SINGULAR RIEMANNIAN STRUCTURES COMPATIBLE WITH π-STRUCTURES

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1. Introduction. G. Legrand [1] studied a generalization of the almost complex structures [2] by considering a linear operator J acting on the complexified space of a differentiable manifold V_m satisfying a relation of the form $J^2 = \lambda^2$(identity) where λ is a nonzero complex constant. Such structures are called π-structures. A π-structure is defined on V_m by the knowledge of two fields, of proper supplementary subspaces T_1 and T_2 of the complexified tangent space $T_x^{\mathbb{C}}$ at $x \in V_m$, such that $\dim(T_1) = n_1$; $\dim(T_2) = n_2$; $n_1 + n_2 = m$. In the remaining case, $\lambda = 0$, H.A. Eliopoulos [3] introduced almost tangent structures and discussed euclidean structures compatible with almost tangent structures [4]. In a similar way the purpose of this paper is to study singular riemannian structures compatible with π-structures, briefly R_π-structures.

2. We define on V_m, equipped with π-structure, a complex metric of class C^∞, that is, a symmetric tensor $G = (g_{ij})$ for which the components, in a system of local coordinates (x^i) are complex functions of (x^i) of class C^∞, with the condition that the rank of G is n_1. We will say that the metric G is compatible

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with π-structure if the scalar product of two arbitrary vectors of $T_x^{\mathbb{C}}$ is proportional to the scalar product of one of the vectors with the transform of the other by J. This means that for any $u, v \in T_x^{\mathbb{C}}$, one has

$$(2.1) \qquad (u, Jv) = \lambda (u, v)$$

where (u, v) denotes the scalar product $g_{ij} u^i v^j$. Relation (2.1) can be written in the form

$$(2.2) \qquad JG = \lambda G.$$

In the above case we shall say that V_m is endowed with a singular riemannian structure subordinate to π-structure, briefly, R_π-structure. Let us refer the space $T_x^{\mathbb{C}}$ to an adapted base. From the relation (2.2) we obtain

$$\begin{Vmatrix} \lambda I_{11} & O_{12} \\ O_{21} & -\lambda I_{22} \end{Vmatrix} \begin{Vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{Vmatrix} = \lambda \begin{Vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{Vmatrix}.$$

It is easy to see that G has the form

$$G = \begin{Vmatrix} G_{11} & O_{12} \\ O_{21} & O_{22} \end{Vmatrix}$$

where $G_{11} = (g_{\alpha\beta})$ is an $n_1 \times n_1$ matrix of rank n_1.

THEOREM 2.1. Given an arbitrary quadratic form on V_m defined by a tensor (a_{ij}) of rank m and a linear operator J on $T_x^{\mathbb{C}}$ such that $J^2 = \lambda^2$(identity), one is able to obtain from (a_{ij}) an R_π-structure.

Proof. Indeed, we shall show that one is able to take for G the matrix given by

$$G = JA + \lambda A \tag{2.3}$$

where $A = (a_{ij})$. We clearly have

$$\begin{aligned} JG &= J^2 A + J\lambda A \\ &= \lambda^2 A + \lambda JA \\ &= \lambda(\lambda A + JA) \\ &= \lambda G. \end{aligned}$$

Moreover, from (2.3) we have that, with respect to an adapted basis, G has the form

$$G = \begin{vmatrix} 2\lambda A_{11} & O_{12} \\ O_{21} & O_{22} \end{vmatrix}$$

where $A_{11} = (a_{\alpha\beta})$ is $n_1 \times n_1$ matrix. This means that $g_{\alpha\beta} = 2\lambda a_{\alpha\beta}$, $g_{\alpha\beta^*} = g_{\alpha^*\beta} = g_{\alpha^*\beta^*} = 0$. Since (a_{ij}) is of rank m, we have $\det(g_{\alpha\beta}) \neq 0$. Moreover, we note that under a change of basis

$$a_{j'k'} = A^h_{j'} A^i_{k'} a_{hi} .$$

In particular we have

$$a_{\alpha'\beta'} = A^h_{\alpha'} A^i_{\beta'} a_{hi} = A^\lambda_{\alpha'} A^\mu_{\beta'} a_{\lambda\mu} ,$$

so that $\det(A'_{11}) = \det(A'_{1})^2 \det(A_{11}) \neq 0$, which shows that $\det(g_{\alpha\beta}) \neq 0$ does not depend on the chosen adapted base. Hence (g_{ij}) is of rank n_1.

