2 + p_{21}^2}. \tag{3.5}$$

Since $|P| > 0$, at least one of the terms in the denominator is not zero. We assume that $p_{22} \neq 0$. Then,

$$\alpha = +\sqrt{\frac{|P|}{p_{22}^2 + p_{21}^2}}, \quad \beta = \frac{p_{21}\alpha + p_{12}\alpha^{-1}}{p_{22}}, \quad \lambda = p_{22}\alpha, \quad \text{and } \mu = p_{21}\alpha.$$

To show uniqueness, suppose that for some S_1 in $\mathfrak{S}$ and D in $\mathfrak{C}$,

$$S_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_1^{-1} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & -\mu_1 \\ \mu_1 & \lambda_1 \end{bmatrix};$$

then we have $P = SC_1 = S_1D$ so that $S_1^{-1}SC_1 = D$. Proceeding as in the proof of the existence of S and C_1, we obtain the equations

$$\left[\frac{\alpha}{\alpha_1} - \frac{\alpha_1}{\alpha}\right]\lambda + \left[\frac{\beta}{\alpha_1} - \frac{\beta_1}{\alpha}\right]\mu = 0 \tag{3.6}$$

and

$$-\left[\frac{\beta}{\alpha_1} - \frac{\beta_1}{\alpha}\right]\lambda + \left[\frac{\alpha}{\alpha_1} - \frac{\alpha_1}{\alpha}\right]\mu = 0. \tag{3.7}$$

A necessary condition for the existence of non-trivial solutions for λ and μ is

$$\left[\frac{\alpha}{\alpha_1} - \frac{\alpha_1}{\alpha}\right]^2 + \left[\frac{\beta}{\alpha_1} - \frac{\beta_1}{\alpha}\right]^2 = 0. \tag{3.8}$$

But (3.8) holds if and only if $\alpha = \alpha_1$ and $\beta = \beta_1$.

The following lemmas are due to J. E. McLaughlin and C. J. Titus. Since Lemma 3.3 has not been published previously, its proof is included here.

LEMMA 3.2. *Let $\mathfrak{B}^{\sim}$ be a vector space in $\mathfrak{B}^{\sim}$ which contains two linearly independent elements. Then there exists a pair of matrices, P and Q, such that $|PQ| > 0$ and $\mathfrak{B}^{\sim} = P\mathfrak{C}Q$*; cf **(9)**.

LEMMA 3.3. *Let $\mathfrak{B}^{\sim}$ be as in Lemma 2.2. Then there exists a unique pair, S and T, of elements of $\mathfrak{S}$ such that*

$$\mathfrak{B}^{\sim} = S\mathfrak{C}T.$$

Proof. Consider the matrices P and Q in Lemma 3.2. We define

$$P_1 = \begin{cases} P & \text{if } |P| > 0, \\ -P & \text{if } |P| < 0, \end{cases} \qquad Q_1 = \begin{cases} Q & \text{if } |P| > 0, \\ -Q & \text{if } |P| < 0. \end{cases}$$

Then, by Lemma 3.1, there exist for P_1 and Q_1 unique factorizations $P_1 = SC_1$, $Q_1 = C_2 T$ where C_1 and C_2 are in $\mathfrak{C}$, and S and T are in $\mathfrak{S}$. Since $\mathfrak{C}$ is a ring, we have $\mathfrak{B}^{\sim} = P\mathfrak{C}Q = SC_1\mathfrak{C}C_2T = S\mathfrak{C}T$. To show uniqueness, we suppose there exist matrices S_1, S_2, T_1, and T_2 in $\mathfrak{S}$ such that $S_1\mathfrak{C}T_1 = S_2\mathfrak{C}T_2$. Then, letting $S_0 = S_1^{-1}S_2$ and $T_0 = T_2T_1^{-1}$, $\mathfrak{C} = S_0\mathfrak{C}T_0$. Therefore, for C_1 in $\mathfrak{C}$, there exists C_2 in $\mathfrak{C}$ such that $C_1 = S_0C_2T_0$. But $C_i (i = 1, 2)$ may be expressed in the form $C_i = \lambda_i I + \mu_i K$, where I is the identity matrix and

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since we may pick C_1 so that either $\lambda_1 = 0$ or $\mu_1 = 0$, it follows that S_0T_0 and S_0KT_0 are in $\mathfrak{C}$; hence

$$(S_0T_0)(S_0KT_0)^{-1} = S_0KS_0^{-1} \in \mathfrak{C} \quad \text{and} \quad (S_0T_0)^{-1}(S_0KT_0) = T_0^{-1}KT_0 \in \mathfrak{C}.$$

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It will be sufficient to show that S_0 must be the identity matrix. The same argument can then be applied to show that T_0 must also be the identity matrix. Let $S_0 K S_0^{-1} = D$, where

$$S_0 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}.$$

Then

$$S_0 K S_0^{-1} = \begin{bmatrix} \beta\alpha^{-1} & -\alpha^2 - \beta^2 \\ \alpha^{-2} & \beta\alpha^{-1} \end{bmatrix} = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}.$$

From $\beta\alpha^{-1} = -\beta\alpha^{-1}$ and $\alpha^{-2} = \alpha^2 + \beta^2$, it follows that $\beta = 0$ and $\alpha = 1$ so that $S_0 = I$.

