PRIME MODULES

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Introduction. Characterizations for prime and semi-prime rings satisfying the right quotient conditions (see § 1) have been determined by A. W. Goldie in (**4** and **5**). A ring R is prime if and only if the right annihilator of every non-zero right ideal is zero. A natural generalization leads one to consider right R-modules having the properties that the annihilator in R of every non-zero submodule is zero and regular elements in R annihilate no non-zero elements of the module. This is the motivation for the definition of prime module in § 1.

By employing some ideas of R. E. Johnson together with those of Goldie, along with some innovations, we are able to generalize Goldie's work (**4**) to modules and add some new results. Specifically, Theorem (3.2) gives an external characterization for a prime R-module M in terms of a completely reducible module containing M. It is interesting to note that in (3.2), although right quotient conditions are assumed in R, it is unnecessary to assume a maximum condition for the submodules of M. An internal type of structure for prime modules (hence prime rings) is also obtained in § 3. It is shown that if R satisfies the right quotient conditions, then an R-module M is Noetherian and prime if and only if M is a subdirect sum of uniform Noetherian prime R-modules.

In (4.4) we characterize the uniform prime modules as those prime modules for which the ring of endomorphisms of the injective envelope is a division ring. For a prime module M over a prime ring R satisfying the right quotient conditions, we know, by (**4**), that $R \subseteq D_n$, a total matrix ring over a division ring D. We obtain D in (4.8) using a uniform prime R-module as given by the structure theorem (3.3). This is not a uniform submodule of M as in Goldie's work, but a module of the form M/J, where J is $\cap$-irreducible.

It is also shown, in § 4, that every prime R-module M contains a uniform B-module N, where B is a prime ring containing R and where N over B has the double centralizer property. This provides the result that every prime ring R with right quotient conditions is contained in a prime ring B, where B is the ring of endomorphisms of a module over an integral domain and where B and R have the same right quotient ring.

In § 5 we consider a finitely generated module M over a right and left Ore domain A. By taking M as a module over $\mathrm{Hom}_A(M, M)$ we obtain a non-trivial example of a uniform prime module.

Received September 1, 1964. This research was supported in part by grants from the National Science Foundation.

https://doi.org/10.4153/CJM-1965-099-5 Published online by Cambridge University Press

In the final section we give a matrix representation for prime rings of the type discussed in § 5.

1. Conventions and definitions. Throughout this paper all R-modules will be right R-modules. In addition, R *will always denote a ring that satisfies the right quotient conditions* of **(4** and **5)**. These are:

(i) every direct sum of non-zero right ideals of R has a finite number of terms;

(ii) the ascending chain condition holds for the annihilator right ideals of R.

Let A and B be rings and let M be an (A, B)-bimodule, i.e. a left A-module and a right B-module. We shall adopt the following notation for the various annihilators to be considered. If X, Y, and Z are subsets of A, M, and B respectively, then

$$X^r = \{m \in M \mid xm = 0 \text{ for all } x \in X\},$$

$$Y^r = \{b \in B \mid yb = 0 \text{ for all } y \in Y\},$$

and

$$Z^l = \{m \in M \mid mz = 0 \text{ for all } z \in Z\}.$$

For $Z \subseteq B$ we shall use *subscripts* to denote annihilators of Z in the ring B. Thus

$$Z_r = \{b \in B \mid zb = 0 \text{ for all } z \in Z\}$$

and

$$Z_l = \{b \in B \mid bz = 0 \text{ for all } z \in Z\}.$$

If M and N are R-modules, then M is an *essential extension* of N if $N \subseteq M$ and if $N \cap P \neq 0$ for every non-zero submodule P of M. If M is an essential extension of N, we write $N \subseteq' M$ and call N a *large submodule* of M. We shall also speak of large right ideals of R by considering R as a right module over itself.

It will be convenient to use the following system when dealing with endomorphisms. If M is an R-module and $H = \mathrm{Hom}_R(M, M)$, then M will be considered as a left H-module (more specifically, M is an (H, R)-bimodule) where, as usual, for $h \in H$ and $m \in M$, hm denotes the image of m under h. Again, writing the elements of $K = \mathrm{Hom}_H(M, M)$ as right operators, M becomes a right K-module. If $M^r = 0$, then the usual correspondence yields an embedding of R in K and we may assume, in this case, that $R \subseteq K$.