3. R_π-adapted bases. Consider at a point x of V_m an adapted basis (e_i) and the corresponding dual basis (θ^i). We have

$$ds^2 = g_{ij}\theta^i\theta^j = g_{\alpha\beta}\theta^\alpha\theta^\beta .$$

Since the quadratic form is of rank n_1, one can always find an orthonormal base $(e_{\alpha'})$ of T_1 by taking suitable linear combinations of (e_α). By doing so

$$ds^2 = \sum_1^{n_1} (\theta^\alpha)^2 .$$

One can also find the set of vectors $(e_{\alpha'}*)$ by suitable linear combinations of $(e_\alpha*)$ such that $Je_{\alpha'}* = -\lambda e_{\alpha'}*$. The vectors $(e_{i'}) = (e_{\alpha'}, e_{\alpha'}*)$ then form an adapted basis for which $(e_{\alpha'})$ are orthonormal. We will say that such a basis is adapted to the subordinate R_π-structure. In the sequal, we shall denote these bases by R_π-adapted bases.

Suppose now that (e_i) and $(e_{j'})$ are two R_π-adapted bases, then

$$g_{k'l'} = A^i_{k'} A^j_{l'} g_{ij} ,$$

where $(A^i_{k'}) = A = \begin{vmatrix} A_{11} & O_{12} \\ O_{21} & A_{22} \end{vmatrix}$ and $(g_{k'1'}) = G = \begin{vmatrix} I_{n_1} & O_{12} \\ O_{21} & O_{22} \end{vmatrix}$.

For the sake of convenience, we shall use A_1 and A_2 for A_{11} and A_{22} respectively. We may then write the above condition in the form

$$(3,1) \qquad G = A \cdot {}^t(AG),$$

where ${}^t(AG)$ is the tranpose of (AG),

$$\text{or} \begin{vmatrix} I_{n_1} & O_{12} \\ O_{21} & O_{22} \end{vmatrix} = \begin{vmatrix} A_1 & O_{12} \\ O_{21} & A_2 \end{vmatrix} \cdot \begin{vmatrix} {}^t(A_1) & O_{12} \\ O_{21} & O_{22} \end{vmatrix} = \begin{vmatrix} A_1\,{}^t(A_1) & O_{12} \\ O_{21} & O_{22} \end{vmatrix},$$

which means that $A_1 \cdot {}^t(A_1) = I_{n_1}$ or A_1 is orthonormal. We thus see that a transformation matrix of any two R_π-adapted bases is of the form

$$R = \begin{vmatrix} A_1 & O_{12} \\ O_{21} & A_2 \end{vmatrix} \quad \text{where } A_1 = (A^i_{k'}) \in O(n_1, \mathbb{C}).$$

Let $O(n_1, n_2)$ be the set of matrices of the form R. This set is the subset of $G(n_1, n_2)$ such that its elements satisfy the relation

$$(3.2) \qquad R\,{}^t(RG) = R\,.$$

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THEOREM 3.1. $O(n_1, n_2)$ <u>is a Lie subgroup of</u> $G(n_1, n_2)$.

<u>Proof</u>. Let R, R^1 belong to $O(n_1, n_2)$. Then

$$
\begin{aligned}
(R.R^1)^t(R R^1 G) &= (R R^1)^t(R^1 G).^t(R) \\
&= R[R^{1\,t}(R^1 G)]^t(R) \\
&= R[G].^t(R) \\
&= (R)^t(G)^t(R) \quad \text{as} \quad G = {}^t(G) \\
&= (R)^t(RG) = G
\end{aligned}
$$

and
$$
\begin{aligned}
(R^{-1})^t(R^{-1} G) &= (R^{-1})^t(G)^t(R^{-1}) \\
&= (R^{-1})(G)^t(R^{-1}) \\
&= (R^{-1})[R^t(RG)]^t(R^{-1}) \\
&= (R^{-1})(R)^t(RG)^t(R^{-1}) \\
&= {}^t(RG)^t(R^{-1}) = {}^t(R^{-1} RG) = G.
\end{aligned}
$$