THEOREM 3.1. *Let $\mathfrak{V}$ be a vector space of elements of $\mathfrak{B}$ such that $\mathfrak{V}$ contains two everywhere linearly independent elements. Then $\mathfrak{V}$ consists of the Jacobian matrices of solutions to an elliptic system $\mathfrak{L}$ of type* (2.1).

Proof. The proof will consist of the construction of the desired elliptic system. By Lemma 3.3, there exist matrices S and T in $\mathfrak{S}$ such that $\mathfrak{V} \subseteq S\mathfrak{C}T$. Let

$$S = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \qquad T = \begin{bmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \end{bmatrix}, \qquad \text{and let } V = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \in \mathfrak{V}.$$

Then there exists C in $\mathfrak{C}$,

$$C = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix},$$

such that

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix} \begin{bmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \gamma\alpha\lambda + \gamma\beta\mu & \left(\alpha\delta + \dfrac{\beta}{\gamma}\right)\lambda + \left(\beta\delta - \dfrac{\alpha}{\gamma}\right)\mu \\ \dfrac{\gamma\mu}{\alpha} & \dfrac{\lambda}{\alpha\gamma} + \dfrac{\delta\mu}{\alpha} \end{bmatrix},$$

so that

$$u_x = \gamma\alpha\lambda + \gamma\beta\mu, \tag{3.9}$$

$$u_y = \left(\alpha\delta + \frac{\beta}{\gamma}\right)\lambda + \left(\beta\delta - \frac{\alpha}{\gamma}\right)\mu, \tag{3.10}$$

$$v_x = \frac{\gamma\mu}{\alpha}, \tag{3.11}$$

and

$$v_y = \frac{\lambda}{\alpha\gamma} + \frac{\delta\mu}{\alpha}. \tag{3.12}$$

Solving (3.11) and (3.12) for λ and μ, we obtain

$$\lambda = -\delta\alpha v_x + \gamma\alpha v_y \tag{3.13}$$

and

$$\mu = (\alpha/\gamma)v_x. \tag{3.14}$$

Substituting into (3.9) and (3.10), we have

$$u_x = \alpha(\beta - \alpha\gamma\delta)v_x + \alpha^2\gamma^2 v_y, \tag{3.15}$$

$$-u_y = \alpha^2(\delta^2 + 1/\gamma^2)v_x - \alpha(\beta + \alpha\gamma\delta)v_y. \tag{3.16}$$

If we set

$$a = \alpha(\beta - \alpha\gamma\delta), \quad b = \alpha^2\gamma^2, \quad c = \alpha^2(\delta^2 + \gamma^{-2}), \quad \text{and} \quad d = -\alpha(\beta + \alpha\gamma\delta),$$

then $4bc - (a + d)^2 = 4\alpha^4 > 0$.

Note that a, b, c, and d depend only on the elements of S and T. Further, if $T = I$, then $\gamma = 1$ and $\delta = 0$, so that in this case, $b = c$ and $a = -d$, and the system of partial differential equations thus determined is a Bers system. On the other hand, if $S = I$, then $\alpha = 1$ and $\beta = 0$, and in this case, $a = d$ and $bc - a^2 \equiv 1$ so that the system of equations becomes a Beltrami system. Conversely, if $\mathfrak{B}$ is the set of Jacobian matrices of solutions of (2.1), a simple computation yields the elements of S and T as functions of the coefficients of the system of partial differential equations. If a, b, c, and d are the coefficients of an elliptic system $\mathfrak{L}$ of type (2.1) and α, β, γ, and δ are the corresponding elements of S and T, it is obvious that the continuity and differentiability properties possessed by *all* the functions a, b, c, and d are also possessed by α, β, γ, and δ. It is easy to show that the converse also holds. Finally, any element of $\mathfrak{S}$ determines two distinct elliptic systems, one Beltrami and one Bers.

Titus and McLaughlin **(9)** have shown that $\mathfrak{C}$ is a maximal real linear vector space in $\mathfrak{B}^\sim$ and that if $\mathfrak{W}$ is any real linear vector space in $\mathfrak{B}^\sim$, then $\mathfrak{W}$ is either one-dimensional and equivalent to the field of real numbers or two-dimensional and isomorphic to $\mathfrak{C}$. It follows that if $\mathfrak{B}^\sim = S\mathfrak{C}T$, for S and T in $\mathfrak{S}$, then $\mathfrak{B}^\sim$ is maximal in $\mathfrak{B}^\sim$ and $\mathfrak{B} = \mathfrak{B}^\sim \cap \mathfrak{B}$ is maximal in $\mathfrak{B}$.

For S and T determined by an elliptic system $\mathfrak{L}$ of type (2.1), it is clear that there exist elements C in $\mathfrak{C}$ such that SCT is not a Jacobian matrix and therefore does not correspond to a solution of $\mathfrak{L}$. If

$$J = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix} \begin{bmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \end{bmatrix} = SCT,$$

a sufficient condition for J to be a Jacobian matrix is that

$$\partial u_1/\partial y = \partial u_2/\partial x \quad \text{and} \quad \partial v_1/\partial y = \partial v_2/\partial x.$$

We use this to impose conditions on C. An easy computation shows that if the pair (λ, μ) satisfy the system

$$
(3.17)\qquad
\begin{aligned}
\partial(\gamma\alpha\lambda + \gamma\beta\mu)/\partial y &= \partial[(\alpha\delta + \beta\gamma^{-1})\lambda + (\beta\delta - \alpha\gamma^{-1})\mu]/\partial x,\\
\partial(\gamma\alpha^{-1}\mu)/\partial y &= \partial[\alpha^{-1}\gamma^{-1}\lambda + \delta\alpha^{-1}\mu]/\partial x,
\end{aligned}
$$

then $J = SCT$ is a Jacobian matrix. For the sake of simplicity, we shall assume that the elements of S and T, hence the coefficients of the corresponding elliptic system $\mathfrak{L}$, have partial derivatives (at least in the L_2 sense). We are led to the following theorem.

