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ITERATION OF PIECEWISE LINEAR MAPS ON AN INTERVAL

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A complete analysis is given of the iterative properties of two piece-piecewise linear maps on an interval, from the point of view of a doubling transformation obtained by functional composition and rescaling. We show how invariant measures may be constructed for such maps and that parameter values where this may be done form a dense set in a one-dimensional subset of parameter space.

1. Introduction

The behavior of first order difference equations

(1) $$x_{n+1} = f(x_n) ,$$

where $f(x)$ has the form shown in Figure 1 (see p. 434), has been the subject of much recent study [2]. Typically when f depends on a parameter λ that decreases the (negative) slope of f at its non-trivial fixed point x^* as λ increases, one obtains a cascade of bifurcations of x^* to stable 2^n-cycles at values λ_n that approach a limit $\lambda_\infty < \infty$ exponentially fast. In fact this rate of convergence,

(1.2) $$\lambda_n - \lambda_\infty \sim \delta^{-n} \quad \text{as} \quad n \to \infty ,$$

was recently found by Feigenbaum [4], [5], to be universal, in the sense

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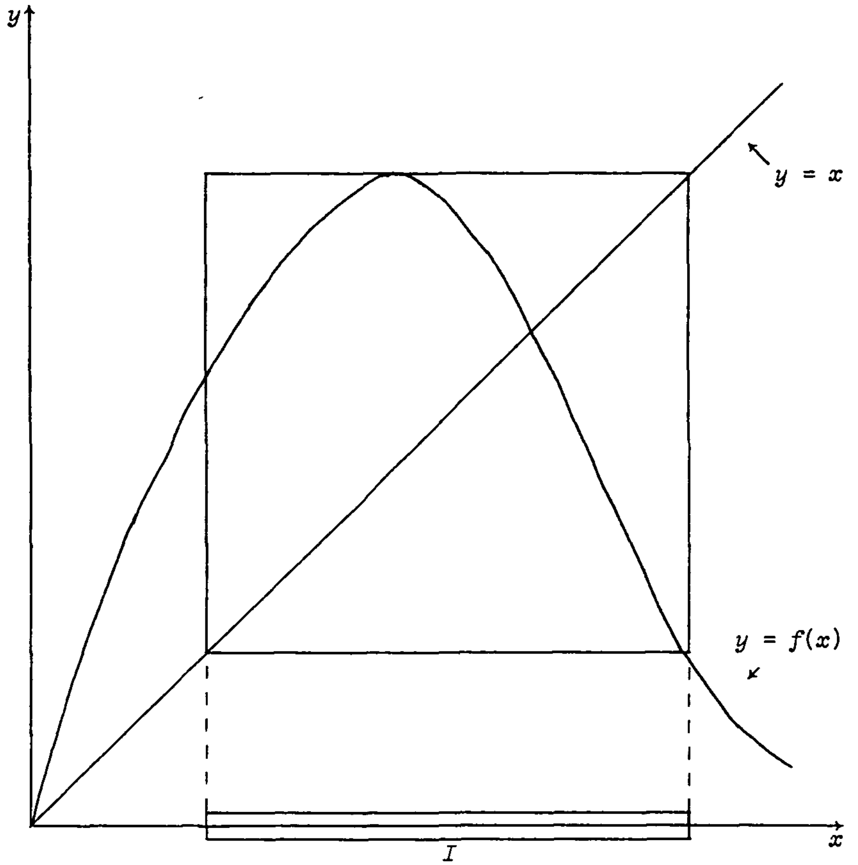

FIGURE 1. Relevant part of the function inside the square where iterates are confined to the interval I .

that δ appears to depend only on the shape of f in the neighborhood of its maximum. There are now elaborate theories of this phenomenon some parts of which have been made rigorous by Collet, Eckmann and Langford [3] when $f(x)$ near its maximum has the shape $|x-x_M|^{1+\varepsilon}$ for ε sufficiently small. In the following section we discuss some aspects of this theory and in Section 3 we give a complete analysis of the piecewise linear case $\varepsilon = 0$.

In the so called chaotic regime $\lambda > \lambda_\infty$ one also has exotic microscopic behavior but there one is also interested in macroscopic properties, such as the existence of "time averages"

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(1.3) $$\overline{F} = \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} F(x_n)$$

where the x_n , $n = 1, 2, \ldots$, are obtained from x_0 by iteration of (1.1), and whether or not there exists a probability measure μ such that

(1.4) $$\overline{F} = \int_E F(x)d\mu(x)$$

for almost all x_0 , where f maps a set E into itself (and $\mu(E) = 1$). This is the classic ergodic problem which has seen renewed interest in recent years, particularly with regard to simple first order difference equations [6]. In this case one is usually after a little more; namely, invariant measures that are absolutely continuous with respect to Lebesgue measure, so that the derivative $\mu'(x)$ gives the density of iterates in truly chaotic situations.

