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RATE OF GROWTH OF FREQUENTLY HYPERCYCLIC FUNCTIONS

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Abstract We study the rate of growth of entire functions that are frequently hypercyclic for the differentiation operator or the translation operator. Moreover, we prove the existence of frequently hypercyclic harmonic functions for the translation operator and we study the rate of growth of harmonic functions that are frequently hypercyclic for partial differentiation operators.

Keywords: frequently hypercyclic operator; frequently hypercyclic vector; Frequent Hypercyclicity Criterion; rate of growth; entire function; harmonic function

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1. Introduction

A (continuous and linear) operator T on a topological vector space X is said to be *hypercyclic* if there exists a vector $x \in X$, also called *hypercyclic*, whose orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. The earliest examples of hypercyclic operators were operators on the space $H(\mathbb{C})$ of entire functions. Birkhoff [**13**] showed that the translation operators $T_a : H(\mathbb{C}) \to H(\mathbb{C})$,

$$T_a f(z) = f(z+a), \quad a \neq 0,$$

are hypercyclic, while MacLane [**27**] obtained the same result for the differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$,

$$Df(z) = f'(z).$$

It is natural to ask how slowly a corresponding hypercyclic function can grow at ∞. Duyos-Ruiz [**18**] showed that translation hypercyclic entire functions can have arbitrarily slow transcendental growth (see also [**16**]). MacLane [**27**] showed that differentiation hypercyclic entire functions can be of exponential type 1, and Duyos-Ruiz [**19**] showed

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that they cannot be of exponential type less than 1. An optimal result on the possible rates of growth for the differentiation operator was subsequently obtained in [**23**] and, independently, in [**28**]. For generalizations of this result we refer the reader to [**12**, **24**].

Analogous investigations have been carried out for harmonic functions. Dzagnidze [**20**] obtained the analogue of Birkhoff's result (see also [**6**]); Armitage [**5**] recently studied corresponding rates of growth. Hypercyclic partial differentiation operators and related growth rates were investigated by Aldred and Armitage [**1**–**3**].

It is the aim of this paper to address the same problems for the notion of frequent hypercyclicity that was recently introduced by Bayart and Grivaux [**10**,**11**]. By definition, a vector is hypercyclic if its orbit meets every non-empty open set. Now, for frequent hypercyclicity one demands that the orbit meets every non-empty open set 'often' in the sense of lower density. We recall that the lower density of a subset A of $\mathbb{N}$ is defined as

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{\#\{n \in A : n \leqslant N\}}{N},$$

where $\#$ denotes the cardinality of a set.

Definition 1.1. Let X be a topological vector space and let $T : X \to X$ be an operator. Then a vector $x \in X$ is called *frequently hypercyclic* for T if, for every non-empty open subset U of X,

$$\underline{\text{dens}}\{n \in \mathbb{N} : T^n x \in U\} > 0.$$

The operator T is called *frequently hypercyclic* if it possesses a frequently hypercyclic vector.

Remark 1.2. A useful alternative formulation is the following [**15**]: a vector $x \in X$ is frequently hypercyclic for an operator T on X if and only if, for every non-empty open subset U of X, there is a strictly increasing sequence (n_k) of positive integers and some $C > 0$ such that

$$n_k \leqslant Ck \quad \text{and} \quad T^{n_k} x \in U \quad \text{for all } k \in \mathbb{N}.$$

The problem of determining possible rates of growth of frequently hypercyclic entire functions has already been studied in [**14**]. The approach chosen there was based on an eigenvalue criterion for frequent hypercyclicity. This criterion allowed a wide class of operators to be treated on the space of entire functions. In this paper we choose a different approach; while we can only apply it to the translation and differentiation operator, at least in the latter case we obtain a better result than in [**14**]. In addition, we show here that the result obtained in [**14**] for the translation operator is the best possible.