Hence RR^1 and R^{-1} both belong to $O(n_1, n_2)$. This shows that $O(n_1, n_2)$ is a subgroup of $G(n_1, n_2)$. Since $O(n_1, n_2)$ is an algebraic subgroup of the Lie group $G(n_1, n_2)$, then necessarily $O(n_1, n_2)$ is itself a Lie group [7]. Let $E_{\mathbb{R}}(V_m)$ be the set of the R_π-adapted bases at the different points of V_m and let $p' : E_{\mathbb{R}}(V_m) \to V_m$ be the canonical mapping which to a base relative to x makes correspond x itself. Furthermore, let p' be the restriction of the canonical mapping $p: E_{\mathbb{C}}(V_m) \to V_m$ [$E_{\mathbb{C}}(V_m)$ being the set of all the complex bases] such that $E_{\mathbb{C}}(V_m)$ has, with respect to p, a natural structure of a principal fibre bundle with base V_m and the structure group $GL(m, \mathbb{C})$. We also know that $O(n_1, n_2)$ is a topological Lie subgroup of $G(n_1, n_2)$ and consequently of $GL(m, \mathbb{C})$. Hence the right translation by $g \in O(n_1, n_2)$ is the restriction to $E_{\mathbb{R}}(V_m)$ of the right translation operated on $E_{\mathbb{C}}(V_m)$. From this it is obviously true that for every $x \in V_m$ there exists a neighbourhood u of x and a differentiable section of $E_{\mathbb{C}}(V_m)$ with values in $E_{\mathbb{R}}(V_m)$. Hence one can deduce

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from a proposition of D. Bernard [5, Proposition 1, 5, 2] that $E_{\mathbb{R}}(V_m)$ is a differentiable principal subfibre bundle of $E_{\mathbb{C}}(V_m)$ with base V_m and structure group $O(n_1, n_2)$.

4. R_π-connections. We will call R_π-connection any infinitesimal connection [2] defined on the fibre bundle $E_{\mathbb{R}}(V_m)$. Given a covering of V_m by neighbourhoods endowed with local cross sections of $E_{\mathbb{R}}(V_m)$ an R_π-connection may be defined in each neighbourhood u by a form W_u with values in the Lie algebra $LO(n_1, n_2)$ of the group $O(n_1, n_2)$. Such a form may be represented by $x \in V_m$ by means of a matrix of order m whose elements are complex valued linear forms at x; it will be denoted locally by $W_u = (W^i_j)$ where $W^i_j \in LO(n_1, n_2)$.

To determine the form of the elements of $LO(n_1, n_2)$ we recall that $O(n_1, n_2)$ consists of matrices R of $GL(m, \mathbb{C})$ such that $R^t(RG) = G$. The Lie algebra of $O(n_1, n_2)$ consists of the set of all the infinetesimal right translations of $O(n_1, n_2)$ defined by a tangent vector at the identity element of $O(n_1, n_2)$. Thus, one can show that $O(n_1, n_2)$ consists of m × m matrices

$$(4.1) \qquad R = \begin{vmatrix} A_1 & O_{12} \\ O_{21} & A_2 \end{vmatrix} \quad \text{where } \overline{RG} + {}^t(RG) = O ,$$

where $\overline{RG}$ is the conjugate of RG. Indeed, let us assume that $\overline{RG} + {}^t(RG) = O$ and $\overline{R^1 G} + {}^t(R^1 G) = O$. For simplicity, we set $RG = X$ and $R^1 G = Y$. Also set $Z = [X, Y] = XY - YX$.

$$\begin{aligned} {}^t(Z) = {}^t(XY) - {}^t(YX) &= {}^t(Y)\,{}^t(X) - {}^t(X)\,{}^t(Y) \\ &= (-\overline{Y})(-\overline{X}) - (-\overline{X})(-\overline{Y}) \\ &= \overline{Y}.\overline{X} - \overline{X}.\overline{Y} \\ &= -\overline{Z} . \end{aligned}$$

