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THE GROUP OF EXTENSIONS AND SPLITTING LENGTH

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This paper is concerned with the internal structure of $\text{Ext}(Q, T)$ where Q is the group of rationals and T a reduced p-primary group of unbounded order. In [**1**] Irwin, Khabbaz, and Rayna define the splitting length of an arbitrary abelian group A, written $l(A)$, to be the least positive integer n, otherwise infinity, such that $A \otimes \ldots \otimes A$ (n factors) splits. The concept of splitting length has been induced on $\text{Ext}(Q, T)$, see [**2**; **5**]. For $E \in \text{Ext}(Q, T)$ where $E : 0 \to T \to X \to Q \to 0$, define $l(E) = l(X)$. In [**2**] it was shown that

$$\text{Ext}\,(Q, T) = F \oplus I = \sum_{n=2}^{n=\infty} C_n, \quad F = \sum_{n=2}^{\infty} C_n$$

where the nonzero elements of F are of finite splitting length, the nonzero elements of I of infinite splitting length, and the nonzero elements of C_n are of splitting length n. Moreover, the C_n, $2 \leqq n \leqq \infty$, were shown to be nonzero for a particular p-primary group T. This was improved in [**5**] where the C_n, $2 \leqq n \leqq \infty$, were shown to be nonzero for an arbitrary p-primary group T and that there are at least c non-isomorphic extensions of T by Q for an arbitrary splitting length n, $2 \leqq n \leqq \infty$, where c is the cardinality of the continuum. In this paper we show that in fact for every splitting length n, $2 \leqq n \leqq \infty$, there are at least $\aleph c$ nonequivalent extensions of T by Q where $\aleph$ is the final rank of a basic subgroup of T.

Throughout the paper we shall use the notation of [**5**]. In addition, the values of the p-height function h will be taken to be either non-negative integers or the symbol ∞.

We begin by recalling a special case of a definition and a theorem from [**6**].

Definition. For a mixed group G of torsion-free rank one with a p-primary torsion subgroup, the *height-slope* of G is defined by

$$\text{h.s. } G = \sup_{\substack{a \in G \\ o(a)=\infty}} \inf_{i} h(p^i a)/i.$$

The following theorem relates height-slope to splitting length.

THEOREM 1. *Let G be a nonsplitting mixed group of torsion-free rank one with*

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height-slope α, its torsion subgroup, $T(G)$, p-primary and $G/T(G)$ is p-divisible. Then $l(G) = n$ if and only if one of the following holds:

(i) *α belongs to the open interval $(n/(n-1), (n-1)/(n-2))$, or*

(ii) *$\alpha = n/(n-1)$ and for some element $a \in G$ of infinite order $h(p^i a) > \alpha(i + f(i))$ for all $i \in Z^+$, where f is a function from $Z^+ \to Z^+$ which is non-decreasing to infinity.*

We now prove two preliminary lemmas.

LEMMA 1. *Let*

$$0 \to A \xrightarrow{\alpha} B$$

be p-pure exact,

$$E : 0 \to A \xrightarrow{\beta} X \to G \to 0$$

an exact sequence, and define $N = \{(-\alpha(a), \beta(a)) \in B \oplus X | a \in A\}$. Then $\eta : X \to B \oplus X/N$ given by $\eta(x) = (o, x) + N$ is a p-height preserving homomorphism.

Proof. Suppose $\eta(x) = (o, x) + N = p^n(b, x_1) + N$. Hence $(-p^n b, x - p^n x_1) \in N$. Let $a \in A$ be such that $\alpha(a) = p^n b$, $\beta(a) = x - p^n x_1$. Since

$$0 \to A \xrightarrow{\alpha} B$$

is p-pure exact we may write $a = p^n a_1$ for some $a_1 \in A$. Now $x - p^n x_1 = \beta(a) = \beta(p^n a_1) = p^n \beta(a_1)$, so that $x = p^n(x_1 + \beta(a_1))$ as desired.

Remark 1. If in Lemma 1, A and B are p-primary groups and X is an extension of A by Q then we can conclude that h.s. X is equal to h.s. $B \oplus X/N$, if h.s. X is finite. We shall use this fact later on in the paper.

Remark 2. The homomorphism η in Lemma 1 does not distinguish between elements of infinite height. For example the image of an element of height ω does at times become of infinite height in the generalized sense.

LEMMA 2. *Let $B = \sum_\gamma B_\gamma$ where the summands are $\aleph$ in number and each B_γ is the direct sum of cyclic p-groups of unbounded order. Then*

$$\text{Ext}\,(Q, B) = F \oplus I = \sum_{n=2}^{n=\infty} C_n$$

contains at least $\aleph c$ nonequivalent extensions for every splitting length n, $2 \leqq n \leqq \infty$.