THEOREM 3.2. *Let S and T be elements of $\mathfrak{S}$,*

$$S = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \qquad T = \begin{bmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \end{bmatrix},$$

and let $\mathfrak{L}$ be the corresponding elliptic system of type (2.1). *Then there exists a corresponding elliptic system $\mathfrak{L}^*$ of the form*

$$
(3.18)\qquad
\begin{aligned}
\lambda_x &= -\gamma\delta\mu_x + \gamma^2\mu_y + A\lambda + B\mu,\\
-\lambda_y &= (\delta^2 + \gamma^{-2})\mu_x - \gamma\delta\mu_y + C\lambda + D\mu,
\end{aligned}
$$

where A, B, C, and D are rational functions of α, β, γ, δ, and their partial derivatives, such that if f is a solution of $\mathfrak{L}$, there exists a unique solution f^* of $\mathfrak{L}^*$ such that the zeros of f^* are precisely the critical points of f. Conversely, if f^* is a solution of $\mathfrak{L}^*$, f^* determines a solution f of $\mathfrak{L}$ uniquely (*up to an additive constant*).

Proof. $\mathfrak{L}^*$ is obtained by simply carrying out the indicated differentiations in eq. (3.17) and solving for λ_x and λ_y. The computations are straightforward but very tedious and will be omitted. One obtains

$$
\begin{aligned}
A &= \alpha^{-1}\alpha_x + \gamma^{-1}\gamma_x,\\
B &= \gamma\gamma_y - \alpha^{-1}\gamma^2\alpha_y - \gamma\delta_x + \gamma\delta\alpha^{-1}\alpha_x,\\
C &= \alpha^{-1}\alpha_y + \gamma^{-1}\gamma_y - 2\delta\gamma^{-1}\alpha^{-1}\alpha_x - \gamma^{-1}\delta_x - \alpha^{-1}\gamma^{-2}\beta_x - \delta\gamma^{-2}\gamma_x - \beta\gamma^{-2}\alpha^{-2}\alpha_x,\\
D &= \alpha^{-1}\beta_y - \alpha^{-1}\gamma^{-1}\delta\beta_x + \gamma^{-2}\alpha^{-1}\alpha_x - \gamma^{-3}\gamma_x - \delta\gamma_y + \delta\gamma\alpha^{-1}\alpha_y\\
&\qquad + \delta\delta_x - \delta^2\alpha^{-1}\alpha_x + \beta\alpha^{-1}\alpha_y - \beta\delta\gamma^{-1}\alpha^{-2}\alpha_x.
\end{aligned}
$$

If $f = u + iv$ and $f^* = \lambda + i\mu$ are corresponding solutions of $\mathfrak{L}$ and $\mathfrak{L}^*$ respectively, it is obvious that the zeros of f^* are precisely the critical points of f.

Elliptic systems of type (3.18) have been studied by Bers and Nirenberg **(2)**. In particular, they have shown that if f^* is a solution of a uniformly elliptic system $\mathfrak{L}^*$ of type (3.18), f^* not identically zero, then the zeros of f^* are isolated and the index of f^* at each zero is positive. Furthermore, f^* is completely determined by its values on any infinite set of points having a limit point in $\mathfrak{T}$.

Note that if $\mathfrak{L}$ is a Beltrami system with constant coefficients, $\mathfrak{L} = \mathfrak{L}^*$. If $\mathfrak{L}$ is a Bers system, $\mathfrak{L}^*$ is of the form

$$(3.19)\qquad \lambda_x = \mu_y + A\lambda + B\mu, \qquad -\lambda_y = \mu_x + C\lambda + D\mu,$$

and if the coefficients of the Bers system are constants, $\mathfrak{L}^*$ is just the Cauchy–Riemann equations. Systems of the form (3.19) were studied by Carleman **(4)**.

4. Some consequences. From the remarks of the preceding section, it follows easily that if $\mathfrak{L}$ is an elliptic system of type (2.1) such that the corresponding system $\mathfrak{L}^*$ has uniformly bounded coefficients on every compact subset of $\mathfrak{D}$, solutions of $\mathfrak{L}$ are pseudo-regular. In an earlier paper **(3)**, it was shown that if $\mathfrak{W}$ is a collection of pseudo-regular functions containing two linearly independent functions such that for f and g in $\mathfrak{W}$, $\xi f + \eta g$ is in $\mathfrak{W}$ for arbitrary complex numbers ξ and η, then $\mathfrak{W}$ consists of solutions to some Beltrami system. One cannot expect so strong a result in the more general systems of type (3.18). The condition that the set $\mathfrak{W}^*$ of solutions to an elliptic system $\mathfrak{L}^*$ of type (3.18) form a vector space over the complex numbers may be shown to be equivalent to requiring that α and β satisfy a system of two non-linear first-order equations. We can, however, prove a weaker theorem.