If M is an R-module, then

$$M^{\blacktriangle} = \{x \in M \mid x^r \subseteq' R\}$$

is a submodule called the *singular submodule* of M. Similarly,

$$R^{\blacktriangle} = \{a \in R \mid a_r \subseteq' R\}$$

is a two-sided ideal in R called the *singular ideal* of R.

As in **(12)**, call an R-module *torsion-free* if, whenever $xc = 0$ for $x \in M$ and c regular in R, then $x = 0$.

The prime rings of **(4)** provide the following natural extension for a definition of prime modules.

(1.1) *Definition.* An R-module M is *prime* if $N^r = 0$ for every non-zero submodule N of M and if one of the following equivalent conditions is satisfied:

(i) $M^{\blacktriangle} = 0$;

(i′) M is torsion-free.

In order to show the equivalence of (i) and (i′) one can proceed as follows. Let J be a non-zero right ideal in R (in general we shall exclude from our discussion the trivial cases $M = 0$ or $R = 0$). Since $M^r = 0$, MJ is a non-zero submodule of M. From $(MJ)J_r = 0$, we have $J_r \subseteq (MJ)^r = 0$, whence R is a prime ring. Then by **(5**, 3.9**)**, an element x of M belongs to $M^{\blacktriangle}$ if and only if x^r contains a regular element of R. Thus $M^{\blacktriangle} = 0$ if and only if M is torsion-free.

This discussion gives us

(1.2.) Proposition. *If M is an R-module such that $N^r = 0$ for every non-zero submodule N, then R is a prime ring.*

Let M be a torsion-free R-module where R is a prime ring and let N be a non-zero submodule of M. Since N^r is a two-sided ideal of R, $N^r = 0$; for otherwise by **(5**, 3.9**)** N^r contains a regular element c, contradicting the fact that M is torsion-free. This proves

(1.3) Proposition. *If M is an R-module and R is a prime ring, then M is prime if and only if either M is torsion-free or $M^{\blacktriangle} = 0$.*

It is clear that the prime rings of **(4)**, when considered as right modules over themselves, are prime modules. Obviously, a prime R-module is faithful and every submodule of a prime R-module is a prime R-module.

It should be noted that different definitions for the term "prime module" occur elsewhere in the literature. Our definition most closely resembles that given by R. E. Johnson in **(9**, p. 353**)**, where a right B-module M is called prime provided $N^r = 0$ for every non-zero submodule N. A somewhat different definition is stated in **(10)**, where a right B-module M is designated as prime if and only if whenever $xJ = 0$ for $x \in M$ and J an ideal in B such that $J_l = 0$, then $x = 0$. Evidently, if M is prime in the sense of **(9)** or (1.1), then it is also prime according to **(10)**. The converse is false, as may be seen by the following example.

Example. A ring B is termed *N-local* if the elements not in the nil-radical N are the units of B. When B satisfies the maximum condition for right ideals, N is nilpotent. Let M be a unitary right B-module, where B is an N-local ring satisfying the maximum condition for right ideals and where $N \neq 0$. Let J be an ideal in B for which $J_l = 0$ and suppose $xJ = 0$ for some $x \in M$.

If $J \subseteq N$, then, since $N^n = 0$ and $N^{n-1} \neq 0$ for some positive integer n, we have $N^{n-1}J = 0$ contrary to $J_l = 0$. Thus J is not contained in N and there exists $a \in J$ such that $a \notin N$. Since B is N-local, a is a unit in B. Then from $xa = 0$ we have $x = 0$, which proves that M is prime in the sense of **(10)**. On the other hand, M is not prime according to **(9)** since B is not a prime ring.

2. Submodules of prime modules. If M is a prime R-module, then as in **(10**. 6.4**)** and **(6**, 3.8**)** each submodule N of M has a unique maximal essential extension cl N in M given by

$$\text{cl } N = \{x \in M \mid xI \subseteq N \text{ for some large right ideal } I \text{ of } R\}.$$

By **(5**, 3.9**)**, a right ideal of R is large if and only if it contains a regular element. It follows that $x \in \text{cl } N$ if and only if $xc \in N$ for some regular $c \in R$.