The problem of constructing invariant measures for two-piece piecewise linear maps on an interval is discussed in Section 5. Some examples are given and it is shown that the set of such ergodic maps is dense in a subset of its parameter space.

Our results and conclusions are summarized in the final section.

2. The doubling transformation

Since our primary concern is with long sequences of iterates of (1.1) (equations (1.2) and (1.3)) it is clear that the only relevant part of f is that part shown in the boxed square of Figure 1. (That is, if x_0 is in the interval I , subsequent iterates will all be in I and if x_0 is not in I , x_n will be in I for some small n .) Furthermore, by a suitable change of variables and scale we need only consider such restricted functions defined on the unit interval. Finally, we will find it convenient to make one further conjugacy

(2.1) $$g^{-1}\cdot f\cdot g(x) \quad \text{with} \quad g(x) = 1 - x$$

which has the effect, as shown in Figure 2 (see p. 436), of turning the boxed part of Figure 1 upside down and back-to-front.

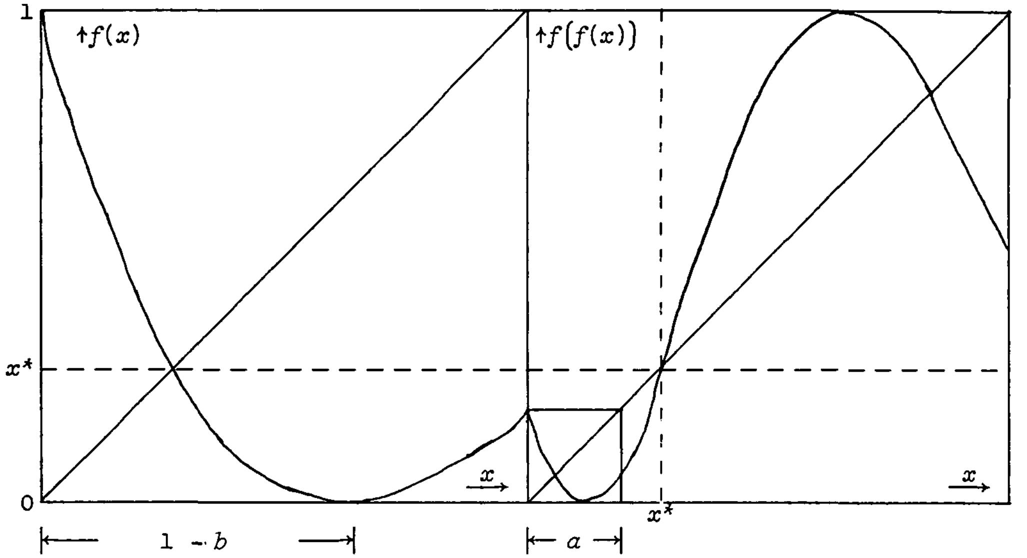

FIGURE 2. Form of conjugated f . The relevant part of $f(f(x))$ is contained in the small square.

The class C of functions we consider, then, is maps f , of the unit interval into itself, such that $f(0) = 1$; f is strictly monotone decreasing to zero at $x = 1 - b$ and strictly monotone increasing for $1-b < x \leq 1$ to $f(1) = a$.

From the point of view of iteration it is now natural to compose such functions with themselves to form functions $f^{(2)}(x) = f(f(x))$, also shown in Figure 2 for the case $a < x^*$.

The interesting feature in this case $(a < x^*)$, as can be seen from Figure 2, is that iterates of $f^{(2)}(x)$ below (above) x^* always remain below (above) x^* . This means that one can have only even cycles of f (including the 2^n bifurcating harmonics of x^*). It also means that $f^{(2)}$ is partitioned naturally into two disjoint pieces so that one can focus attention on either one, say the piece corresponding to $0 < x < x^*$. (In fact these two pieces are conjugate to one another [7].) For the same reasons as given above, iterates in this region are soon confined to the part of $f^{(2)}$ in the small boxed square which resembles a scaled down version of a function in our class C . Scaling the small square up to the unit square in fact amounts to defining a transformation $T : f \to \bar{f}$ on C by

$$\bar{f}(x) = \frac{1}{a} f\big(f(ax)\big) \quad \big(a = f(1)\big) . \tag{2.2}$$

This transformation can obviously be repeated, but for reasons given above it only makes sense to do so if

$$0 < \bar{a} = \bar{f}(1) < \bar{x}^* = \bar{f}(\bar{x}^*) . \tag{2.3}$$

The interesting situation of course, is when (2.2) itself iterates to a fixed point (of T).

Transformations similar to (2.2) have been discussed by Feigenbaum [4], [5], Collet, Eckmann and Lanford [3] who refer to it as the doubling transformation, and others [2].

A discussion of our version will be given elsewhere [7]. Here we will mainly be concerned with the piecewise linear subset of C .