THEOREM 4.1. *Let S and T be elements of $\mathfrak{S}$ and let $\mathfrak{L}^*$ be the corresponding elliptic system of type* (3.18). *If S is a constant matrix, the solutions of $\mathfrak{L}^*$ form a vector space over the complex numbers.*

Proof. We need only show that if S is a constant matrix and $\lambda + i\mu$ is a solution of $\mathfrak{L}^*$, then $-\mu + i\lambda = i(\lambda + i\mu)$ is also a solution. Let S be a constant matrix. Then $\mathfrak{L}^*$ is of the form

$$\begin{aligned} \lambda_x &= -\gamma\delta\mu_x + \gamma^2\mu_y + \gamma^{-1}\gamma_x\,\lambda + \gamma(\gamma_y - \delta_x)\mu, \\ -\lambda_y &= (\delta^2 + \gamma^{-2})\mu_x - \gamma\delta\mu_y + \gamma^{-2}(\gamma\gamma_y - \gamma\delta_x - \delta\gamma_x)\lambda \\ &\qquad - (\gamma^{-3}\gamma_x + \delta\gamma_y - \delta\delta_x)\mu. \end{aligned} \tag{4.1}$$

Solving for μ_x and μ_y in (4.1), we obtain

$$\begin{aligned} -\mu_x &= -\gamma\delta\lambda_x + \gamma^2\lambda_y - \gamma^{-1}\gamma_x\,\mu + \gamma(\gamma_y - \delta_x)\lambda, \\ \mu_y &= (\delta^2 + \gamma^{-2})\lambda_x - \gamma\delta\lambda_y - \gamma^{-2}(\gamma\gamma_y - \gamma\delta_x - \delta\gamma_x)\mu \\ &\qquad - (\gamma^{-3}\gamma_x + \delta\gamma_y - \delta\delta_x)\lambda. \end{aligned} \tag{4.2}$$

Therefore, $-\mu + i\lambda$ is a solution of $\mathfrak{L}^*$.

If $f = u + iv$ is a solution of an elliptic system of type (2.1), it is well known that f is quasi-conformal a.e. and the dilatation D of f is given by

$$E(f) = \frac{{u_x}^2 + {u_y}^2 + {v_x}^2 + {v_y}^2}{u_x\, v_y - u_y\, v_x} = \frac{||J(f)||}{|J(f)|} = D + \frac{1}{D}.$$

In general, $E(f)$ depends on f and cannot be expressed solely as a function of the coefficients of $\mathfrak{L}$. A simple computation, however, shows that if $\mathfrak{L}$ is either a Bers system or a Beltrami system, $E(f)$ depends only on the coefficients of $\mathfrak{L}$. The following theorem shows that these systems are the only ones with this property.

THEOREM 4.2. *Let $\mathfrak{W}$ be the set of solutions of an elliptic system $\mathfrak{L}$ of type* (2.1).

$\mathfrak{V}$ the set of Jacobian matrices of elements of $\mathfrak{W}$, and let S and T be the elements of $\mathfrak{S}$ such that $\mathfrak{V} \subset S\mathfrak{C}T$. For $f \in \mathfrak{W}$, $E(f)$ depends only on the coefficients of $\mathfrak{L}$ if and only if at least one of the matrices S and T is the identity matrix.

Proof. Let $J(f) = SCT$, where

$$S = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \qquad C = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}, \qquad \text{and } T = \begin{bmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \end{bmatrix}.$$

Then

$$E(f) = \frac{||SCT||}{|SCT|} = \frac{||SCT||}{\lambda^2 + \mu^2}$$

and

$$\begin{aligned} ||SCT|| = {} & (\gamma^2\alpha^2 + \alpha^2\delta^2 + 2\alpha\beta\delta\gamma^{-1} + \beta^2\gamma^{-2} + \alpha^{-2}\gamma^{-2})\lambda^2 \\ & + 2(\alpha\beta\gamma^2 + \alpha\beta\delta^2 - \alpha^2\delta^2\gamma^{-1} + \beta^2\delta\gamma^{-1} - \alpha\beta\gamma^{-2} + \delta\alpha^{-2}\gamma^{-1})\lambda\mu \\ & + (\beta^2\gamma^2 + \beta^2\delta^2 - 2\alpha\beta\delta\gamma^{-1} + \alpha^2\gamma^{-2} + \gamma^2\alpha^{-2} + \delta^2\alpha^{-2})\mu^2. \end{aligned}$$

In order for $E(f)$ to depend only on S and T, the coefficient of the $\lambda\mu$ term must vanish and the coefficient of the λ^2 term must equal that of the μ^2 term. These conditions are equivalent to the equations

$$\alpha^2\beta(\gamma^4 + \gamma^2\delta^2 - 1) = \gamma\delta(\alpha^4 - \alpha^2\beta^2 - 1), \tag{4.3}$$

$$(\alpha^4 - \alpha^2\beta^2 - 1)(\gamma^4 + \gamma^2\delta^2 - 1) + 4\alpha^3\beta\gamma\delta = 0. \tag{4.4}$$

Suppose $\beta \neq 0$. Then from (4.3) and (4.4)

$$\gamma\delta\alpha^{-3}\beta^{-1}[(\alpha^4 - \alpha^2\beta^2 - 1)^2 + 4\alpha^6\beta^2] = 0, \tag{4.5}$$

so that we must have $\delta = 0$. If $\delta = 0$, it follows from (4.3) that $\gamma = 1$ so $T = I$, and $\mathfrak{L}$ is a Bers system. Conversely, if $\beta = 0$, $\alpha = 1$ and $S = I$ so that $\mathfrak{L}$ is a Beltrami system.