Proof. If $\aleph c = c$ then the result follows from Theorem 2 of [**5**] and the remark following the theorem. So suppose $\aleph > c$. Again by Theorem 2, for each γ,

$$\text{Ext}\,(Q, B_\gamma) = \sum_{n=2}^{n=\infty} C_n$$

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where all the C_n are nonzero. Let $E_1 \in \text{Ext}(Q, B_\alpha)$, $E_2 \in \text{Ext}(Q, B_\delta)$ where

$$E_1 : 0 \to B_\alpha \xrightarrow{\rho_1} X_\alpha \to Q \to 0, \quad E_2 : 0 \to B_\delta \xrightarrow{\rho_2} X_\delta \to Q \to 0$$

and $l(E_1) = n = l(E_2)$. Then

$$E_1' : 0 \to \sum_{\gamma \neq \alpha} B_\gamma \oplus B_\alpha \xrightarrow{(1, \rho_1)} \sum_{\gamma \neq \alpha} B_\gamma \oplus X_\alpha \to Q \to 0$$

$$E_2' : 0 \to \sum_{\gamma \neq \delta} B_\gamma \oplus B_\delta \xrightarrow{(1, \rho_2)} \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta \to Q \to 0$$

are elements of $\text{Ext}(Q, B)$ of splitting length n. Moreover, E_1', E_2' are non-equivalent extensions. Suppose the contrary. Then there exists an isomorphism

$$\theta : \sum_{\gamma \neq \alpha} B_\gamma \oplus X_\alpha \to \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta$$

such that $\theta(1, \rho_1) = (1, \rho_2)$. Now θ induces an isomorphism

$$\hat{\theta} : X_\alpha \to \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta \Big/ \theta\Big(\sum_{\gamma \neq \alpha} B_\gamma\Big) = B_\alpha \oplus \frac{X_\delta}{B_\delta}.$$

This implies that X_α splits, a contradiction. Thus, E_1' and E_2' are nonequivalent. Since the summands $B\gamma$ are $\aleph$ in number our conclusion follows.

Remark 3. If in Lemma 2 the B_γ are all equal then we can conclude that the same group is appearing in at least $\aleph c$ nonequivalent extensions of B by Q. This indicates that the cardinality of $\text{Ext}(Q, B)$ arises largely from this fact.

THEOREM 2. *Let T be a reduced p-primary group of unbounded order with B a basic subgroup of T with final rank $\aleph$. Then* $\text{Ext}(Q, T) = F \oplus I = \sum_{n=2}^{n=\infty} C_n$ *contains at least $\aleph c$ nonequivalent extensions of splitting length n, $2 \leqq n \leqq \infty$.*

Proof. By [**5**], $\text{Ext}(Q, T)$ contains at least c nonequivalent extensions for every splitting length n, $2 \leqq n \leqq \infty$, and so we may assume $\aleph > c$. Since the final rank of B is $\aleph$ we may write $B = \sum_\gamma B_\gamma$ where the B_γ are $\aleph$ in number and each B_γ is the direct sum of cyclic p-groups of unbounded order. The exact sequence

$$0 \to B \xrightarrow{i} T \to T/B \to 0$$

gives us the exact sequence

$$0 = \text{Hom}\ (Q, T) \to \text{Hom}\ (Q, T/B) \to \text{Ext}\ (Q, B) \xrightarrow{i_*} \text{Ext}\ (Q, T) \to$$
$$\text{Ext}\ (Q, T/B) = 0.$$

By Lemma 2, $\text{Ext}(Q, B)$ contains at least $\aleph c$ nonequivalent extensions for every splitting length n, $2 \leqq n \leqq \infty$. By Lemma 1 the extensions having a middle group with a finite height slope are mapped under i_* into ones with the same height slope. Thus, by Theorem 1, i_* preserves the splitting length of those extensions having middle groups with finite height slope. It remains to show that they are nonequivalent in $\mathrm{Ext}(Q, T)$. Let

$$E_1, E_2 \in \mathrm{Ext}\,(Q, B)$$

$$E_1 : 0 \to \sum_{\gamma \neq \alpha} B_\gamma \oplus B_\alpha \xrightarrow{\rho_1} \sum_{\gamma \neq \alpha} B_\gamma \oplus X_\alpha \to Q \to 0$$

$$E_2 : 0 \to \sum_{\gamma \neq \delta} B_\gamma \oplus B_\delta \xrightarrow{\rho_2} \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta \to Q \to 0$$

and h.s. $X_\alpha < \infty$, h.s. $X_\delta < \infty$. The images of E_1 and E_2 under i_* are

$$i_*(E_1) : 0 \to T \xrightarrow{\tilde{\rho}_1} \frac{T \oplus \sum_{\gamma \neq \alpha} B_\gamma \oplus X_\alpha}{N_1} \to Q \to 0$$

$$i_*(E_2) : 0 \to T \xrightarrow{\tilde{\rho}_2} \frac{T \oplus \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta}{N_2} \to Q \to 0$$

where $N_i = \{(-b, \rho_i(b)) | b \in B\}$ $i = 1, 2$. If $i_*(E_1) = i_*(E_2)$ then there exists an isomorphism

$$\theta : \frac{T \oplus \sum_{\gamma \neq \alpha} B_\gamma \oplus X_\alpha}{N_1} \to \frac{T \oplus \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta}{N_2}$$

such that $\theta\tilde{\rho}_1 = \tilde{\rho}_2$. This induces an isomorphism

$$\hat{\theta} : \frac{T \oplus \sum_{\gamma \neq \alpha} B_\gamma \oplus X_\alpha}{N_1} \Big/ \tilde{\rho}_1(B_\alpha) \to \frac{T \oplus \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta}{N_2} \Big/ \theta\tilde{\rho}_1(B_\alpha) =$$

$$\frac{T \oplus \sum_{\gamma \neq \delta} B_\gamma \oplus X_\delta}{N_2} \Big/ \tilde{\rho}_2(B_\alpha).$$