Similarly, each right ideal J in R has a unique maximal essential extension in R.

(2.1) LEMMA. *Let M be a prime R-module and let N be a proper submodule of M. Then M/N is a prime R-module if and only if* cl $N = N$.

Proof. From the remarks above, $x \in \text{cl } N$ if and only if $(x + N)c$ is zero in M/N for some regular $c \in R$. Consequently, if $x \in \text{cl } N$ and M/N is prime (hence torsion-free by (1.3)), then $x \in N$ and we have $\text{cl } N \subseteq N$, whence $\text{cl } N = N$.

Conversely, if $\text{cl } N = N$ and $(x + N)c \subseteq N$ for some regular $c \in R$, then $x \in N$ so that M/N is a prime R-module.

(2.2) THEOREM. *Let M be a prime R-module and suppose N is an $\cap$-irreducible submodule of M which is not large. Then M/N is a prime module.*

Proof. Let P be a non-zero submodule of M such that $N \cap P = 0$. Then we have $N \subseteq \text{cl } N \cap (N \oplus P)$. On the other hand, let $x = n + p \in \text{cl } N \cap (N \oplus P)$, where $n \in N$, $p \in P$. Since $n + p \in \text{cl } N$, there is a regular element c in R such that $(n + p)c \in N$. Hence $pc \in N \cap P = 0$ and since M is torsion-free, $p = 0$. Thus $x \in N$, and we have proved that $N = \text{cl } N \cap (N \oplus P)$. Since N is $\cap$-irreducible, $N = \text{cl } N$ and by (2.1), M/N is a prime module.

(2.3) LEMMA. *If M is a prime R-module, then for every $x \in M$,* $\text{cl } (x^r) = x^r$ *in R.*

Proof. We know that $x^r \subseteq \text{cl } (x^r)$. Let $a \in \text{cl } (x^r)$. Then $ac \in x^r$ for some regular $c \in R$. Therefore $xac = 0$ and since M is torsion-free, $xa = 0$. Hence $a \in x^r$ and the lemma is proved.

A submodule U of M is *uniform* if $U \neq 0$ and every pair of non-zero submodules of U has non-zero intersection. Similarly one defines uniform right ideal of R.

The proof of the following theorem is patterned after that given for rings in **(6**, p. 68**)**.

(2.4) THEOREM. *Let M be a right R-module such that $M^{\blacktriangle} = 0$. If J is a uniform right ideal in R, then*

$$J^{l} = \{x \in M \mid x^{r} \cap J \neq 0\}.$$

Proof. Let $N = \{x \in M \mid x^{r} \cap J \neq 0\}$. Take $x \in N$, $j \in J$, and let K be any non-zero right ideal in R. If $jK = 0$, then $xjK = 0$ and hence $K \subseteq (xj)^{r}$. Thus $(xj)^{r} \cap K \neq 0$. On the other hand, if $jK \neq 0$, then $(x^{r} \cap J) \cap jK \neq 0$ since J is uniform. Therefore K contains an element k such that $jk \neq 0$ and $xjk = 0$ and again we have $(xj)^{r} \cap K \neq 0$. This proves that $xj \in M^{\blacktriangle}$ for all $x \in N, j \in J$, and hence $NJ \subseteq M^{\blacktriangle}$. Since $M^{\blacktriangle} = 0$, we have $N \subseteq J^{l}$. The proof that $J^{l} \subseteq N$ is direct.

3. Internal and external characterizations of prime modules. If M is a prime R-module, then R is a prime ring with right quotient conditions and by Goldie's Theorem R has a right quotient ring S that is a simple ring with minimum condition (the term right (left) quotient ring, whenever it is used in this paper, refers to the *classical* quotient rings described in **(4**, p. 604**)**). As in **(12**, p. 134**)**, the mapping $m \to m \otimes 1$, $m \in M$, is an R-isomorphism of M into the tensor product $M \otimes_{R} S$, every element of MS has the form mc^{-1} where $m \in M$ and c is regular in R, and $M \otimes_{R} S \cong MS$ under the correspondence $m \otimes s \to ms$, $s \in S$. As usual, we consider M as a submodule of MS and identify $m \in M$ with $m1$. In similar fashion if N is a submodule of M, we have

$$N \subseteq N \otimes_{R} S \cong NS = \{nc^{-1} \mid n \in N, \ c \text{ regular in } R\}.$$