If $\mathfrak{D}$ and $\mathfrak{D}_1$ are topologically equivalent domains and h is a homeomorphism of $\mathfrak{D}$ onto $\mathfrak{D}_1$, then for f defined in $\mathfrak{D}$, h induces a function $\tilde{f}$ in $\mathfrak{D}_1$, $\tilde{f} = f \circ h^{-1}$. It follows that if h is C^1 and $\mathfrak{L}$ is an elliptic system defined in $\mathfrak{D}$, h induces an elliptic system $\mathfrak{L}^\sim$ in $\mathfrak{D}_1$. Furthermore, if $\mathfrak{W}$ is the set of solutions of $\mathfrak{L}$, h maps $\mathfrak{W}$ into a collection $\mathfrak{W}_1$ of light interior functions defined in $\mathfrak{D}_1$. (It is not true, in general, that $\mathfrak{W}_1$ will consist of solutions to $\mathfrak{L}^\sim$. In an earlier paper **(3)**, it was shown that if h is conformal, a necessary and sufficient condition for $\mathfrak{W}_1$ to be the set of solutions to $\mathfrak{L}^\sim$ is that $\mathfrak{L}^\sim$ (hence $\mathfrak{L}$) be a Bers system.) These considerations, together with the matrix representation concept, suggest the following factorization theorems.

THEOREM 4.3. *Let $\mathfrak{L}$ be an elliptic system of type* (2.1), *$\mathfrak{W}$ the set of solutions of $\mathfrak{L}$, and S and T the corresponding elements of $\mathfrak{S}$. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be the Bers and Beltrami systems corresponding to S and T respectively, and let h be a univalent solution of $\mathfrak{L}_2$. If $\mathfrak{L}_1$ is the Bers system induced in $\mathfrak{D}_1 = h(\mathfrak{D})$ by the Beltrami*

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function h, then for $f \in \mathfrak{W}$ there exists a Bers function g satisfying $\mathfrak{L}^{\sim}{}_1$ such that $f = g \circ h$.

Proof. In view of the matrix representation, we can find functions α, β, γ, and δ such that $\mathfrak{L}$ is of the form

$$(4.6) \qquad \begin{aligned} U_x &= \alpha(\beta - \alpha\gamma\delta) V_x + \alpha^2\gamma^2 V_y, \\ -U_y &= \alpha^2(\delta^2 + \gamma^{-2}) V_x - \alpha(\beta + \alpha\gamma\delta) V_y. \end{aligned}$$

Then $\mathfrak{L}_1$ is of the form

$$(4.7) \qquad U_x = \alpha\beta V_x + \alpha^2 V_y, \qquad -U_y = \alpha^2 V_x - \alpha\beta V_y,$$

and $\mathfrak{L}_2$ is of the form

$$(4.8) \qquad U_x = -\gamma\delta V_x + \gamma^2 V_y, \qquad -U_y = (\delta^2 + \gamma^{-2}) V_x - \gamma\delta V_y.$$

Let $f = u + iv$ be a solution of $\mathfrak{L}$ and let $h = p + iq$ be a solution of $\mathfrak{L}_2$. Then $\mathfrak{L}^{\sim}{}_1$ is of the form

$$(4.9) \qquad \phi_p = \tilde{\alpha}\tilde{\beta}\psi_p + \tilde{\alpha}^2\psi_q, \qquad -\phi_q = \tilde{\alpha}^2\psi_p - \tilde{\alpha}\tilde{\beta}\psi_q,$$

where $\tilde{\alpha} = \alpha \circ h^{-1}$ and $\tilde{\beta} = \beta \circ h^{-1}$. The proof of the theorem will be accomplished by showing that if $g(p, q) = r + is$ is defined by $g = f \circ h^{-1}$, then g is a solution of $\mathfrak{L}^{\sim}{}_1$. For g so defined,

$$r(p(x, y), q(x, y)) = u(x, y) \quad \text{and} \quad s(p(x, y), q(x, y)) = v(x, y).$$

Using the chain rule,

$$r_p p_x + r_q q_x = u_x, \quad r_p p_y + r_q q_y = u_y, \quad s_p p_x + s_q q_x = v_x,$$

and

$$s_p p_y + s_q q_y = v_y.$$

Since $h = p + iq$ is a univalent solution of $\mathfrak{L}_2$, $p_x q_y - p_y q_x \neq 0$ and we can solve for r_p and r_q. We have

$$(4.10) \qquad r_p = (p_x q_y - p_y q_x)^{-1}(u_x q_y - u_y q_x);$$

and since $u + iv$ is a solution of $\mathfrak{L}$,

$$\begin{aligned} (4.11) \quad r_p &= (p_x q_y - p_y q_x)^{-1}\{[\alpha(\beta - \alpha\gamma\delta)v_x + \alpha^2\gamma^2 v_y]q_y - [\alpha^2(\delta^2 + \gamma^{-2})v_x \\ &\qquad - \alpha(\beta + \alpha\gamma\delta)v_y]q_x\} \\ &= (p_x q_y - p_y q_x)^{-1}\{[\alpha(\beta - \gamma\alpha\delta)q_y - \alpha^2(\delta^2 + \gamma^{-2})q_x]v_x \\ &\qquad + [\alpha^2\gamma^2 q_y + \alpha(\beta + \alpha\gamma\delta)q_x]v_y\}. \end{aligned}$$