However, this is not possible since the first group splits while the second does not. This completes the proof.

Remark 4. By Lemma 1 (or a direct computation) the kernel of i_* contains nontrivial extensions with middle groups having infinite height slope.

Before we pose a concluding question we show that $|\mathrm{Ext}(Q, T)| = \aleph^{\aleph_0}$ where $\aleph$ is the final rank of a basic subgroup of T (see also [**3**]). Let B be basic in T and $\aleph$ the final rank of B. Write $B = \sum_n B_n$ where $B_n = \sum Z(p^n)$. Choose $m \in Z$ such that $r(p^m B) = \aleph$ and $B = B' \oplus B''$ where $B' = \sum_{i=1}^m B_i$ and $B'' = \sum_{i=m+1}^\infty B_i$. Thus $\aleph = r(p^m B) = |B''|$. Then $\mathrm{Ext}(Q, B) \simeq \mathrm{Ext}(Q, B'')$ since B' is bounded. Thus $|\mathrm{Ext}(Q, B)| = |\mathrm{Ext}(Q, B'')|$. The exact sequence $0 \to B \to T \to T/B \to 0$ yields the exact sequence $\mathrm{Ext}(Q, B) \to \mathrm{Ext}(Q, T) \to 0$. So $|\mathrm{Ext}(Q, B)| \geqq |\mathrm{Ext}(Q, T)|$. Consider the exact sequence $0 \to B'' \to D \to D/B'' \to 0$ where D is the divisible hull of B''. This last sequence induces the sequence

$\mathrm{Hom}(Q, D/B'') \to \mathrm{Ext}(Q, B'') \to 0$ but

$$|\mathrm{Hom}(Q, D/B'')| \leqq |D/B''|^{\aleph_0} = |D|^{\aleph_0} = |B''|^{\aleph_0}.$$

Hence $|\mathrm{Ext}(Q, B'')| \leqq |\mathrm{Hom}(Q, D/B'')| \leqq |B''|^{\aleph_0}$, i.e., $|\mathrm{Ext}(Q, T)| \leqq |B''|^{\aleph_0}$. Now from the exact sequence $0 \to B'' \to \bar{B}'' \to \bar{B}''/B'' \to 0$ we get $0 \to \mathrm{Hom}(Q, \bar{B}''/B'') \to \mathrm{Ext}(Q, B'')$. Therefore

$$|\mathrm{Ext}(Q, B'')| \geqq |\mathrm{Hom}(Q, \bar{B}''/B'')| \geqq |\bar{B}''/B''| = |\bar{B}''| = |B''|^{\aleph_0},$$

the last equalities holding because $|B''| = \aleph$. Moreover, by Szele there is an exact sequence $T \to B \to 0$ which implies $\mathrm{Ext}(Q, T) \to \mathrm{Ext}(Q, B) \to 0$; thus $|\mathrm{Ext}(Q, T)| \geqq |\mathrm{Ext}(Q, B)| = |\mathrm{Ext}(Q, B'')| \geqq |B''|^{\aleph_0}$. Therefore $|\mathrm{Ext}(Q, T)| = |B''|^{\aleph_0} = \aleph^{\aleph_0}$.

Let $B = \sum_{i=1}^{\infty} Z(p^i)$. We know that $\mathrm{Ext}(Q, T)$ is isomorphic to

$$\mathrm{Ext}\left(Q, \sum_{\aleph^{\aleph_0}} B\right),$$

both being rational vector spaces of dimension $\aleph^{\aleph_0}$. By Lemma 2,

$$\mathrm{Ext}\left(Q, \sum_{\aleph^{\aleph_0}} B\right) = \sum_{n=2}^{n=\infty} C_n$$

contains $\aleph^{\aleph_0}$ nonequivalent extensions for each splitting length n, $2 \leqq n \leqq \infty$. Since $|\mathrm{Ext}(Q, T)| = \aleph^{\aleph_0}$ hence for some n, $2 \leqq n \leqq \infty$, $\mathrm{Ext}(Q, T) = \sum_{n=0}^{n=\infty} C_n$ contains $\aleph^{\aleph_0}$ nonequivalent extensions. The question we pose then is: does $\mathrm{Ext}(Q, T)$ contain $\aleph^{\aleph_0}$ nonequivalent extensions for all n, $2 \leqq n \leqq \infty$.

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