3. Iteration of two-piece, piecewise linear maps

In the piecewise linear case, the class C defined in the previous section reduces to two-piece, piecewise linear maps shown in Figure 3 (see p. 438) and characterized uniquely by the two parameters a and b . We will denote such functions by $[a; b]$.

The doubling transformation $T : [a, b] \to [\bar{a}; \bar{b}]$ defined in general by (2.2), in this case is found, after some simple algebra, to be given by

$$\begin{aligned} \bar{a} &= (1-b)^{-2}[1-b(1-b)/a] , \\ \bar{b} &= 1 - b(1-b)/a . \end{aligned} \tag{3.1}$$

Repeated application of the transformation T results in a sequence of maps $[a_n; b_n]$ defined by

$$\begin{aligned} b_{n+1} &= 1 - b_n\big(1-b_n\big)/a_n , \\ a_{n+1} &= b_{n+1}/\big(1-b_n\big)^2 \end{aligned} \tag{3.2}$$

with $a_0 = a$ and $b_0 = b$.

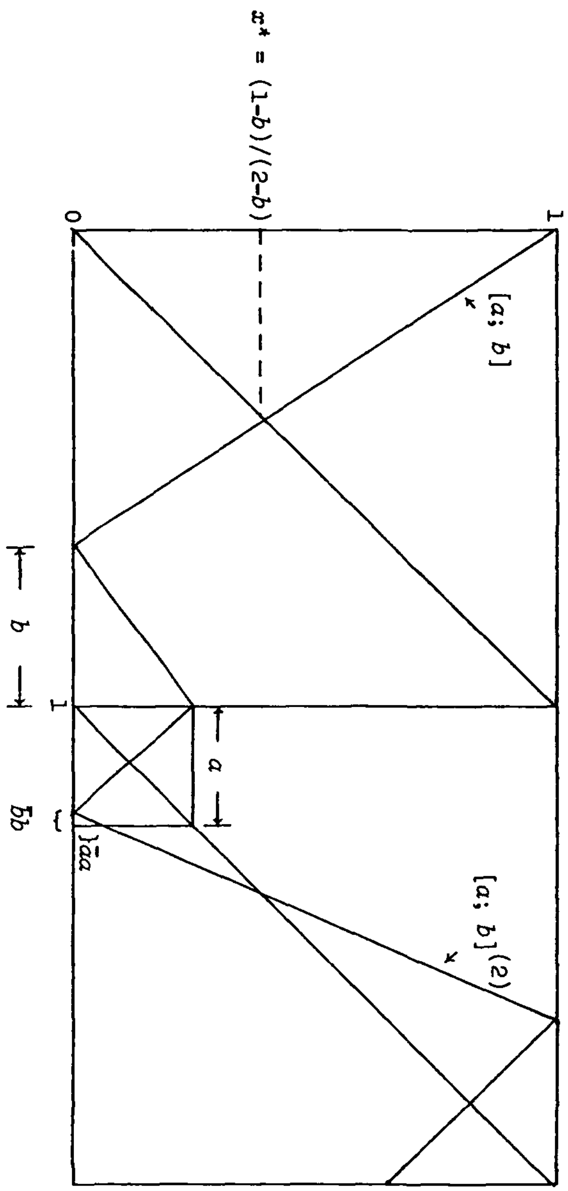

FIGURE 3. Piecewise linear version of maps of the type shown in Figure 2.

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It turns out that these recurrence relations can be solved exactly. Thus if one defines

(3.3) $$\mu_n = b_n/a_n \quad\text{and}\quad c_n = 1 - b_n$$

it is an easy matter to show that equations (3.2) become

(3.4) $$\mu_n = c_{n+1}/c_n \quad\text{and}\quad \mu_{n+1} = c_n^2$$

which yield the second order non-linear equation

(3.5) $$c_{n+1} = c_n c_{n-1}^2$$

with initial conditions

(3.6) $$c_0 = 1 - b_0 \quad\text{and}\quad c_1 = \mu_0 c_0 = (1-b_0)b_0/a_0 \ .$$

Before proceeding, it should be stressed again that it only makes sense to transform $[a_n; b_n]$ according to (3.2) when the requirement

(3.7) $$a_n < x_n^* = (1-b_n)/(2-b_n)$$

is satisfied. In terms of the c_n's this is equivalent to the condition

(3.8) $$c_{n+1} + c_n^2 > 1 \ .$$

If this condition is violated when $n \geq N$ the iteration defined by (3.2) and (3.5) should cease at $n = N - 1$.