Substituting for v_x and v_y, the term in brackets becomes

$$\begin{aligned} (4.12) \quad & [\alpha(\beta - \alpha\gamma\delta)q_y + \alpha^2(\delta^2 + \gamma^{-2})q_x](s_p p_x + s_q q_x) \\ &\qquad + [\alpha^2\gamma^2 q_y - \alpha(\beta + \alpha\gamma\delta)q_x](s_p p_y + s_q q_y). \end{aligned}$$

Since $p + iq$ is a solution of $\mathfrak{L}_2$,

$$p_x q_y - p_y q_x = (\delta^2 + \gamma^{-2})q_x^2 - 2\gamma\delta q_x q_y + \gamma^2 q_y^2,$$

and (4.12) reduces to

$$(4.13) \qquad (p_x q_y - p_y q_x)(\alpha\beta s_p + \alpha^2 s_q).$$

Noting that $\tilde{\alpha}(p, q) = \alpha(x, y)$ and $\tilde{\beta}(p, q) = \beta(x, y)$, we obtain

$$(4.14) \qquad r_p = \tilde{\alpha}\tilde{\beta}s_p + \tilde{\alpha}^2 s_q.$$

Using the same procedure, we also obtain

$$(4.15) \qquad -r_q = \tilde{\alpha}^2 s_p - \tilde{\alpha}\tilde{\beta}s_q.$$

Therefore $r + is$ is a solution of $\mathfrak{L}^{\sim}{}_1$, and this completes the proof.

If $\mathfrak{L}$ in the theorem above consists of the Cauchy–Riemann equations, then $\mathfrak{L}_1$ and $\mathfrak{L}_2$ will also consist of the Cauchy–Riemann equations and in this case the theorem is trivial. One can, however, relate analytic functions with Bers and Beltrami functions.

THEOREM 4.4. *Let $\mathfrak{L}$ be a Bers system defined in $\mathfrak{D}$ and let h be a univalent solution of $\mathfrak{L}$. Then there exists a uniquely determined Beltrami system $\mathfrak{L}_1$ defined in $\mathfrak{D}_1 = h(\mathfrak{D})$ such that if f is analytic in $\mathfrak{D}$, there exists a Beltrami function g satisfying $\mathfrak{L}_1$, and such that $f = g \circ h$. Conversely, if g is any solution of $\mathfrak{L}_1$, $g \circ h$ is analytic in $\mathfrak{D}$.*

Proof. Let $\mathfrak{L}$ be the Bers system

$$U_x = \alpha\beta V_x + \alpha^2 V_y, \qquad -U_y = \alpha^2 V_x - \alpha\beta V_y, \qquad \alpha > 0,$$

and let $h = p + iq$ be a homeomorphic solution of $\mathfrak{L}$. Define the functions γ and δ in $\mathfrak{D}_1 = h(\mathfrak{D})$ by

$$\gamma(p, q) = 1/[\alpha(x(p, q), y(p, q))], \qquad \delta(p, q) = -\beta(x(p, q), y(p, q))$$

and let $\mathfrak{L}_1$ be the Beltrami system

$$U_p = \gamma\delta V_p + \gamma^2 V_q, \qquad -U_q = (\delta^2 + 1/\gamma^2) V_p - \gamma\delta V_q.$$

If g is a solution of $\mathfrak{L}_1$, it is easy to verify that the composite function $f = g \circ h$ is analytic in $\mathfrak{D}$. Conversely, if f is analytic in $\mathfrak{D}$ and we define $g = f \circ h^{-1}$, a simple computation similar to that in the proof of the preceding theorem shows that g is a solution of $\mathfrak{L}_1$.

The following corollary is known but is included for the sake of completeness.

COROLLARY 4.1. *If h is a univalent Bers function, h^{-1} is a Beltrami function.*

Proof. In the preceding theorem, choose f to be the identity mapping.

Note that Theorem 4.3 may be applied to the problem of mapping a second-order elliptic equation into canonical form. Let A, B, and C be real-valued C^1 functions in $\mathfrak{D}$ such that $AC - B^2 > 0$, $A > 0$. If we define functions α, γ, and δ by $\alpha^4 = AC - B^2$, $\alpha^2\gamma^2 = A$, $-2\alpha^2\gamma\delta = B$, and $\alpha^2(\delta^2 + \gamma^{-2}) = C$, it is easy to verify that these functions are well defined, provided we pick γ to be positive. A simple computation shows that if $h = p + iq$ is a homeomorphic solution of the Beltrami system

$$U_x = -\gamma\delta V_x + \gamma^2 V_y, \qquad -U_y = (\delta^2 + 1/\gamma^2)V_x - \gamma\delta V_y,$$

h maps the elliptic equation

$$C\phi_{xx} + 2B\phi_{xy} + A\phi_{yy} + D\phi_x + E\phi_y + F\phi = 0 \tag{4.16}$$

into the form

$$\psi_{pp} + \psi_{qq} + H\psi_p + K\psi_q + L\psi = 0, \tag{4.17}$$

and if D, E, and F are bounded and continuous, and α^2 and the Jacobian determinant of h are bounded away from zero, then H, K, and L are bounded and continuous in $h(\mathfrak{D})$.