The present approach consists in studying the frequent hypercyclicity of the sequence (T^n) of iterates of the given operator T on a space of functions of restricted growth. For this we shall need the notion of frequent hypercyclicity for an arbitrary sequence of mappings [**15**].

Definition 1.3. Let X and Y be topological spaces and let $T_n : X \to Y$, $n \in \mathbb{N}$, be mappings. Then an element $x \in X$ is called *frequently universal* for the sequence (T_n) if,

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for every non-empty open subset U of Y,

$$\underline{\text{dens}}\{n \in \mathbb{N} : T_n x \in U\} > 0.$$

The sequence (T_n) is called *frequently universal* if it possesses a frequently universal element.

In [**15**], a Frequent Universality Criterion was obtained that generalizes the Frequent Hypercyclicity Criterion of Bayart and Grivaux [**11**]. We state it here only for Fréchet spaces. A collection of series $\sum_{k=1}^{\infty} x_{k,j}$, $j \in I$, in a Fréchet space X is said to be *unconditionally convergent, uniformly in $j \in I$,* if, for every continuous seminorm p on X and every $\varepsilon > 0$, there is some $N \geqslant 1$ such that for every finite set $F \subset \mathbb{N}$ with $F \cap \{1, 2, \ldots, N\} = \emptyset$ and every $j \in I$ we have that $p(\sum_{k \in F} x_{k,j}) < \varepsilon$.

Theorem 1.4 (Frequent Universality Criterion). *Let X be a Fréchet space, let Y be a separable Fréchet space and let $T_n : X \to Y$, $n \in \mathbb{N}$, be operators. Suppose that there are a dense subset Y_0 of Y and mappings $S_n : Y_0 \to X$, $n \in \mathbb{N}$, such that, for all $y \in Y_0$,*

(i) *$\sum_{n=1}^{k} T_k S_{k-n} y$ converges unconditionally in Y, uniformly in $k \in \mathbb{N}$,*

(ii) *$\sum_{n=1}^{\infty} T_k S_{k+n} y$ converges unconditionally in Y, uniformly in $k \in \mathbb{N}$,*

(iii) *$\sum_{n=1}^{\infty} S_n y$ converges unconditionally in X,*

(iv) *$T_n S_n y \to y$.*

Then the sequence (T_n) is frequently universal.

We note that the sums in (i) can be understood as infinite series by adding zero terms.

The spaces $H(\mathbb{C})$ of entire functions and $\mathcal{H}(\mathbb{R}^N)$ of harmonic functions on $\mathbb{R}^N$, $N \geqslant 2$, are Fréchet spaces when endowed with the topology of local uniform convergence.

As usual, throughout this paper constants $C > 0$ can take different values for different occurrences. By $\mathbb{R}_+$ we denote the set of real numbers $x > 0$. We write $a_n \sim b_n$ for positive sequences (a_n) and (b_n) if a_n/b_n and b_n/a_n are bounded.

2. Frequently hypercyclic entire functions for the differentiation operator

For hypercyclicity with respect to the differentiation operator D, the rate of growth $\mathrm{e}^r/\sqrt{r}$ turns out to be critical. Indeed, one has the following precise result [**23, 28**]: if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is any function with $\varphi(r) \to \infty$ as $r \to \infty$, then there is a D-hypercyclic entire function f with

$$|f(z)| \leqslant \varphi(r)\frac{\mathrm{e}^r}{\sqrt{r}} \quad \text{for } |z| = r \text{ sufficiently large;}$$

however, there is no D-hypercyclic entire function f that satisfies

$$|f(z)| \leqslant C\frac{\mathrm{e}^r}{\sqrt{r}} \quad \text{for } |z| = r > 0,$$

where $C > 0$.

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Before beginning our investigation into frequent hypercyclicity we want to extend this result to growth rates in terms of L^p-averages. For an entire function f and $1 \leqslant p < \infty$ we consider

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|^p \,\mathrm{d}t\right)^{1/p}$$

and

$$M_\infty(f,r) = \sup_{|z|=r} |f(z)|,$$

for $r > 0$. The growth result stated above is in terms of M_∞. It extends directly to all M_p.

