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Bounded approximate identities and tensor products

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The question has been raised [R.J. Loy, *Bull. Austral. Math. Soc.* 2 (1970), 253-260] as to whether the existence of a bounded (left) approximate identity in the tensor product $A \otimes_\alpha B$ of Banach algebras A and B (for α a crossnorm on $A \otimes B$) implies the existence of a bounded (left) approximate identity in A and B. This is known [David A. Robbins, *Bull. Austral. Math. Soc.* 6 (1972), 443-445] to be the case for α equal to the greatest crossnorm. This paper answers the general question affirmatively.

Let A and B denote Banach algebras, α a crossnorm $\geq \lambda$ (λ the "least" crossnorm [4]) and $A \otimes_\alpha B$ the completion of $A \otimes B$ with respect to α. The purpose of this note is to prove the following

THEOREM. *If $A \otimes_\alpha B$ has a bounded (left) approximate identity then each of A and B has a bounded (left) approximate identity.*

This theorem answers the problem posed by Loy in [2] and extends the result of [3] in which the special case of the theorem when α is the greatest crossnorm is proved.

Let (z_n) be a bounded left approximate identity in $A \otimes_\alpha B$, say $\sup_n \|z_n\|_\alpha \leq K$. Then for each n, $z_n = \sum_{i=1}^{\infty} t_i^{(n)}$ where

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$t_i^{(n)} \in A \otimes B$ and the series is absolutely convergent in $A \otimes_\alpha B$, [1]. In the interest of simplicity of notation we suppress several indices which might help in identification and write $t_i^{(n)} = \sum_{j=1}^{K(i)} x_j \otimes y_j$.

Fix $a \in A$ with $\|a\|_A = 1$ and let $f \in A^*$ be such that $\|f\| = 1 = \langle f, a \rangle$. Define the operator $T_a : A \to A$ by $T_a(x) = x \cdot a$. It is clear that $\|T_a\| \le 1$. Let I denote the identity operator on B and $T_a \otimes I$ the tensor product of the maps T_a and I . Then $T_a \otimes I$ is continuous on $A \otimes_\lambda B$ and $\|T_a \otimes I\|_\lambda \le 1$, [4].

It follows that

$$\left\|T_a \otimes I\left(t_i^{(n)}\right)\right\|_\lambda = \left\|\sum_{j=1}^{K(i)} x_j \cdot a \otimes y_j\right\|_\lambda \le \left\|t_i^{(n)}\right\|_\lambda ,$$

and by definition

$$\left\|\sum_{j=1}^{K(i)} \langle f, x_j \cdot a \rangle y_j\right\|_B \le \left\|\sum_{j=1}^{K(i)} x_j \cdot a \otimes y_j\right\|_\lambda .$$

Therefore

$$\left\|\sum_{j=1}^{K(i)} \langle f, x_j \cdot a \rangle y_j\right\|_B \le \left\|t_i^{(n)}\right\|_\lambda$$

for all i and so

$$\sum_{i=1}^{\infty} \left\|\sum_{j=1}^{K(i)} \langle f, x_j \cdot a \rangle y_j\right\|_B \le \sum_{i=1}^{\infty} \left\|t_i^{(n)}\right\|_\lambda \le \sum_{i=1}^{\infty} \left\|t_i^{(n)}\right\|_\alpha \le K .$$

In particular, then, $\sum_{i=1}^{\infty} \sum_{j=1}^{K(i)} \langle f, x_j \cdot a \rangle y_j$ converges in B , say to q_n , and $\sup_n \|q_n\|_B \le K$. We claim (q_n) is a bounded left approximate identity in B .

To see this, let $b \in B$. Then $a \otimes b \in A \otimes_\alpha B$, so by assumption given $\varepsilon > 0$ there is an n_0 such that if $n \ge n_0$ then

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$\|z_n\cdot(a\otimes b) - a\otimes b\|_\alpha < \varepsilon$.

Let $n \geq n_0$. Then

$$\|z_n\cdot(a\otimes b) - a\otimes b\|_\lambda \leq \|z_n\cdot(a\otimes b) - a\otimes b\|_\alpha < \varepsilon ,$$

implying

$$\left\|\sum_{i=1}^{\infty}\sum_{j=1}^{K(i)}\langle f, x_j\cdot a\rangle y_j \cdot b - \langle f, a\rangle b\right\|_B < \varepsilon .$$

Since $\langle f, a\rangle = 1$ we have $\|q_n\cdot b-b\|_B < \varepsilon$ for $n \geq n_0$, implying (q_n) is a left approximate identity in B for which $\sup_n \|q_n\|_B \leq K$.

Trivial modifications of the proof show that A also has a bounded left approximate identity and that the theorem holds with "left" replaced by "right".

References

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