Every R-module M has a unique (up to isomorphism) maximal essential extension $\tilde{M}$ that is simultaneously the unique minimal injective extension of M. We call $\tilde{M}$ the injective envelope of M. In our case we have

(3.1) THEOREM. *If M is a prime R-module, then the injective envelope $\tilde{M}$ of M is MS, where S is the right quotient ring of R.*

Proof. If $x = mc^{-1} \in MS$, then $xc = m \in M$, which implies that MS is an essential extension of M.

Suppose $M \subseteq M'$, where M' is any essential extension of M. If x is any non-zero element of M', there exists $b \in R$ such that $0 \neq xb \in M$. Let $J = \{a \in R \mid xa \in M\}$. Using the method of proof given in **(6**, p. 63**)** one can show that J is a large right ideal in R and therefore J contains a regular element c. Thus $xc = m \in M$ and $x = mc^{-1} \in MS$. Then MS is the maximal essential extension $\tilde{M}$ of M.

The following theorem gives a characterization for prime R-modules.

(3.2) THEOREM. *An R-module M is prime if and only if M is contained in a completely reducible unitary right S-module where S is a simple ring with minimum condition for right ideals and a right quotient ring for R.*

Proof. If M is prime, then by our previous discussion and (**8**, p. 47), the right S-module MS fulfils the conditions stated in the theorem.

Conversely, suppose M is contained in a unitary right S-module M' having the indicated properties. From (**4** and **5**), R is a prime ring with right quotient conditions. Let $xc = 0$, where $x \in M$ and c is regular in R. Then c^{-1} exists in S and we have $x = x1 = (xc)c^{-1} = 0$. Hence M is torsion-free over R and, by (1.3), M is prime.

The preceding result may be interpreted as an "external" characterization of prime modules. If M is Noetherian, in the sense that the submodules satisfy the maximum condition, an "internal" type structure for prime modules can be obtained.

We employ the notion of irredundant subdirect sum as given by Levy (**13**, p. 65). As pointed out in (**13**), a module M is an irredundant subdirect sum of modules $\{M_\alpha\}$ if and only if there exists a set of submodules $\{P_\alpha\}$ of M such that $M_\alpha \cong M/P_\alpha$, $\bigcap_\alpha P_\alpha = 0$, and, for each β, $\bigcap_{\alpha\neq\beta} P_\alpha \neq 0$. We shall use this criterion to establish

(3.3) THEOREM. *An R-module M is a Noetherian prime R-module if and only if M is a finite irredundant subdirect sum of uniform Noetherian prime R-modules.*

Proof. If M satisfies the maximum condition for submodules, we can write $0 = N_1 \cap \ldots \cap N_t$, where each N_i is an $\cap$-irreducible submodule of M and where, for each k, $\bigcap_{i\neq k} N_i \neq 0$. Setting $M_i = M/N_i$, M is an irredundant subdirect sum of the M_i. If $t = 1$, then $M = M_1$ and M is uniform. If $t > 1$, then the N_i are not large and by (2.2) each M/N_i is a prime module. It follows readily that each M/N_i is uniform and Noetherian.

Conversely, suppose that M is an irredundant subdirect sum of uniform prime Noetherian R-modules $M_1, \ldots, M_t$. Then there exist submodules $P_1, \ldots, P_t$ of M such that, for all i, $M_i \cong M/P_i$, $\bigcap_i P_i = 0$, and, for each k, $\bigcap_{i\neq k} \mathrm{P}_i \neq 0$.

If $t = 1$ the result is trivial. If $t > 1$, then the P_i are proper submodules and, from (2.1), $\mathrm{cl}\, P_i = P_i$ for each i. Let $xa = 0$, where $x \in M$ and a is regular in R. Then $xa \in P_i$ for all i and we have $x \in \mathrm{cl}\, P_i = P_i$, $1 \leqslant i \leqslant t$. Therefore $x \in \bigcap_i P_i = 0$ and M is torsion-free and prime.

The fact that M is Noetherian follows from a result of Grundy (**7**, p. 242).