Theorem 2.1. *Let $1 \leqslant p \leqslant \infty$.*

(a) *For any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$ there is a D-hypercyclic entire function f with*

$$M_p(f,r) \leqslant \varphi(r)\frac{\mathrm{e}^r}{\sqrt{r}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

(b) *There is no D-hypercyclic entire function f that satisfies*

$$M_p(f,r) \leqslant C\frac{\mathrm{e}^r}{\sqrt{r}} \quad \text{for } r > 0,$$

where $C > 0$.

Proof. Since

$$M_p(f,r) \leqslant M_\infty(f,r), \quad r > 0,$$

part (a) follows from the corresponding classical result.

Part (b) follows as in the proof given in [**23**]. By the Cauchy estimates we have that

$$|f^{(n)}(0)| \leqslant \frac{n!}{r^n} M_1(f,r).$$

Since $M_1(f,r) \leqslant M_p(f,r)$, we find that

$$|f^{(n)}(0)| \leqslant C\frac{n!}{n^{n+1/2}}\mathrm{e}^n, \quad n \geqslant 1.$$

Now Stirling's Formula implies that $(f^{(n)}(0))$ is bounded, so that f cannot be hypercyclic. $\square$

After these preliminaries we study growth rates for frequent hypercyclicity. As an auxiliary result we shall need the following estimate; it is certainly known in some form or other, but we have not been able to find a reference. We deduce it here from a much stronger result of Barnes [**9**].

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Lemma 2.2. *Let $0 < \alpha \leqslant 2$ and $\beta \in \mathbb{R}$. Then there is some $C > 0$ such that, for all $r > 0$,*

$$\sum_{n=0}^{\infty} \frac{r^{\alpha n}}{(n+1)^{\beta}(n!)^{\alpha}} \leqslant C r^{(1-\alpha-2\beta)/2} \mathrm{e}^{\alpha r}.$$

Proof. Barnes [**9**, pp. 289–292] studied the functions

$$E_{\alpha}(z;\theta,\beta) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\theta)^{\beta}\Gamma(\alpha n+1)}$$

for $\alpha, \theta > 0$, $\beta \in \mathbb{R}$; for the special case of $\beta = 0$ these are the Mittag–Leffler functions (see also [**21**, § 18.1]). Barnes derived asymptotic expansions for his functions which, for $0 < \alpha \leqslant 2$ and $z \in \mathbb{R}$, $z \to \infty$, yield that

$$E_{\alpha}(r;\theta,\beta) = \alpha^{\beta-1} r^{-\beta/\alpha} \exp(r^{1/\alpha})(1 + O(r^{-1/\alpha})).$$

Now, by Stirling's Formula we have that

$$\frac{(n!)^{\alpha}}{\Gamma(\alpha n+1)} \sim \alpha^{-\alpha n}(n+1)^{(\alpha-1)/2}.$$

Thus, we obtain that

$$\sum_{n=0}^{\infty} \frac{(r/\alpha^{\alpha})^n}{(n+1)^{\beta+(1-\alpha)/2}(n!)^{\alpha}} \leqslant C r^{-\beta/\alpha} \exp(r^{1/\alpha}).$$

A change of variables and parameters implies the claimed estimate. □

Our first main result gives growth rates for which D-frequently hypercyclic functions exist. In the following, we shall set $1/(2p) = 0$ for $p = \infty$.

Theorem 2.3. *Let $1 \leqslant p \leqslant \infty$, and set $a = 1/(2\max\{2,p\})$. Then, for any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$, there is an entire function f with*

$$M_p(f,r) \leqslant \varphi(r)\frac{\mathrm{e}^r}{r^a} \quad \text{for } r > 0 \text{ sufficiently large}$$

that is frequently hypercyclic for the differentiation operator.