In an earlier paper **(3)**, it was shown that if $\mathfrak{L}$ is a Bers system with C^1 coefficients σ and τ which has a harmonic mapping as a solution, then all solutions of $\mathfrak{L}$ are harmonic and $\tau + i\sigma$ is analytic. We want to show that if $\mathfrak{L}$ is an elliptic system of type (2.1) such that all solutions are harmonic, then $\mathfrak{L}$ is a Bers system. Before proceeding, however, a few preliminary remarks are necessary. If $\mathfrak{L}$ is an elliptic system of type (2.1), it follows from the extended Riemann mapping theorem **(2)** that $\mathfrak{L}$ has as many linearly independent solutions as we want. Linear independence of two solutions $f = u + iv$ and $g = p + iq$ does not, however, preclude the possibility that at some point z_0 in $\mathfrak{D}$, $v_x q_y - v_y q_x = 0$. One can show that a necessary and sufficient condition for the Jacobian of some real linear combination $\alpha f + \beta q$ to vanish at z_0 is that $v_x q_y - v_y q_x = 0$ at that point. If $\mathfrak{W}$ is the set of all functions analytic in $\mathfrak{D}_1$, it is easy to show that for $z_0 \in \mathfrak{D}$ one can find $f = u + iv$ and $g = p + iq$ in $\mathfrak{W}$ such that $v_x q_y - v_y q_x \neq 0$ at $z = z_0$. (It follows easily that the same statement is true for Beltrami functions.) I have been unable to prove the theorem for the general case where $\mathfrak{W}$ consists of the solutions to an elliptic system $\mathfrak{L}$ of type (2.1). The following lemma, however, is an immediate consequence of the remarks on analytic functions.

LEMMA 4.1. *Let $\mathfrak{D}$ be a simply connected domain and let $\mathfrak{L}$ be an elliptic system of type* (2.1) *defined in $\mathfrak{D}$ and such that all solutions of $\mathfrak{L}$ are harmonic. Then for z_0 in $\mathfrak{D}$ there exist solutions $f = u + iv$ and $g = p + iq$ such that $v_x q_y - v_y q_x \neq 0$ at $z = z_0$.*

The proof is trivial and will be omitted.

LEMMA 4.2. *Let $\mathfrak{D}$ be a simply connected domain and let $\mathfrak{L}$ be an elliptic system of type* (2.1) *with C^1 coefficients and such that all solutions of $\mathfrak{L}$ are harmonic mappings. Let $\mathfrak{L}^*$ be the corresponding system of type* (3.18). *Then for $z_0 \in \mathfrak{D}$ and $\epsilon > 0$ there exists a solution $\lambda + i\mu$ of $\mathfrak{L}^*$ and a point z_1 in $N(z_0, \epsilon)$ (the ϵ-neighbourhood of z_0) such that z_1 is a zero of $\lambda + i\mu$ but is not a critical point.*

Proof. Let $f = \tilde{u} + i\tilde{v}$, $g = u + iv$, and $h = p + iq$ be linearly independent solutions of $\mathfrak{L}$ such that $v_x q_y - v_y q_x \neq 0$ at $z = z_0$, and choose $\delta \leqslant \epsilon$ such that $v_x q_y - v_y q_x \neq 0$ in $N(z_0, \delta)$. At $z = z_0$, the equations $\tilde{v}_x = lv_x + mq_x$, $\tilde{v}_y = lv_y + mq_y$ uniquely determine l and m so that the function $F = f - lg - mh$ has a critical point at z_0, and the corresponding solution of $\mathfrak{L}^*$, $\lambda + i\mu$, has a zero at z_0. We may, however, have the unhappy situation that at z_0 we also have $\tilde{v}_{xx} = lv_{xx} + mq_{xx}$ and $\tilde{v}_{yy} = lv_{yy} + mq_{yy}$. In this case, $\lambda + i\mu$ will also have a critical point at z_0. Define functions $\tilde{l}$ and $\tilde{m}$ in $N(z_0, \delta)$ by the equations $\tilde{v}_x = \tilde{l}v_x + \tilde{m}q_x$, $\tilde{v}_y = \tilde{l}v_y + \tilde{m}q_y$. Since solutions of $\mathfrak{L}$ are at least C^2, $\tilde{l}$ and $\tilde{m}$ are at least C^1. It is trivial to verify that $\tilde{l}$ and $\tilde{m}$ also satisfy the equations $\tilde{u}_x = \tilde{l}u_x + \tilde{m}p_x$ and $\tilde{u}_y = \tilde{l}u_y + \tilde{m}p_y$. If at some point in $N(z_0, \delta)$ the equations

$$\tilde{v}_{xx} = \tilde{l}v_{xx} + \tilde{m}q_{xx}, \quad \tilde{v}_{yy} = \tilde{l}v_{yy} + \tilde{m}q_{yy}, \quad \tilde{u}_{xx} = \tilde{l}u_{xx} + \tilde{m}p_{xx},$$

and

$$\tilde{u}_{yy} = \tilde{l}u_{yy} + \tilde{m}p_{yy}$$

also hold, then at this point

$$\tilde{l}_x v_x + \tilde{m}_x q_x = 0, \quad \tilde{l}_x u_x + \tilde{m}_x p_x = 0, \quad \tilde{l}_y v_y + \tilde{m}_y q_y = 0,$$

and

$$\tilde{l}_y u_y + \tilde{m}_y p_y = 0,$$

and it follows that at this point $\tilde{l}_x = \tilde{l}_y = \tilde{m}_x = \tilde{m}_y = 0$. But if this happens at every point in $N(z_0, \delta)$, we must have $\tilde{l}$ and $\tilde{m}$ constant, and this contradicts the assumption that f, g, and h are linearly independent. Let z_1 be a point in $N(z_0, \delta)$ such that the above equations do not hold at $z = z_1$. Then for $l = \tilde{l}(z_1)$ and $m = \tilde{m}(z_1)$, the solution of $\mathfrak{L}^*$ corresponding to $F = f - lg - mh$ has a zero at z_1 but does not have a critical point at that point.