4. Endomorphism rings of uniform modules.

4a. A characterization of uniform prime modules. We begin with the following general result.

https://doi.org/10.4153/CJM-1965-099-5 Published online by Cambridge University Press

(4.1.) Proposition. *Let B be a ring that has a right quotient ring $Q(B)$. If N is a right $Q(B)$-module, then*

$$\operatorname{Hom}_B (N, N) = \operatorname{Hom}_{Q(B)}(N, N).$$

Proof. Obviously $\operatorname{Hom}_{Q(B)}(N, N) \subseteq \operatorname{Hom}_B(N, N)$. If $h \in \operatorname{Hom}_B(N, N)$, then for $x \in M$ and $a, b \in B$ with b regular we have

$$(hx)(ab^{-1}) = (h(xab^{-1}b))b^{-1} = (hxab^{-1})bb^{-1} = h(xab^{-1}).$$

Thus $h \in \operatorname{Hom}_{Q(B)}(N, N)$ and we have equality.

(4.2.) Corollary. *The module N is an injective B-module if and only if N is an injective $Q(B)$-module.*

Now suppose M is a prime R-module and let S be the right quotient ring of R. Since MS is the injective envelope of M and $M^\blacktriangle = 0$, then by **(10**, § 7**)** each $h \in \operatorname{Hom}_R(M, M)$ has a unique extension $h^* \in \operatorname{Hom}_R(MS, MS)$. We may, therefore, assume that

$$\operatorname{Hom}_R(M, M) \subseteq \operatorname{Hom}_R(MS, MS).$$

Indeed by **(12**, 1.6**)**, h^* is given by

$$h^*(mc^{-1}) = (h(m))c^{-1}$$

for all $m \in M$, c regular in R.

(4.3) Theorem. *Let U be a uniform submodule of a prime R-module M. Then* $\operatorname{Hom}_R(U, U)$ *is an integral domain and* $\operatorname{Hom}_R(US, US)$ *is a division ring containing* $\operatorname{Hom}_R(U, U)$.

Proof. If $u \in U^\blacktriangle$, then the large right ideal u^r of R contains a regular element and since M is torsion-free, we have $u = 0$. Thus $U^\blacktriangle = 0$. Moreover, since U is uniform, every pair of non-zero submodules of U has non-zero intersection. Hence U is an irreducible R-module in the sense of **(11**, p. 262**)**. Using **(11**, Theorem 1.7**)** we have that $\operatorname{Hom}_R(U, U)$ is an integral domain.

Since US is the injective envelope of U, we may write $\operatorname{Hom}_R(U, U) \subseteq \operatorname{Hom}_R(US, US)$. Now the injective envelope of an irreducible module is irreducible and, by **(11**, Theorem 1.7**)**, $\operatorname{Hom}_R(US, US)$ is a division ring.

Uniform prime R-modules may be characterized in the following way.

(4.4) Theorem. *A prime R-module M is uniform if and only if* $\operatorname{Hom}_R(\tilde{M}, \tilde{M})$ *is a division ring.*

Proof. The "only if" part was proved above. To prove the converse, suppose $N_1 \cap N_2 = 0$, where N_1 and N_2 are non-zero submodules of M. Let $N = N_1 \oplus N_2$ and let h be the projection of N on N_1. If h^* is the extension of h to $\tilde{M}$, then, since $\operatorname{Hom}_R(\tilde{M}, \tilde{M})$ is a division ring, $(h^*)^r = 0$. This contradicts the fact that $hN_2 = 0$. Hence M is uniform.

4b. The quotient ring of R.

(4.5) LEMMA. *Let M be a prime R-module and let J be a uniform right ideal in R. Then there exists an $x \in M$ such that $x^r \cap J = 0$.*

Proof. If $x^r \cap J \neq 0$ for every $x \in M$, then, by (2.4), $MJ = 0$. But this contradicts the fact that M is a prime R-module.