Proof. Since

$$M_p(f,r) \leqslant M_2(f,r) \quad \text{for } 1 \leqslant p < 2,$$

we need only prove the result for $p \geqslant 2$.

Thus, let $2 \leqslant p \leqslant \infty$. We shall make use of the Frequent Universality Criterion. Assuming without loss of generality that $\inf_{r>0} \varphi(r) > 0$, we consider the space

$$X = \Big\{ f \in H(\mathbb{C}) : \|f\|_X := \sup_{r>0} \frac{M_p(f,r) r^{1/(2p)}}{\varphi(r)\mathrm{e}^r} < \infty \Big\}.$$

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It is not difficult to see that $(X, \|\cdot\|_X)$ is a Banach space that is continuously embedded in $H(\mathbb{C})$; one need only note that on account of the Cauchy estimates we have that, for $0 < \rho < r$,

$$\sup_{|z|\leqslant\rho} |f(z)| \leqslant \frac{r}{r-\rho} M_1(f,r) \leqslant \frac{r}{r-\rho} M_p(f,r).$$

In addition, we define $Y_0 \subset H(\mathbb{C})$ as the set of polynomials, and we consider the mappings

$$T_n : X \to H(\mathbb{C}), \qquad T_n = D^n|_X,$$

which are continuous, and

$$S_n : Y_0 \to X, \quad S_n = S^n \quad \text{with} \quad Sf(z) = \int_0^z f(\zeta)\,\mathrm{d}\zeta.$$

Then we have, for any polynomial f and any $k \in \mathbb{N}$, that

$$\sum_{n=1}^{k} T_k S_{k-n} f = \sum_{n=1}^{k} D^n f;$$

this converges unconditionally in $H(\mathbb{C})$, uniformly for $k \in \mathbb{N}$, because $\sum_{n=1}^{\infty} D^n f$ is a finite series. Moreover, we have that

$$T_n S_n f = f \quad \text{for any } n \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} T_k S_{k+n} f = \sum_{n=1}^{\infty} S_n f.$$

Thus, the conditions (i)–(iv) in Theorem 1.4 are satisfied if we can show that $\sum_{n=1}^{\infty} S_n f$ converges unconditionally in X, for any polynomial f; note that X is continuously embedded in $H(\mathbb{C})$. It suffices to consider $f(z) = z^k$, $k \in \mathbb{N}_0$, in which case

$$\sum_{n=1}^{\infty} S_n f(z) = \sum_{n=1}^{\infty} \frac{k!}{(k+n)!} z^{k+n}.$$

Therefore, all we need to show is that

$$\sum_{n=1}^{\infty} \frac{z^n}{n!}$$

converges unconditionally in X. To this end, let $\varepsilon > 0$ and $N \in \mathbb{N}$. By the Hausdorff–Young Inequality (see, for example, [**25**]) we obtain for any finite set $F \subset \mathbb{N}$ that

$$M_p\biggl(\sum_{n\in F} \frac{z^n}{n!}, r\biggr) \leqslant \biggl(\sum_{n\in F} \frac{r^{qn}}{(n!)^q}\biggr)^{1/q},$$

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where q is the conjugate exponent of p. Hence, if $F \cap \{0, 1, \dots, N\} = \emptyset$, then

$$\bigg\| \sum_{n \in F} \frac{z^n}{n!} \bigg\|_X \leqslant \bigg(\sup_{r>0} \frac{r^{q/(2p)}}{\varphi(r)^q \mathrm{e}^{qr}} \sum_{n>N} \frac{r^{qn}}{(n!)^q} \bigg)^{1/q}.$$

