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A NOTE ON SKEW-SYMMETRIC DETERMINANTS

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A short proof, based on the Schur complement, is given of the classical result that the determinant of a skew-symmetric matrix of even order is the square of a polynomial in its coefficients.

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Let

$$A=\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & & \cdots \\ -a_{1n} & -a_{2n} & -a_{3n} & \cdots & 0 \end{pmatrix}$$

be an n by n skew-symmetric matrix $\{A^T=-A)$, in which the $n(n-1)/2$ elements

$$a_{ij}\ (1\leqq i\leqq j\leqq n) \tag{1}$$

above the diagonal are indeterminates.

There are two classical results about a skew-symmetric matrix A:

(I) *When n is odd, then* $\det A=0$.

(II) *When n is even, then* $\det A=(p_n(A))^2$, *where $p_n(A)$ is a polynomial of degree $n/2$ in the indeterminates* (1); *$p_n(A)$ is determined up to a factor* ± 1.

The statement (I) follows at once from the observation that

$$\det A=\det A^T=\det(-A)=(-1)^n\det A.$$

Theorem (II) is more difficult to establish. It is traditionally proved by means of Jacobi's theorem on the adjugate determinant ([**4**, pp. 105–107]); a direct demonstration can be given which, however, involves somewhat complicated manipulations with permutations ([**3**, pp. 125–128]). P. M. Cohn [**1**, p. 209] uses an argument based on the canonical form.

The proof presented in this note uses only some simple facts about triangular block matrices, in particular the result that

$$\det\begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix} = (\det X)(\det Y), \tag{2}$$

where X and Y are square matrices, not necessarily of the same order.

When $n=2$, the truth of Theorem (II) is evident. For in this case

$$A = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}. \tag{3}$$

Hence $\det A = v^2 = (p_2(A))^2$, where we have defined

$$p_2(A) = v.$$

Using induction on the set of even integers we assume that (II) holds for skew-symmetric matrices of order $n-2$.

An arbitrary skew-symmetric matrix of even order n (>2) can be partitioned thus:

$$A = \begin{pmatrix} B & C \\ -C^T & V \end{pmatrix}, \tag{4}$$

where

$$B = \begin{pmatrix} 0 & a_{12} & \dots & a_{1,n-2} \\ -a_{12} & 0 & \dots & a_{2,n-2} \\ \dots & \dots & \dots & \dots \\ -a_{1,n-2} & -a_{2,n-2} & \dots & 0 \end{pmatrix}$$

is a skew-symmetric matrix of order $n-2$, and

$$C = \begin{pmatrix} a_{1,n-1} & a_{1n} \\ a_{2,n-1} & a_{2n} \\ \dots & \dots \\ a_{n-2,n-1} & a_{n-2,n} \end{pmatrix} \quad V = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}$$

are of orders $n-2 \times 2$ and 2×2 respectively, and we have used the abbreviation

$$v = a_{n-1,n}.$$

Let

$$P = \begin{pmatrix} I_{n-2} & CV^{-1} \\ 0 & I_2 \end{pmatrix}.$$

A straightforward calculation shows that

$$PA=\begin{pmatrix} B-CV^{-1}C & 0 \\ -C^T & V \end{pmatrix}. \tag{5}$$

Since V^{-1} is skew-symmetric, so are $CV^{-1}C^T$ and

$$B-CV^{-1}C^T,$$

which is known as the Schur complement of V in A [**2**, p. 22]. By the inductive hypothesis we have that

$$\det(B-CV^{-1}C^T)=[p_{n-2}(B-CV^{-1}C^T)]^2. \tag{6}$$

Since $\det P=1$, we deduce from (5) that

$$\det A=\det V\det(B-CV^{-1}C^T),$$

whence by (3) and (6)

$$\det A=[vp_{n-2}(B-CV^{-1}C^T)]^2. \tag{7}$$

Although p_{m-2} is a polynomial in its arguments, the presence of V^{-1} in the argument leaves it open that

$$vp_{n-2}(B-CV^{-1}C^T)$$

may be a rational function of the indeterminates (1) whose denominator is, at worst, a power of v. More precisely, let

$$vp_{n-2}(B-CV^{-1}C^T)=v^{-m}f_0+v^{-m+1}f_1+\cdots+f_m+vf_{m+1}, \tag{8}$$

where $f_0, f_1, \ldots$ are polynomials in the indeterminates a_{ij} other than v $(=a_{n-1,n})$, and where $f_0\neq 0$. From first principles, $\det A$ is a polynomial in all the indeterminates, including v; so no negative power of v appear in (7).

Therefore on substituting (8) in (7) and comparing powers of v on both sides of the equation we conclude that $m=0$. Thus $vp_{n-2}(B-CV^{-1}C^T)$ is, after all, a polynomial in the a_{ij}, and we may define

$$p_n(A)=vp_{n-2}(B-CV^{-1}C^T).$$

This concludes the proof.

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