(4.6) THEOREM. *Let M be a prime R-module and let J be a uniform right ideal in R. Then there exists an $x \in M$ such that $I = \mathrm{Hom}_R(xJ, xJ)$ is a right Ore domain. The ring $D = \mathrm{Hom}_R(xJS, xJS)$ is a right quotient division ring for I.*

Proof. By (4.5) there is an $x \in M$ such that $x^r \cap J = 0$. Then $xJ \neq 0$, for otherwise, $J \subseteq x^r$, a contradiction. It then follows that the correspondence $j \to xj$, $j \in J$, is an R-isomorphism of the R-module J onto the R-module xJ. Consequently, $I \cong \mathrm{Hom}_R(J, J)$ and by **(4**, Theorem 4**)**, I is a right Ore domain. From **(4**, pp. 606–607**)** we have that JS is a minimal right ideal in S and $\mathrm{Hom}_S(JS, JS)$ is the quotient ring of $\mathrm{Hom}_R(J, J)$. However, the S-modules JS and xJS are isomorphic and the last part of the theorem follows at once.

(4.7) COROLLARY. *The ring S is isomorphic to a total matrix ring D_n, where $D = \mathrm{Hom}_R(xJS, xJS)$.*

Proof. From **(4)**, $D = \mathrm{Hom}_R(JS, JS)$.

We can also use (4.6) to associate the quotient ring S of R with the decomposition of a Noetherian prime R-module as given in (3.3). Specifically we have

(4.8) THEOREM. *Let M be a Noetherian prime R-module. Then M is a subdirect sum of uniform Noetherian prime R-modules $M_1, \ldots, M_t$ and, for each i, there exists $x_i \in M_i$ such that $I_i = \mathrm{Hom}_R(x_iJ, x_iJ)$ is a right Ore domain where J is a uniform right ideal of R. Moreover, if S is the quotient ring of R, then $S \cong D_n$, where D is the quotient ring of I_i.*

4c. A prime ring containing R. In the following discussion let xJ, I, and D be as in (4.6). Since S satisfies the maximum condition for right ideals, JS is finitely generated over S and consequently xJS is a finitely generated right S-module. Then by **(1**, 58.14 and 59.7**)**, xJS is a finitely generated injective left D-module and, from **(1**, 59.6**)**, the pair (D, xJS) has the double centralizer property, i.e. $S \cong \mathrm{Hom}_D(xJS, xJS)$. Since D is a quotient ring for I, we may use (4.1) to obtain $S \cong \mathrm{Hom}_I(xJS, xJS)$.

If $\alpha \in \mathrm{Hom}_I(xJ, xJ)$, define α^* by $(xjc^{-1})\alpha^* = ((xj)\alpha)c^{-1}$ for all $j \in J$ and c regular in R. It follows directly that α^* is single-valued, preserves sums, and is an extension of α to xJS.

We show that α^* is an I-endomorphism of xJS. Let $h \in I$. By the remarks preceding (4.3), we may identify h with $h^* \in D$, where

$$h^*(xjc^{-1}) = (h(xj))c^{-1}.$$

Then

$$(h^*(xjc^{-1}))\alpha^* = ((h(xj))c^{-1})\alpha^* = ((h(xj))\alpha)c^{-1}.$$

Since $\alpha \in \mathrm{Hom}_I(xJ, xJ)$,

$$((h(xj))\alpha)c^{-1} = (h((xj)\alpha))c^{-1} = h^*(((xj)\alpha)c^{-1}) = h^*((xjc^{-1})\alpha^*).$$

Hence $\alpha^* \in \mathrm{Hom}_I(xJS, xJS)$.

One can show that the correspondence $\alpha \to \alpha^*$ is an isomorphism of $\mathrm{Hom}_I(xJ, xJ)$ into $\mathrm{Hom}_I(xJS, xJS)$. We may, therefore, write

$$R \subseteq B \subseteq S,$$

where $B = \mathrm{Hom}_I(xJ, xJ)$. Therefore, S is a quotient ring for B and by **(4)**, B is a prime ring with right quotient conditions. This gives us the following two theorems.