We choose $R > 0$ such that $\varphi(r)^q \geqslant 1/\varepsilon$ for $r \geqslant R$. Then we have that

$$\sup_{r \leqslant R} \frac{r^{q/(2p)}}{\varphi(r)^q \mathrm{e}^{qr}} \sum_{n>N} \frac{r^{qn}}{(n!)^q} \leqslant \frac{R^{q/(2p)}}{\inf_{r>0} \varphi(r)^q} \sum_{n>N} \frac{R^{qn}}{(n!)^q} \to 0 \quad \text{as } N \to \infty;$$

moreover, Lemma 2.2 implies that

$$\sup_{r \geqslant R} \frac{1}{\varphi(r)^q} \frac{r^{q/(2p)}}{\mathrm{e}^{qr}} \sum_{n>N} \frac{r^{qn}}{(n!)^q} \leqslant C\varepsilon \quad \text{for any } N \in \mathbb{N},$$

where C is a constant only depending on q; note that $\frac{1}{2}(1-q) + q/(2p) = 0$.

This shows that

$$\bigg\| \sum_{n \in F} \frac{z^n}{n!} \bigg\|_X^q \leqslant (1 + C)\varepsilon,$$

if $\min F > N$ and N is sufficiently large, so that $\sum_{n=1}^{\infty} z^n/n!$ converges unconditionally in X. □

The following result gives lower estimates on the possible growth rates.

Theorem 2.4. *Let $1 \leqslant p \leqslant \infty$, and set $a = 1/(2\min\{2, p\})$.*

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be any function with $\psi(r) \to 0$ as $r \to \infty$. Then there is no D-frequently hypercyclic entire function f that satisfies

$$M_p(f, r) \leqslant \psi(r) \frac{\mathrm{e}^r}{r^a} \quad \textit{for } r > 0 \textit{ sufficiently large.} \tag{2.1}$$

Proof. First, for $p = 1$ the result follows immediately from Theorem 2.1 (b) (one may even take $\psi(r) \equiv C$ here). Moreover, since

$$M_2(f, r) \leqslant M_p(f, r) \quad \text{for } 2 < p \leqslant \infty,$$

it suffices to prove the result for $p \leqslant 2$.

Thus, let $1 < p \leqslant 2$. We obviously may assume that ψ is decreasing. Suppose that f satisfies (2.1). With the help of the Hausdorff–Young Inequality (see [**25**]) we obtain

$$\bigg(\sum_{n=0}^{\infty} \bigg(\frac{|f^{(n)}(0)|}{n!} r^n \bigg)^q \bigg)^{1/q} \leqslant M_p(f, r) \leqslant \psi(r) \frac{\mathrm{e}^r}{r^{1/(2p)}} \tag{2.2}$$

for $r > 0$ sufficiently large, where q is the conjugate exponent of p. Thus, we have that, for large r,

$$\sum_{n=0}^{\infty} |f^{(n)}(0)|^q \frac{r^{qn + q/(2p)} \mathrm{e}^{-qr}}{\psi(r)^q (n!)^q} \leqslant 1. \tag{2.3}$$

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Using Stirling's Formula we see that the function

$$g(r)=\frac{r^{qn+q/(2p)}\mathrm{e}^{-qr}}{(n!)^q}$$

has its maximum at $a_n := n + 1/(2p)$ with $g(a_n) \sim 1/\sqrt{n}$ and an inflection point at $b_n := a_n + \sqrt{n/q + n/(2pq)}$; we have used here that $q/(2p) - \frac{1}{2}q = -\frac{1}{2}$. On $I_n := [a_n, b_n]$, g therefore dominates the linear function h that satisfies $h(a_n) = g(a_n)$, $h(b_n) = 0$.