THEOREM 4.5. *Let $\mathfrak{D}$ and $\mathfrak{L}$ be as in Lemma* 4.2. *Then $\mathfrak{L}$ is a Bers system.*

Proof. Let S and T be the elements of $\mathfrak{S}$ determined by the coefficients of $\mathfrak{L}$. For f a solution of $\mathfrak{L}$, let $\lambda + i\mu$ be the function determined by $J(f) = SCT$. As we have already seen, $\lambda + i\mu$ must satisfy the system

$$\begin{aligned} \lambda_x &= -\gamma\delta\mu_x + \gamma^2\mu_y + A\lambda + B\mu, \\ -\lambda_y &= (\delta^2 + \gamma^{-2})\mu_x - \gamma\delta\mu_y + C\lambda + D\mu. \end{aligned} \tag{4.18}$$

It is easy to verify that since f is harmonic, $\lambda + i\mu$ must also satisfy the system

$$\begin{aligned} \lambda_x &= \gamma\delta\mu_x + (\delta^2 + \gamma^{-2})\mu_x + \tilde{A}\lambda + \tilde{B}\mu, \\ -\lambda_y &= \gamma^2\mu_x + \gamma\delta\mu_y + \tilde{C}\lambda + \tilde{D}\mu, \end{aligned} \tag{4.19}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ are continuous rational functions of α, β, γ, δ, and their partial derivatives. If z_1 is a zero but not a critical point of $\lambda + i\mu$, (4.18) and (4.19) require that $\delta = 0$ and $\gamma = 1$ at z_1. The continuity of γ and δ and Lemma 4.2 then ensure that $\delta \equiv 0$ and $\gamma \equiv 1$ so T is the identity and $\mathfrak{L}$ is a Bers system.

Note that if $\mathfrak{L}$ is a Bers system whose solutions are harmonic mappings, the associated system $\mathfrak{L}^*$ is of the form

$$(4.20) \qquad \lambda_x = \mu_y + \frac{\sigma_x}{2\sigma}\lambda - \frac{\sigma_y}{2\sigma}\mu, \qquad -\lambda_y = \mu_x - \frac{\sigma_y}{2\sigma}\lambda - \frac{\sigma_x}{2\sigma}\mu.$$

Bers and Nirenberg **(2)** have shown that if $g = \lambda + i\mu$ is a solution of a system $\mathfrak{L}^*$ of type (3.18), there exist a complex-valued function $s(z)$ and an analytic function $h(z)$ such that $g(z) = e^{s(z)}h(z)$. In general, $s(z)$ depends on $g(z)$. It is easy to verify that $s(z)$ must satisfy the equation $gs_{\bar{z}} = g_{\bar{z}}$. If, however, $\mathfrak{L}^*$ is of the form (4.20), we are more fortunate.

THEOREM 4.6. *Let $\mathfrak{L}^*$ be a system of type* (4.20) *where σ is a positive harmonic function, and let $s = p + iq$ be a solution of the system*

$$(4.21) \qquad p_x = q_y + \frac{\sigma_x}{2\sigma}, \qquad -p_y = q_x - \frac{\sigma_y}{2\sigma}.$$

If $g = \lambda + i\mu$ is a solution of $\mathfrak{L}^*$, $h(z) = e^{-s(z)}g(z)$ is analytic. Conversely, if h is analytic in $\mathfrak{D}$, $e^{s(z)}h(z)$ is a solution of $\mathfrak{L}^*$.

Proof. For g and s as above and $h(z) = e^{-s(z)}g(z)$, it is trivial to verify that $h_{\bar{z}} \equiv 0$ and h is analytic. Conversely, if h is analytic, let

$$\lambda + i\mu = g(z) = e^{s(z)}h(z).$$

Then

$$g_{\bar{z}} = e^{s(z)}h(z)s_{\bar{z}} = (\lambda + i\mu)s_{\bar{z}} = (\lambda + i\mu)\left(\frac{\sigma_x}{4\sigma} + i\frac{\sigma_y}{4\sigma}\right)$$

and g is a solution of $\mathfrak{L}^*$.

Note that if $p + iq$ is a solution of (4.21), q is harmonic, and if $\mathfrak{D}$ is simply connected, every harmonic function q determines a solution of (4.21). In particular, if we pick $q = 0$, $p = \frac{1}{2}\ln\sigma$ and $e^p = \sqrt{\sigma}$, it follows that if h is any analytic function, $\sqrt{\sigma}\, h$ is a solution of $\mathfrak{L}^*$. It is easy to see that all such solutions can be represented in this form. Furthermore, if $\mathfrak{L}$ is the Bers system associated with $\mathfrak{L}^*$ and $h = \phi + i\psi$ is analytic, the solution (unique up to an additive constant) of $\mathfrak{L}$ determined by $\sqrt{\sigma}\, h$ can easily be represented as a line integral. For example, if $\phi = 1$ and $\psi = 0$, the solution of $\mathfrak{L}$ determined will be $\int^x \sigma\, dx + iy$.

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