(4.9) THEOREM. *If M is a prime R-module, then M contains a uniform submodule N and R is contained in a prime ring B such that the pair (N, B) has the double centralizer property. The submodule N may be chosen to be of the form xJ, where $x \in M$ and J is a uniform right ideal of R.*

(4.10) THEOREM. *Every prime ring R with right quotient conditions is contained in a prime ring B which has the same quotient ring as R and satisfies the following properties:*

(a) *B is the ring of endomorphisms of a left module N over an integral domain;*
(b) *the pair (N, B) has the double centralizer property.*

We conclude this section by noting that if M is a finitely generated prime R-module and S is the right quotient ring of R, then from **(1**, §§ 58, 59**)**, MS is a finitely generated left H-module where $H = \mathrm{Hom}_R(MS, MS)$. Moreover, H is a self-injective semi-simple ring and $\mathrm{Hom}_R(M, M) \subseteq H$. We do not know whether H is a quotient ring for $\mathrm{Hom}_R(M, M)$.

5. An application. Throughout this section M will denote a finitely generated torsion-free left A-module, where A is a right and left Ore domain with identity and quotient ring Q. In **(2)** the authors proved that $R = \mathrm{Hom}_A(M, M)$ is a prime ring with right and left quotient ring of the form Q_n, a total matrix ring over Q. Indeed, as in **(2)**, we may assume that M is contained in the left Q-vector space $Q \otimes_A M$, and, making the usual identifications, we write

$$R = \mathrm{Hom}_A(M, M) \subseteq \mathrm{Hom}_Q(Q \otimes_A M, Q \otimes_A M) = S \cong Q_n$$

for some positive integer n. In the following we shall discuss properties of M

when considered as an R-module. Of particular interest is the fact that M is a prime R-module of the type discussed in § 1. This is a consequence of (1.3) and

(5.1) PROPOSITION. *The singular submodule* $M^{\blacktriangle}$ *of the* R*-module* M *is zero.*

Proof. If $x \in M$, then the large right ideal x^r of R contains a regular element c. Now c has an inverse in the simple ring S and hence $x = 0$.

(5.2) PROPOSITION. *The* R*-module* M *is uniform.*

Proof. Let x and y be non-zero elements of M. Since S is a dense ring of linear transformations acting on $Q \otimes_A M$, we have $xab^{-1} = y$ for suitable $a, b \in R$, b regular. Then $xa = yb \neq 0$. Thus the intersection of two non-zero R-submodules of M is non-zero.

(5.3) PROPOSITION. *The* R*-module* $Q \otimes_A M$ *is the injective envelope of the* R*-module* M.

Proof. Let x and y be non-zero elements of $Q \otimes_A M$ and M respectively. Then, as in the proof of (5.2), $xa = yb \neq 0$ for some $a, b \in R$, which implies that $Q \otimes_A M$ is an essential extension of M as an R-module. Employing **(1**, 58.14 and 59.7**)**, we have that $Q \otimes_A M$ is an injective right S-module and by (4.2) $Q \otimes_A M$ is an injective R-module. Hence $Q \otimes_A M$ is the injective envelope of M and

$$\mathrm{Hom}_S(Q \otimes_A M, Q \otimes_A M) = \mathrm{Hom}_R(Q \otimes_A M, Q \otimes_A M) = Q.$$

Since the injective envelope is unique up to isomorphism and since M is a prime R-module, we may write

$$Q \otimes_A M \cong M \otimes_R S \cong M \otimes_R Q_n.$$

From (4.3) M is a left module over the integral domain $C = \mathrm{Hom}_R(M, M)$. In addition we may write $A \subseteq C \subseteq Q$, where Q is a right quotient ring for C. Then $R = \mathrm{Hom}_C(M, M)$ and the (C, R)-bimodule M has the double centralizer property on both sides.

The discussion above gives us a non-trivial example of a uniform prime module. Furthermore, if A is commutative and Noetherian, then M is finitely generated over the Noetherian ring R. From (3.3), any subdirect sum of such modules is again a Noetherian prime module.

6. Matrix representations. In this section a different characterization for $\mathrm{Hom}_A(M, M)$, where M is a left A-module of the type discussed in § 5, is given. Actually we work in a more general setting and obtain this characterization as a special case.

In the following let A be an *arbitrary* ring with identity. It is known that if M is a finitely generated unitary free left A-module, then $\mathrm{Hom}_A(M, M)$ has a faithful representation as A_n, a total matrix ring over A. We shall provide a faithful matrix representation for $\mathrm{Hom}_A(M, M)$ in the case where M is *contained* in a free module. Large classes of such modules are given by Levy **(12)** and Gentile **(3)**. Our work is a generalization of that in **(8**, p. 25**)**, where a characterization for $\mathrm{Hom}_A(M, M)$ is obtained in the case where M is cyclic.