Now let $m \geqslant 1$. If m is sufficiently large and $m < n \leqslant 2m$, then $I_n \subset [m, 3m]$. Hence, for these n we have

$$\int_m^{3m}\frac{r^{qn+q/(2p)}\mathrm{e}^{-qr}}{\psi(r)^q(n!)^q}\geqslant\int_{I_n}\frac{h(r)}{\psi(r)^q}\,\mathrm{d}r\geqslant C\frac{1}{\psi(m)^q}\frac{1}{\sqrt{n}}\sqrt{\frac{n}{q}+\frac{1}{2pq}}\geqslant C\frac{1}{\psi(m)^q}.$$

Integrating (2.3) over $[m, 3m]$, we thus obtain that, for m sufficiently large,

$$\frac{1}{m}\sum_{n=m+1}^{2m}|f^{(n)}(0)|^q\leqslant C\psi(m)^q;$$

hence,

$$y_m=\frac{1}{m}\sum_{n=m+1}^{2m}|f^{(n)}(0)|^q\to 0.$$

Notice that

$$\sum_{n=1}^{m}y_n=\sum_{n=1}^{m}\frac{1}{n}\sum_{n<j\leqslant 2n}|f^{(j)}(0)|^q\geqslant\sum_{j=2}^{m}|f^{(j)}(0)|^q\biggl(\sum_{j/2\leqslant n<j}\frac{1}{n}\biggr)\geqslant\frac{1}{2}\sum_{j=2}^{m}|f^{(j)}(0)|^q.$$

Since $y_m \to 0$ implies $(1/m)\sum_{n=1}^{m} y_n \to 0$, one obtains that

$$\frac{1}{m}\sum_{n=0}^{m}|f^{(n)}(0)|^q\to 0.$$

Hence, we have

$$\begin{aligned}\underline{\mathrm{dens}}\{n\in\mathbb{N}:|f^{(n)}(0)|>1\}&=\liminf_{m\to\infty}\frac{1}{m}\#\{n\leqslant m:|f^{(n)}(0)|>1\}\\&\leqslant\liminf_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m}|f^{(n)}(0)|^q\to 0,\end{aligned}$$

which shows that f is not frequently hypercyclic for the differentiation operator. □

Figure 1 shows our present knowledge of possible or impossible growth rates e^r/r^a for frequent hypercyclicity with respect to D.

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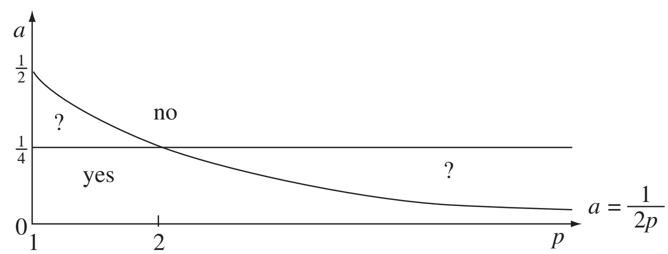

Figure 1. Possible or impossible growth rates e^r/r^a for frequent hypercyclicity with respect to D.

3. Frequently hypercyclic entire functions for translation operators

We turn to the translation operators

$$T_a f(z) = f(z+a), \quad a \neq 0.$$

Duyos-Ruiz [**18**] has shown that for any chosen transcendental growth rate there is a T_a-hypercyclic entire function with slower growth. We shall see here that this behaviour does not extend to frequent hypercyclicity. In fact, we have the following sharp result.

Theorem 3.1. *Let $a \in \mathbb{C}$, $a \neq 0$.*

(a) *For any $\varepsilon > 0$, there is an entire function f that is frequently hypercyclic for T_a such that*

$$M_\infty(f, r) \leqslant C e^{\varepsilon r} \quad \text{for } r > 0,$$

where $C > 0$.

(b) *Let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ with $\liminf_{r\to\infty} \varepsilon(r) = 0$. Then there exists no entire function f that is frequently hypercyclic for T_a with*

$$M_1(f, r) \leqslant C e^{\varepsilon(r) r} \quad \text{for } r \text{ sufficiently large.}$$

Proof. For (a) we combine ideas from the proofs of [**15**, Theorem 4.2] and our Theorem 2.3. By considering the change of variable $z \mapsto z/a$ it is easy to see that we may assume that $a = 1$. For fixed $\varepsilon > 0$ we consider the Banach space

$$X = \left\{ f \in H(\mathbb{C}) : \|f\|_X := \sup_{r>0} \sup_{|z|=r} \frac{|f(z)|}{e^{\varepsilon r}} < \infty \right\}$$

and, for $n \in \mathbb{N}$, the continuous mappings

$$T_n : X \to H(\mathbb{C}), \qquad T_n f(z) = f(z+n).$$

The result will follow if we can show that (T_n) is frequently universal, for which we shall apply the Frequent Universality Criterion.