Returning now to the solution of (3.5), the substitution

(3.9) $$c_n = c_0^{\alpha_n} c_1^{\beta_n}$$

gives the same linear equation for α_n and β_n to solve, namely

(3.10) $$x_{n+1} = x_n + 2x_{n-1}$$

with appropriate initial conditions

$$x_0 = 1 \ , \quad x_1 = 0 \quad (\text{for } \alpha_n)$$

and

(3.11) $$x_0 = 0 \ , \quad x_1 = 1 \quad (\text{for } \beta_n).$$

The solution of (3.10) has "eigenvalues" $\delta = 2$ and -1, resulting in

(3.12) $$\alpha_n = \tfrac{1}{3}[2^n+2(-1)^n] , \quad \beta_n = \tfrac{1}{3}[2^n-(-1)^n]$$

or, from (3.9),

(3.13) $$c_n = [c_0c_1]^{2^n/3}\left[c_0^2/c_1\right]^{(-1)^n/3}$$

By definition $0 \le c_0 = 1 - b_0 < 1$ but $c_1 = (1-b_0)b_0/a_0$, while always non-negative, may exceed or equal unity. When this happens, however, the second functional iterate of $[a_0; b_0]$ has the form shown in Figure 4. In this case $[a_1; b_1]$ has a stable fixed point, or equivalently $[a_0; b_0]$ has a stable two cycle.

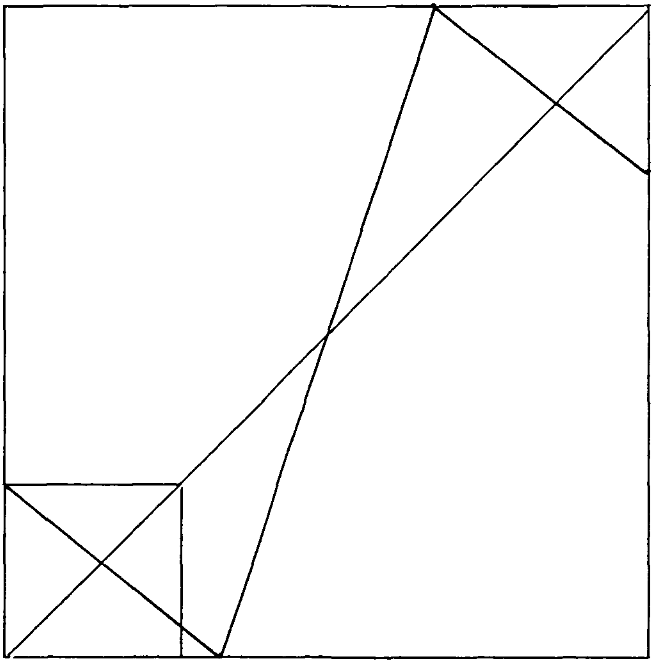

FIGURE 4. The second functional iterate of $[a_0; b_0]$ when $(1-b_0)b_0 > a_0$.

When $c_1 < 1$, on the other hand, c_n approaches zero exponentially fast with increasing n in which case (3.8) is violated for large n and in the first instance when $n = N$ say.

We are now in a position to completely classify the behavior of two-piece piecewise linear maps $[a; b]$ on the unit interval. This is done in Figure 5.

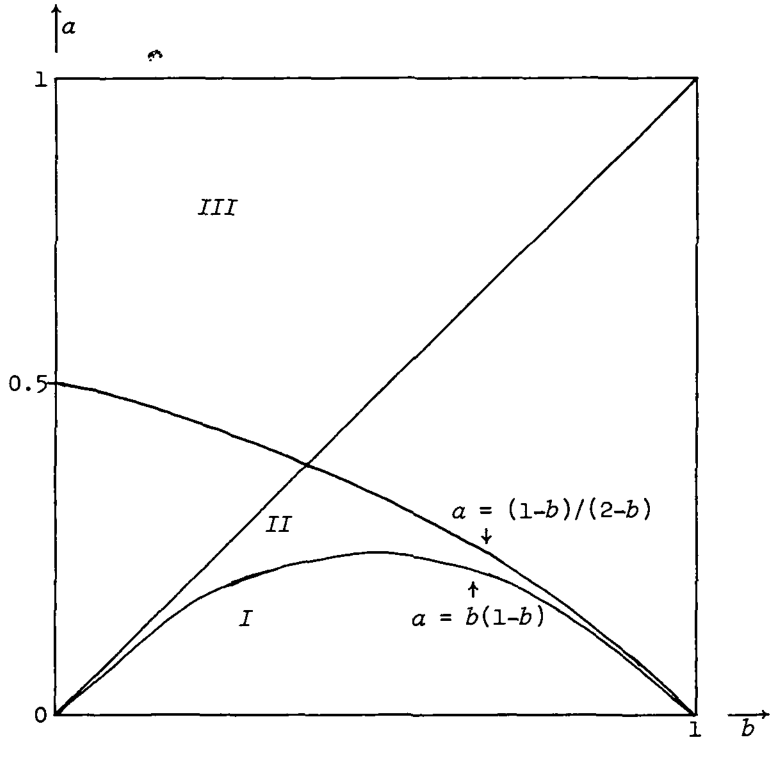

FIGURE 5. Regions for the piecewise linear map $[a; b]$:

I. $a < b(1-b)$: stable two-cycle;

II. $b(1-b) < a < (1-b)/(2-b)$: maps transformed to maps in region *III* under the doubling transformation;

III. $(1-b)/(2-b) < a < 1$; $a \geq b$.

There are only three relevant regions in the unit square, in the a, b plane. Firstly

$$\text{Region } I : \; a < b(1-b) \tag{3.14}$$

corresponds as above to $c_1 \geq 1$; that is, a stable two cycle. Secondly, we have

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(3.15) $$\text{Region } II : \ b(1-b) < a < (1-b)/(2-b) .$$

Equation (3.7) holds in this region but only for $N - 1$ steps of the iteration. After N steps, condition (3.8) is violated giving (a_N, b_N) in the

(3.16) $$\text{Region } III : \ 1 \geq a \geq (1-b)(2-b)^{-1} .$$

Note also from (3.2) that $a_n \geq b_n$ so that points in region II are in fact transformed under (3.2) to that part of region III satisfying $a \geq b$.

So while the doubling transformation can be carried through exactly for functions $[a; b]$ there is unfortunately no fixed point function of the type mentioned in the previous section. The "pseudo fixed point"

(3.17) $$c^* = \lim_{n \to a} c_n = 0 \quad \left(\text{for} \quad (a_0, b_0) \in II \cup III\right)$$

also makes little sense since in terms of parameters a_n and b_n the limiting values corresponding to (3.17) are a^* unbounded and $b^* = 1$ which do not give a legitimate piecewise linear function.

Nevertheless the fact that functions of the type $[a; b]$ either have a stable two cycle or iterate under the doubling transformation to chaos in region III has some interesting consequences concerning the existence of invariant measures for such functions. In fact we show in the following sections that for fixed b , the set of a for which $[a; b]$ has an absolutely continuous invariant measure is dense on a subset of the interval $(b(1-b), 1)$.

4. Reconstruction of invariant measures under doubling

Before discussing the piecewise linear case let us first consider the general doubling transformation $f \to \bar{f}$ defined by

(4.1) $$\bar{f}(x) = \frac{1}{a} f\left(f(ax)\right) , \quad x \in [0, 1] ,$$

and let us suppose that f is ergodic with absolutely continuous measure μ , so that for any integrable $g(x)$ on $[0, 1]$, (4.2) $$\lim_{N\to\infty} N^{-1} \sum_{k=1}^{N} g(x_k) = \int_0^1 g(x)\mu'(x)dx$$

for almost all x_0 where

(4.3) $$x_{k+1} = \bar{f}(x_k) .$$

Define

(4.4) $$y_k = ax_k$$

so that from (4.1) and (4.3),

(4.5) $$y_{k+1} = f(f(y_k)) .$$

Next define z_n , $n = 1, 2, \ldots$, by

(4.6) $$z_{2n-1} = y_n \quad \text{and} \quad z_{2n} = f(y_n) , \quad n = 1, 2, \ldots ,$$

and note that

$$z_{2n} = f(z_{2n-1})$$

and

$$z_{2n+1} = y_{n+1} = f(f(y_n)) = f(z_{2n}) .$$

That is

(4.7) $$z_{k+1} = f(z_k) , \quad k = 1, 2, \ldots .$$

Now choose the function g in (4.2) to be

(4.8) $$g(x) \equiv G(ax) + G(f(ax))$$

where G is integrable.

From (4.4) and (4.6) one obtains

(4.9) $$\sum_{k=1}^{N} \{G(ax_k)+G(f(ax_k))\} = \sum_{k=1}^{N} G(z_{2k-1}) + \sum_{k=1}^{N} G(z_{2k}) = \sum_{k=1}^{2N} G(z_k)$$

and hence, from (4.2),

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(4.10) $$\lim_{N\to\infty} (2N)^{-1} \sum_{k=1}^{2N} G(z_k) = \int_0^1 \tfrac{1}{2}[G(ax)+G(f(ax))]\mu'(x)dx \; .$$

By assumption

(4.11) $$f(0) = 1 \; , \quad f(a) \equiv c > x^* = f(x^*) > a$$

so that $f(y)$ on $0 < y < a$ is invertible. Denoting the inverse of $f/\{(0, a)\}$ by h and changing variables in (4.10) one easily shows that

(4.12) $$\lim_{N\to\infty} (2N)^{-1} \sum_{k=1}^{2N} G(z_k) = \int_0^1 G(z)m'(z)dz$$

where z_k is obtained from (4.7) and $(g'(z) < 0)$

(4.13) $$m'(z) = \begin{cases} \frac{1}{2a}\,\mu'(z/a) & , \quad 0 < z < a \; , \\ 0 & , \quad a < z < c \; , \\ -\frac{1}{2a}\,\mu'\big(h(z)/a\big)h'(z) & , \quad c < z < 1 \; . \end{cases}$$

It follows that f is ergodic and m is the invariant measure for f .