Let M be a finitely generated unitary left A-module with generators $e_1, \ldots, e_n$. Write $E = (e_1, \ldots, e_n)$ and let E^t denote the corresponding transpose. For $V = x_1 e_1 + \ldots + x_n e_n \in M$, $x_i \in A$, we write $V = XE^t$, where $X = (x_1, \ldots, x_n)$. If $\theta \in \mathrm{Hom}_A(M, M)$, then $V\theta = (XE^t)\theta = XDE^t$, where $D \in A_n$. Of course, the mapping $\theta \to D$ is not necessarily onto A_n, nor is it necessarily one-to-one. A matrix $D \in A_n$ is termed *allowable* if one of the following equivalent conditions is satisfied:

(i) the mapping $\phi : XE^t \to XDE^t$ of M into M is single-valued;

(ii) whenever $XE^t = 0$, then $XDE^t = 0$.

The set W of allowable matrices forms a ring and to each element of W there corresponds an element of $\mathrm{Hom}_A(M, M)$. Let

$$T = \{D \in W | DE^t = 0\}.$$

Then the correspondence $D \to \phi$, where ϕ is the mapping defined in (i) above, is a homomorphism of W onto $\mathrm{Hom}_A(M, M)$ with kernel T. We have proved

(6.1) THEOREM. *Let M be a finitely generated unitary left A-module and let $E = (e_1, \ldots, e_n)$, where the e_i generate M. Then* $\mathrm{Hom}_A(M, M) \cong W/T$, *where W is the ring of allowable matrices and $T = \{D \in W | DE^t = 0\}$.*

Now suppose M is contained in a free left A-module N with basis $f_1, \ldots, f_k$. If $F = (f_1, \ldots, f_k)$, then $E^t = QF^t$ where Q is an $n \times k$ matrix with elements in A. For $V = XE^t = XQF^t$ we have $V\theta = XDE^t = XDQF^t$. The mapping $\theta \to DQ$ is a 1–1 mapping of $\mathrm{Hom}_A(M, M)$ onto the set Z of distinct matrices of the form DQ, where D is allowable (note that if $DQ = BQ$, B allowable, then $D - B \in T$). If $\theta_1 \to DQ$ and $\theta_2 \to BQ$, then $\theta_1\theta_2 \to DBQ$. Define addition and multiplication in Z by

$$DQ + BQ = (D + B)Q, \qquad DQ \cdot BQ = DBQ.$$

It is clear that addition is well defined. To see that multiplication is well defined, let $DQ = D'Q$ and $BQ = B'Q$, where D' and B' are allowable. Then, for all i, $D_i Q = D'_i Q$ where $D_i(D'_i)$ is row i of $D(D')$. Thus $(D_i - D'_i)E^t = 0$ and, since B is allowable, $(D_i - D'_i)BE^t = 0$ for all i. Hence $DBE^t = D'BE^t$ and $DBQ = D'BQ = D'B'Q$. Then Z is a ring isomorphic to $\mathrm{Hom}_A(M, M)$ under the correspondence $\theta \to DQ$.

This establishes part (a) of the next theorem. The proof of part (b) is direct.

(6.2) THEOREM. *Let M be a finitely generated unitary left A-module with generators $e_1, \ldots, e_n$ and suppose M is contained in a free left A-module N*

with basis $f_1, \ldots, f_k$. *Let* $E = (e_1, \ldots, e_n)$, $F = (f_1, \ldots, f_k)$, *and define* Q *by* $E^t = QF^t$. *Then*

(a) $\mathrm{Hom}_A(M, M) \cong Z$, *where* Z *is the ring of matrices of the form* DQ, *for* D *allowable, and where multiplication and addition in* Z *are defined by*

$$DQ \cdot BQ = DBQ, \qquad DQ + BQ = (D + B)Q;$$

(b) $\mathrm{Hom}_A(M, M) \cong W'/T'$, *where* W' *is the set of* $D \in W$ *such that if* $XQ = 0$, *then* $XDQ = 0$ *and where* T' *is the set of* $D \in W'$ *such that* $DQ = 0$.

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