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For $m, k \in \mathbb{N}$ we consider the entire functions

$$f_{m,k}(z) = z^m \left(\frac{\sin(z/k)}{z/k}\right)^{m+2},$$

and we set

$$Y_0 = \operatorname{span}\left\{ f_{m,k} : m, k \in \mathbb{N},\ \frac{m+2}{k} \leqslant \frac{\varepsilon}{2} \right\}.$$

Since $f_{m,k}(z) \to z^m$ in $H(\mathbb{C})$ as $k \to \infty$, we have that Y_0 is dense in $H(\mathbb{C})$.

Finally, we define, for $n \in \mathbb{N}$, the mappings

$$S_n : Y_0 \to X, \qquad S_n f(z) = f(z-n),$$

which are easily seen to be well defined.

Now, for all $m, k, l \in \mathbb{N}$,

$$\sum_{n=1}^{l} T^l S_{l-n} f_{m,k}(z) = k^{m+2} \sum_{n=1}^{l} \frac{\sin^{m+2}((z+n)/k)}{(z+n)^2};$$

since the series

$$\sum_{n=1}^{\infty} \frac{\sin^{m+2}((z+n)/k)}{(z+n)^2}$$

converge absolutely (hence unconditionally in $H(\mathbb{C})$), condition (i) of the Frequent Universality Criterion follows. Condition (iv) is immediate, and for conditions (ii) and (iii) it suffices to show that, whenever $(m+2)/k \leqslant \frac{1}{2}\varepsilon$,

$$\sum_{n=1}^{\infty} S_n f_{m,k}(z) = k^{m+2} \sum_{n=1}^{\infty} \frac{\sin^{m+2}(z-n)/k}{(z-n)^2}$$

converges absolutely in X.

To see this, let $n \in \mathbb{N}$. We first consider the case when $|z-n| \leqslant 1$. Writing

$$C_{m,k} = \max_{|z| \leqslant 1} \left| \frac{\sin^{m+2}(z/k)}{z^2} \right|,$$

we find that

$$\frac{|S_n f_{m,k}(z)|}{\mathrm{e}^{\varepsilon|z|}} \leqslant k^{m+2} \frac{C_{m,k}}{\mathrm{e}^{\varepsilon(n-1)}}.$$

Next, if $1 \leqslant |z-n| \leqslant \frac{1}{2}n$, then $|\operatorname{Re} z| \geqslant \frac{1}{2}n$, so that

$$\frac{|S_n f_{m,k}(z)|}{\mathrm{e}^{\varepsilon|z|}} \leqslant k^{m+2} \frac{\exp((m+2)|\operatorname{Im} z|/k)}{\exp(\varepsilon(|\operatorname{Re} z| + |\operatorname{Im} z|)/2)} \leqslant \frac{k^{m+2}}{\exp(\varepsilon|\operatorname{Re} z|/2)} \leqslant \frac{k^{m+2}}{\mathrm{e}^{\varepsilon n/4}}.$$

Finally, if $|z-n| \geqslant \frac{1}{2}n$, then

$$\frac{|S_n f_{m,k}(z)|}{\mathrm{e}^{\varepsilon|z|}} \leqslant k^{m+2} \frac{\exp((m+2)|\operatorname{Im} z|/k)}{\mathrm{e}^{\varepsilon|z|}|z-n|^2} \leqslant 4\frac{k^{m+2}}{n^2}.$$

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Combining these results we have that

$$\sum_{n=1}^{\infty} \|S_n f_{m,k}\|_X < \infty,$$

which had to be shown.