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Hence ${}^{t}(Z) + \bar{Z} = O$ which implies that $[X, Y] \in LO(n_1, n_2)$. With respect to an R_π-adapted basis (4.1) can be written as

$$\begin{vmatrix} A_1 & O_{12} \\ O_{21} & O_{22} \end{vmatrix} + \begin{vmatrix} {}^{t}(A_1) & O_{12} \\ O_{21} & O_{22} \end{vmatrix} = \begin{vmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{vmatrix}$$

or

$$A_1 + {}^{t}(A_1) = O \tag{4.2}$$

$E_{\mathbb{R}}(V_m)$ being a subbundle of the fibre bundle $E_{\mathbb{C}}(V_m)$, one concludes that any R_π-connection defines canonically a linear connection with which it can be identified.

Conversely, let us consider a complex linear connection and a covering of V_m by open sets, each equipped with a local form, with values in the Lie algebra of $GL(m, \mathbb{C})$, represented by a matrix (W^i_j) whose elements are complex valued local Pfaffian forms. In order that the given connection be able to be identified with an R_π-connection it is necessary and sufficient that (W^i_j) belongs to the Lie algebra of the structure group $O(n_1, n_2)$ of $E_{\mathbb{R}}(V_m)$, that is to say that the following conditions be satisfied:

$$W^{\beta *}_{\alpha} = W^{\alpha}_{\beta *} = O \tag{4.3}$$

$$W^{\beta}_{\alpha} + W^{\alpha}_{\beta} = O\,. \tag{4.4}$$

As shown by G. Legrand [1] the condition (4.3) expresses that the tensor $J = (F^i_j)$ has absolute differential zero (which is a necessary and sufficient condition that the given connection be a π-connection).

The condition (4.4) expresses that the submatrix (W^{α}_{β}) belongs to the Lie algebra of the group $O(n_1, \mathbb{C})$. In order to interpret (4.4) we introduce the absolute differential of the tensor, assuming the condition (4.3). We have

$$\nabla g_{ij} = - W^{k}_{i} g_{kj} - W^{k}_{j} g_{ik} .$$

We also recall that with respect to R_π-adapted basis $g_{\alpha\beta} = \delta_{\alpha\beta}$. Hence we have

$$\nabla g_{\alpha\beta} = - (W^{\lambda}_{\alpha} g_{\lambda\beta} + W^{\lambda^*}_{\alpha} g_{\lambda^*\beta}) - (W^{\lambda}_{\beta} g_{\alpha\lambda} + W^{\lambda^*}_{\beta} g_{\alpha\lambda^*}) = -(W^{\beta}_{\alpha} + W^{\alpha}_{\beta}) = 0$$

$$\nabla g_{\alpha\beta^*} = - (W^{\lambda}_{\alpha} g_{\lambda\beta^*} + W^{\lambda^*}_{\alpha} g_{\lambda^*\beta^*}) - (W^{\lambda}_{\beta^*} g_{\alpha\lambda} + W^{\lambda^*}_{\beta^*} g_{\alpha\lambda^*}) = 0$$

$$\nabla g_{\alpha^*\beta^*} = - (W^{\lambda}_{\alpha^*} g_{\lambda\beta^*} + W^{\lambda^*}_{\alpha^*} g_{\lambda^*\beta^*}) - (W^{\lambda}_{\beta^*} g_{\alpha^*\lambda} + W^{\lambda^*}_{\beta^*} g_{\alpha^*\lambda^*}) = 0 .$$

Hence $\nabla g_{ij} = 0$.

THEOREM 4.1. The absolute differential of the metric tensor in an R_π-connection is zero.

Combining this result with $\nabla F^{i}_{j} = 0$, we have

THEOREM 4.2. A complex linear connection can be identified with an R_π-connection iff the tensors (F^{i}_{j}) and (g_{ij}) have absolute differential zero.

We will say that a complex linear connection defined on a complex metric (g_{ij}) is euclidean if $\nabla g_{ij} = 0$. The preceding theorm expresses that one is able to identify the R_π-connection with euclidean π-connection.