5. The piecewise linear case

Turning now to the piecewise linear case we can, in view of the results of Section 4, and the observations of Section 3, restrict our search for invariant measures to the region III , $a > x^*$. Thus, if one can find an invariant measure for an $[a; b]$ in this region (with $a > b$) we can run the transformation (3.2) backwards to find a map in region II having an invariant measure that can be obtained by repeated application of (4.13).

With regard to ergodicity it is known that whenever a map $[a; b]$ has a cycle that includes the minimum it has a piecewise constant and unique absolutely continuous invariant measure [1]. The first problem then is to construct maps $[a; b]$ that iterate from the minimum back to the minimum. This is most easily done by fixing b and iterating backwards from the minimum on the "b-piece". The odd iterates obtained in this way and denoted by a_1, a_3, ... then provide a countably infinite set of maps $[a_{2k-1}; b]$, $k = 1, 2, \ldots$, with the desired property, as shown in Figure 6 (see p. 445).

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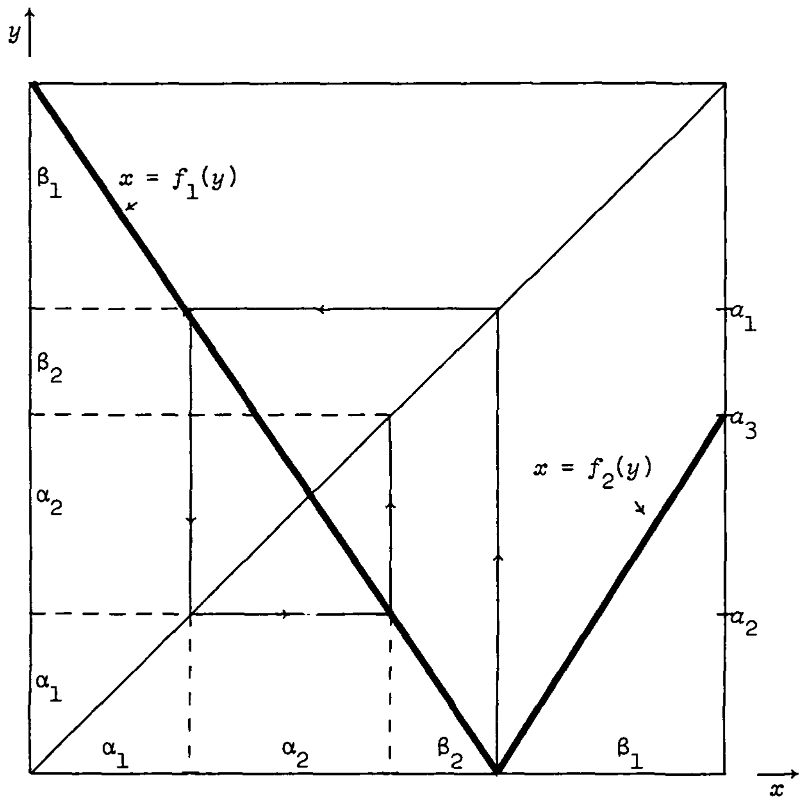

FIGURE 6. Backwards iteration giving rise to maps having stable odd cycles including the minimum.

In more detail, the "b-piece" is the function

(5.1) $$f(x) = 1 - \frac{x}{1-b} \quad (0 \leq x \leq 1-b)$$

and by iterating backwards from the minimum we mean solving the recurrence relation

(5.2) $$f(a_{n+1}) = a_n \;, \quad n = 1, 2, \ldots \quad (a_1 = 1 - b) \;.$$

The solution of this recurrence is

(5.3) $$a_n = (2-b)^{-1}(1-b)[1-(b-1)^n]$$

so that

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$$(5.4)\qquad a_{2k-1} = (2-b)^{-1}(1-b)\left[1+(1-b)^{2k-1}\right] , \quad k = 1, 2, \ldots .$$

It will be noted that $a_{2k-1} > (2-b)^{-1}(1-b) = x^*$ so this set of functions is in region *III* .

In order to construct the invariant measure for $[a_{2p-1}; b]$ for example, we iterate backwards from the minimum $2p - 1$ times and partition the intervals $0 \le y \le 1$ and $0 \le x \le 1$ into the sets

$$(5.5)\qquad I_k = (a_{2k-1}, a_{2k-3}) , \quad k = 1, 2, \ldots, p \quad (a_{-1} \equiv 1) ,$$

$$J_k = (a_{2k-2}, a_{2k}) , \quad k = 1, 2, \ldots, p-1 \quad (a_0 \equiv 0) ,$$

and

$$(5.7)\qquad J_p = (a_{2p-2}, a_{2p-1}) .$$

We then seek a piecewise linear invariant measure of the form

$$(5.8)\qquad \mu'(y) = \sum_{k=1}^{p} \{\beta_k \chi_{I_k}(y) + \alpha_k \chi_{J_k}(y)\}$$

where χ_I is the characteristic function of the set I , that is,

$$(5.9)\qquad \chi_I(y) = \begin{cases} 1 & \text{if } y \in I , \\ 0 & \text{if } y \notin I . \end{cases}$$

The case $p = 2$ is illustrated in Figure 6.

