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AN ELEMENTARY PROOF OF A FIXED POINT THEOREM OF J. LEWITTES AND D. L. McQUILLAN

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In (3), J. Lewittes establishes a connection between the number of fixed points of an automorphism of a compact Riemann surface and Weierstrass points on the surface; Lewittes' techniques are analytic in nature. In (4), D. L. McQuillan proved the result by purely algebraic methods and extended it to arbitrary algebraic function fields in one variable over algebraically closed ground fields, but with restriction to tamely ramified places. In this paper we will give a different proof of the theorem and show that it is an elementary consequence of the Riemann-Hurwitz relative genus formula. Moreover, we can remove the tame ramification restriction.

For notation, definitions, and standard theorems used here, we refer the reader to references (1), (2) and (5).

Let k be an algebraically closed field and let M be an algebraic function field of one variable over k. Let $\mathrm{Aut}(M)$ denote the full automorphism group of M over k and let $P(M)$ denote the set of places of M. It is well known that there is a natural representation of $\mathrm{Aut}(M)$ into the symmetric group on $P(M)$. Let $g(M)$ denote the genus of M.

We shall prove the following

THEOREM. *Assume that $\sigma \neq id$ is an element in* $\mathrm{Aut}(M)$ *and that σ has at least five fixed points in $P(M)$. Then each of these fixed points has a non-gap $\leq g(M)$.*

We will need to use the following easily proved

LEMMA. *Let M and L be algebraic function fields of one variable over k and assume that M is a finite algebraic extension of L. Let $\mathfrak{P} \in P(M)$ and assume that it is totally ramified over L. Set $\mathfrak{p} = L \cap \mathfrak{P}$. If μ is an integer and if $x \in \mathbb{L}_L(\mathfrak{p}^{\mu})$, then $x \in \mathbb{L}_M(\mathfrak{P}^{e\mu})$ where e is the ramification index of $\mathfrak{P}$ over L.*

Proof of the Theorem. Assume that σ is a non-trivial element of $\mathrm{Aut}(M)$ and let L denote the fixed field of σ. Without loss of generality we may assume that the order d of σ is a prime number. Since $[M:L] = d$ and since d is prime, each of the fixed points $\mathfrak{P}$ of σ is totally ramified over L, i.e., $e_{\mathfrak{P}} = d$.

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Furthermore, the set of fixed points coincides with the set of ramification points in M over L. Let ν denote the number of fixed points of σ and let $\mathfrak{D}$ denote the different of M over L. Since M is separable over L we may apply the Riemann-Hurwitz formula (see (2)) to obtain

$$\begin{aligned} 2g(M) &= d(2g(L)-2)+\deg \mathfrak{D}+2 \\ &\geq d(2g(L)-2)+\sum_{\mathfrak{P}}(e_{\mathfrak{P}}-1)+2 \\ &= d(2g(L)-2)+\nu d-\nu+2. \end{aligned}$$

Hence we have

(A) $$2g(M)\geq 2dg(L)+(d-1)(\nu-2).$$

If $d\geq 3$, then since $\nu\geq 5$ we obtain from ineq. (A) the inequality

(B) $$2g(M)\geq 2dg(L)+2d.$$

If $d=2$, we must argue according to whether char k is or is not equal to 2. If char $k\neq 2$, then we have

$$\deg \mathfrak{D}=\sum_{\mathfrak{P}}(e_{\mathfrak{P}}-1)=\nu$$

since in this case the ramification points are tamely ramified and $e_{\mathfrak{P}}=2$ for each such point. From the relative genus formula we obtain

(C) $$2g(M)=4g(L)+\nu-2.$$

Hence ν is an even integer >5 and we obtain from eq. (C)

(D) $$2g(M)\geq 4g(L)+4.$$

Now if char $k=2$, then each fixed point $\mathfrak{P}$ of σ is wildly ramified and we have

$$\operatorname{ord}_{\mathfrak{P}} \mathfrak{D}\geq e_{\mathfrak{P}}.$$

Hence the relative genus formula yields

$$\begin{aligned} 2g(M) &= 4(g(L)-1)+\deg \mathfrak{D}+2 \\ &\geq 4g(L)-2+\sum_{\sigma(\mathfrak{P})=\mathfrak{P}} e_{\mathfrak{P}} \\ &= 4g(L)-2+2\nu. \end{aligned}$$

Therefore, since $\nu\geq 5$ we obtain

(E) $$2g(M)\geq 4g(L)+4.$$

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We see from inequalities (B), (D) and (E) that regardless of char k we have

$$g(M) \geq d(g(L)+1). \tag{F}$$

If $\mathfrak{P}$ is any place of M and $\mathfrak{p} = L \cap \mathfrak{P}$, then by the Riemann-Roch Theorem we have

$$l_L(\mathfrak{p}^{g(L)+1}) \geq 2.$$

Hence $\mathbb{L}_L(\mathfrak{p}^{g(L)+1})$ contains a non-constant function, say x. If $\mathfrak{P}$ is a fixed point of σ, then by the lemma we have $x \in \mathbb{L}_M(\mathfrak{P}^{d(g(L)+1)})$. Therefore, by inequality (F), $\mathfrak{P}$ has a non-gap $\leq g(M)$. ■

COROLLARY. *If the regular gap sequence of M is $(1, 2, \ldots, g(M))$, then each fixed point of an automorphism which satisfies the conditions of the theorem is a Weierstrass point; in particular, this is the case if* char $k = 0$.

It is interesting to note that if char $k \neq 0$, one can deduce from Boseck's computations in (1) that if the fixed field L of $\sigma \in \operatorname{Aut}(M)$ satisfies $g(L) = 0$ and if σ fixes at least five places, then these places are Weierstrass points. However, if $g(L) > 0$, the evidence is that the result is probably true but no proof is known.

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