5. Holonomy groups of the R_π-connections. Let us be given in an R_π-structure an R_π-connection. Any horizontal path constructed in $E_{\mathbb{C}}(V_m)$ relative to the complex linear connection

coinciding with the R_π-connection and beginning at an R_π-adapted basis b ends at an R_π-adapted basis. One concludes from this that the holonomy group at b[2] of the complex linear connection is a subgroup of $O(n_1, n_2)$.

Conversely, let V_m be a differentiable manifold endowed with a complex linear connection and let us suppose that at the point x of V_m there exists a complex basis b such that the holonomy group ψ_b of the connection at b is a subgroup of $O(n_1, n_2)$. Let us consider at the point x the tensors (g_{ij}) and (F^i_j) defined on the whole manifold. Now at the point x we have $F^i_h F^h_j = \lambda^2 \delta^i_j$ and

$$\begin{aligned} F^i_k g_{ij} - \lambda g_{jk} &= F^\alpha_\beta g_{\alpha\gamma} - \lambda g_{\gamma\beta} \\ &= \lambda \delta^\alpha_\beta \delta_{\alpha\gamma} - \lambda \delta_{\gamma\beta} \\ &= \lambda \delta_{\beta\gamma} - \lambda \delta_{\gamma\beta} \\ &= \lambda - \lambda \\ &= 0. \end{aligned}$$

which implies that $JG - \lambda G = O$ or $JG = \lambda G$. These relations remain true at any point of V_m. One thus has defined on V_m an R_π-structure. Since the tensors (g_{ij}) and (F^i_j) are invariant under ψ_b they have absolute differential zero [2]. Thus the given connection is able to be identified with an R_π-connection. We may thus state the following.

THEOREM 5.1. <u>In order that a differentiable manifold has an R_π-structure it is necessary and sufficient that there exists a complex linear connection whose holonomy group is a subgroup of $O(n_1, n_2)$.</u>

6. <u>Note on characteristic forms.</u> An R_π-connection determines canonically a π-connection. We can thus associate to it characteristic forms defined by:

$$\psi_1 = \lambda \Omega^\alpha_\alpha, \quad \psi_2 = -\lambda \Omega^{\alpha*}_{\alpha*},$$

where $\Omega^i_j = d\pi^i_j + \pi^i_h \wedge \pi^h_j$ is a tensor 2-form. If the connection is defined with respect to π-adapted bases by (π^i_j), we have

$$\psi_1 = \lambda d(\pi^\alpha_\alpha), \quad \psi_2 = -\lambda d(\pi^{\alpha*}_{\alpha*}).$$

This given connection being an R_π-connection, we have $\pi^\beta_\alpha + \pi^\alpha_\beta = 0$, or $\pi^\alpha_\alpha + \pi^\alpha_\alpha = 0$, or $\pi^\alpha_\alpha = 0$. Hence $\psi_1 = \lambda d(\pi^\alpha_\alpha) = 0$. This leads to the following theorem.

THEOREM 6.1. The first characteristic form ψ_1 is zero for any R_π-connection.

7. R_π-structures and infinitesimal transformations. It has been proved by G. Legrand [1] that with a π-structure, one can associate a tensor field t (two times covariant and one time contravariant) in V_m called the torsion of π-structure. He has also shown that this torsion can be regarded as the torsion of a suitable π-connection. Furthermore, one can deduce from Section 4 of this paper that any R_π-connection defines canonically a π-connection with which it can be identified. We conclude that the torsion t of a π-structure can also be regarded as the torsion of a suitable R_π-connection. Let us recall that if π-structure is integrable then t is identically equal to zero. The converse is true only if π-structure is of class C^ω. It is obviously true that this condition of integrability is same for R_π-structure.

We consider the set $M(V_m)$ of all the contravariant differentiable vector fields (infinitesimal transformations) in V_m. We also consider the following bilinear operators in $M(V_m)$, associating to $u, v \in M(V_m)$ a field $\omega \in M(V_m)$ [9].