In order for $\mu(x)$ to be invariant with respect to $[a_{2p-1}, b]$ we require, [6],

$$(5.10)\qquad \mu'(y) + \mu'(f_1(y))f_1'(y) = 0 , \quad a_{2p-1} < y < 1 ,$$

and

$$(5.11)\qquad \mu'(y) + \mu'(f_1(y))f_1'(y) + \mu'(f_2(y))f_2'(y) = 0 , \quad 0 < y < a_{2p-1} ,$$

where from Figure 6 (for general p now)

$$(5.12)\qquad f_1(y) = (1-b)(1-y)$$

and

(5.13) $$f_2(y) = s^{-1}y + b^{-1}(1-b)$$

where

(5.14) $$s = (1-b)[b(2-b)]^{-1}[1+(1-b)^{2p-1}] .$$

In terms of the α_k and β_k defined by (5.8), equations (5.10) and (5.11) become

(5.15) $$\beta_k = (1-b)\alpha_k , \quad k = 1, 2, \ldots, p ,$$

(5.16) $$\alpha_k = s^{-1}\beta_1 + (1-b)\beta_{k+1} , \quad k = 1, 2, \ldots, p-1 ,$$

and

(5.17) $$\alpha_p = s^{-1}\beta_1 + (1-b)\alpha_p .$$

The solution of these equations is

(5.18) $$\alpha_k = [1-b+(1-b)^{2p}]^{-1}\beta_1[1+(1-b)^{2(p-k)+1}] , \quad k = 1, 2, \ldots, p ,$$

with β_k given by (5.15). It is to be noted that these equations are internally consistent and that the (apparently arbitrary) constant β_1 is determined by the obvious normalization condition,

(5.19) $$\int_0^1 \mu'(y)dy = 1 .$$

So by the simple device of iterating backwards from the minimum we have constructed a countably infinite number of maps $[a_{2p-1}; b]$, $p = 1, 2, \ldots$, with explicit and unique invariant measures (5.8), (5.15) and (5.18).

This set of maps can be further enlarged to a dense set of maps $[a; b]$ for $x^* < a < a_1$ by essentially showing that for two maps $[a'; b]$ and $[a''; b]$, $a' < a''$ having cycles containing the minimum there is a third map $[a; b]$ with $a' < a < a''$ that also has a cycle containing the minimum.

As a special case and as a first step in an induction argument,

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consider the maps $[a_3;\ b]$ and $[a_1;\ b]$ defined above. By construction, the third iterate of $x_0 = 1$ under the action of $[a_3;\ b]$ $([a_1;\ b])$ intersects the line $y = a_3$ at point P_3 (P_1) as shown in Figure 7. It follows that the third iterate of $x_0 = 1$ under the action of $[a;\ b]$ for $a_3 < a < a_1$ intersects $y = a_3$ at P which is in the interior of the segment P_3P_1 and such that the line aP intersects $y = 0$ at the point Q, also shown in Figure 7. Now since the distance from the origin to Q is large and negative when a is close to $a_3(+)$ and approaches the point $(1,\ 0)$ as a approaches a_1, it follows by continuity that there is an $a_6 \in (a_3,\ a_1)$ such that Q coincides with the minimum M at $x = 1 - b$. It is then obvious that the sixth iterate of $[a_6;\ b]$ from $x_0 = 1$ is at

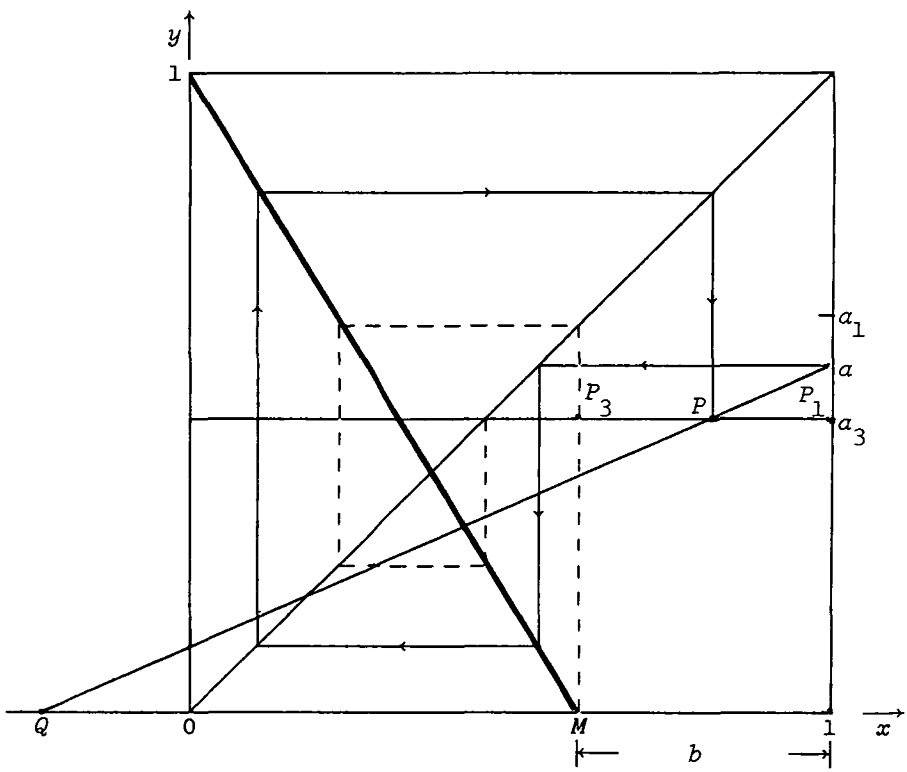

FIGURE 7. The construction of a dense set of a values for which $[a;\ b]$ has an absolutely continuous invariant measure.

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the minimum and hence that the minimum is part of an eight cycle.

In order to complete the proof of denseness we argue inductively. That is, suppose we have maps $[a_k; b]$ and $[a_l; b]$ with $a_k < a_l$ that iterate $x_0 = 1$ to the minimum in k and l steps respectively, and that there exist non-negative integers n and m such that

(5.20) $$n(l+2) = m(k+2) + k \equiv N .$$

(The above special case corresponds to $k = 3$, $l = 1$, $m = 0$, $n = 1$.) This implies that under the action of $[a_k; b]$ $([a_l; b])$ the Nth iterate from $x_0 = 1$ is at the minimum and intersects $y = a_k$ at P_k (at unity and intersects the line $y = a_k$ at $x = 1$). Choosing an $a \in (a_k, a_l)$ and iterating N times from $x_0 = 1$, the continuity argument above establishes the existence of an $[a_{N+k}; b]$ (iterating $x_0 = 1$ to the minimum in $N + k$ steps) with

(5.21) $$a_k < a_{N+k} < a_l .$$

Denseness will then be established once we have shown that given k, l, n, m and N defined by (5.20), there exists n' and m' and n'' and m'' such that

(5.22) $$n'(l+2) = m'(N+k+2) + N + k = N'$$

and

(5.23) $$n''(N+k+2) = m''(k+2) + k = N'' .$$

Since (5.22) ((5.23)) is equation (5.20) with k replaced by $N + k$ (l replaced by $N + k$) the above argument establishes the existence of $[a_{N+k+N'}; b]$ and $[a_{N''+k}; b]$ having cycles containing the minimum and satisfying

(5.24) $$a_k < a_{N''+k} < a_{N+k} < a_{N+k+N''} < a_l .$$

Repetition of this argument establishes denseness, and our proof is completed by noting that (5.22) and (5.23) follow from (5.20) by choosing

(5.25) $$n' = n(m+2)(l+3) + 2 , \quad m' = (l+3)(m+1) - 1$$

and

(5.26) $$n'' = k + 3 \ , \quad m'' = (m+2)k + 3(m+1) \ .$$

6. Discussion

In this paper we have presented a different version of the so called doubling transformation of maps on an interval [2] and have discussed in detail the case of two piece-piecewise linear maps. This represents an extreme case ($\varepsilon = 0$) of functions having the form $|x - x_m|^{1+\varepsilon}$ in the neighborhood of their (single) extremum. For small $\varepsilon > 0$, some aspects of the doubling-type transformation, and its relevance to Feigenbaum universality, have been made rigorous by Collet, Eckmann and Lanford [3]. When $\varepsilon = 0$, however, there is no universality as such, since transitions or bifurcations in this case, take place solely from stable fixed points, to stable two-cycles and then to chaos. In other words, there is no cascade of 2^n-cycle harmonics prior to chaos. A detailed analysis of this behavior for the piecewise linear case is given from the doubling transformation point of view in Section 3 of this paper.

Another important aspect in the study of maps on an interval is the ergodic problem, or more precisely the existence of invariant measures which are absolutely continuous with respect to Lebesgue measure. We have examined this problem in detail for the piecewise linear case and have shown how such invariant measures may be constructed. We have also shown that parameter values where this may be done form a dense set in a one-dimensional subset of parameter values describing such functions. It is quite likely, we feel, that the set of parameter values where there exists an invariant measure, in fact has positive (Lebesgue) measure.

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