(i) $\omega = [u, v]$ (bracket of Poisson), defined with the help of local coordinates x^j and of the corresponding local components

u^j, v^j of u and v by

$$\omega^j = v^k \frac{\partial u^j}{\partial x^k} - u^k \frac{\partial v^j}{\partial x^k} \qquad (j = 1, \ldots, m);$$

(ii) $\omega = [u, v]_\pi$, π being briefly denoted by any R_π-connection in V_m, defined by $\omega^j = v^k \nabla_k u^j - u^k \nabla_k v^j$, where ∇_j is the covariant derivative with respect to π;

(iii) $\omega = T(u, v)$, T being the torsion of the R_π-connection defined by $\omega^j = T^j_{k\ell} u^k v^\ell$. At the point $x \in V_m$, $T(u, v)$ depends only on the vectors u, v at x, whereas the bracket depends on the fields u, v.

The relation $[u, v]_\pi = [u, v] + T(u, v)$ is obviously true. We claim that

$$(7.1) \qquad [Ju, Jv]_\pi = J[Ju, v]_\pi + J[u, Jv]_\pi - \lambda^2 [u, v]_\pi .$$

Indeed, in terms of the local coordinates x^j, let F^j_k be the components of the tensor field representing J(x), then $\nabla_j F_\ell{}^k = 0$. The j-th component of $[Ju, Jv]_\pi$ is equal to

$$F^j_k F^p_\ell (v^\ell \nabla_p u^k - u^\ell \nabla_p v^k)$$

$$= -\lambda^2 (v^p \nabla_p u^j - u^p \nabla_p v^j) + (F^j_k v^p \nabla_p (F^k_\ell u^\ell) - F^j_k F^p_\ell u^\ell \nabla_p v^k)$$

$$+ (F^j_k F^p_\ell v^\ell \nabla_p u^k - F^j_k u^p \nabla_p (F^k_\ell v^\ell))$$

which is the j-th component of the second member of (7.1). Replacing in (7.1) $[u, v]_\pi$ by $[u, v] + T(u, v)$ we get

$$(7.2) \qquad [Ju, Jv] = J[Ju, v] + J[u, Jv] - \lambda^2 [u, v] + JT(Ju, v) + JT(u, Jv) - T(Ju, Jv) - \lambda^2 T(u, v).$$

Let us in particular arrange in such a manner that T is the torsion t of the R_π-structure. Now in this case the operators J and $T(u, v) = t(u, v)$ in $M(V_m)$ are related by

(7.3) $$t(u, Jv) = t(Ju, v) = -Jt(u, v).$$

These relations are consequences of the definition of the tensor t.

Operating J on the relations (7.3), we get

$$Jt(u, Jv) = Jt(Ju, v) = -J^2t(u, v) = -\lambda^2 t(u, v).$$

Also replacing u by Ju in (7.3), we get

$$t(Ju, Jv) = t(J^2u, v) = \lambda^2 t(u, v).$$

Substituting these values in (7.2) and replacing T by t, we have

$$[Ju, Jv] = J[Ju, v] + J[u, Jv] - \lambda^2[u, v] - 4\lambda^2 t(u, v)$$

or

(7.4) $$-4\lambda^2 t(u, v) = [Ju, Jv] - J[Ju, v] - J[u, Jv] + \lambda^2[u, v].$$

Thus the condition of integrability $t = 0$ can be formulated as follows:

In order that the R_π-structure in V_m be without torsion (integrable, if the structure is of class C^ω) it is necessary and sufficient that $[Ju, Jv] = J[Ju, v] + J[u, Jv] - \lambda^2[u, v]$ for all vector fields $u, v \in M(V_m)$.

The Nijenhuis tensor is defined [8] by

$$N(u, v) = [Ju, Jv] + J^2[u, v] - J[Ju, v] - J[u, Jv]$$

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for any vector fields u, v. For R_π-structure we have $J^2 = \lambda^2$, then

$$N(u, v) = [Ju, Jv] + \lambda^2[u, v] - J[Ju, v] - J[u, Jv];$$

comparing this relation with (7.4), we have

$$N(u, v) = -4\lambda^2 t(u, v). \tag{7.5}$$

Hence we conclude this section by stating the following theorem.

THEOREM 7.1. In order that the R_π-structure in V_m be completely integrable it it necessary and sufficient (only if the structure is of class C^ω) that the Nijenhuis tensor be equal to zero.

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