We turn to the proof of (b), again assuming that $a = 1$. Let f be an entire function that is frequently hypercyclic for T_1; by adding a constant, if necessary, we may assume that $f(0) = 1$. Then there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ with $n_k = O(k)$ such that, for all $k \in \mathbb{N}$,

$$|f(z + n_k) - z| < \tfrac{1}{2} \quad \text{for } |z| \leqslant \tfrac{1}{2}.$$

It follows from Rouché's Theorem that f has a zero in $|z - n_k| < \frac{1}{2}$. If $N(r)$ denotes the number of zeros of f in $|z| < r$, counting multiplicity, then, for all $k \in \mathbb{N}$, we consequently have

$$N(n_k + 1) \geqslant k. \tag{3.1}$$

Now assume in addition that there is $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ with $\liminf_{r\to\infty} \varepsilon(r) = 0$ and some $C > 0$ such that

$$M_1(f, r) \leqslant C\mathrm{e}^{\varepsilon(r)r} \quad \text{for } r \text{ sufficiently large.}$$

Applying Jensen's Formula [**17**, pp. 280–282]

$$\log|f(0)| + \int_0^R \frac{N(r)}{r}\,\mathrm{d}r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(R\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t$$

and Jensen's Inequality for concave functions,

$$M_1(\log|f|, R) \leqslant \log M_1(f, R),$$

we have that, for large R,

$$N(R)\log 2 \leqslant \int_R^{2R} \frac{N(r)}{r}\,\mathrm{d}r \leqslant \log C + 2R\varepsilon(2R).$$

Let R_ν be such that $\varepsilon(2R_\nu) \to 0$. Then there are $k_\nu \in \mathbb{N}$ with

$$n_{k_\nu} + 1 \leqslant R_\nu \leqslant n_{k_\nu+1}.$$

We obtain that

$$N(n_{k_\nu} + 1)\log 2 \leqslant N(R_\nu)\log 2 \leqslant \log C + 2R_\nu\varepsilon(2R_\nu) \leqslant \log C + 2n_{k_\nu+1}\varepsilon(2R_\nu);$$

hence,

$$\frac{N(n_{k_\nu} + 1)}{n_{k_\nu+1}} \to 0. \tag{3.2}$$

On the other hand, it follows from (3.1) that

$$\frac{N(n_{k_\nu} + 1)}{n_{k_\nu+1}} \geqslant \frac{k_\nu}{n_{k_\nu+1}} \geqslant \frac{1}{2}\frac{k_\nu + 1}{n_{k_\nu+1}},$$

so that, by (3.2), $\sup_k n_k/k = \infty$, which is a contradiction. This proves the claim. □

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The theorem implies that T_a has frequently hypercyclic entire functions of order 1 and any given positive type, but not of type 0. The positive part was also obtained, by a different method, in [**14**, Corollary 3.5].

The theorem can also be phrased more succinctly in the following way.

Corollary 3.2. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be arbitrary and let $1 \leqslant p \leqslant \infty$. Then there exists a T_a-frequently hypercyclic entire function f with*

$$M_p(f, r) \leqslant C\phi(r) \quad \text{for } r \text{ sufficiently large}$$

for some $C > 0$ if and only if

$$\liminf_{r\to\infty} \frac{\log(\phi(r))}{r} > 0.$$

4. Frequently hypercyclic harmonic functions for differentiation operators

We next study harmonic functions on $\mathbb{R}^N$, $N \geqslant 2$, where we again start by considering differentiation operators. Here they take the form

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}},$$

where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ and $|\alpha| = \alpha_1 + \dots + \alpha_N$.

Following [**1**], we first consider the special operators $\partial/\partial x_k$, and we study rates of growth in terms of the L^2-norm on spheres. More precisely, let $S(r)$ be the sphere of radius r centred at the origin 0 of $\mathbb{R}^N$, and let σ be the normalized $(N-1)$-dimensional measure on $S(r)$. For $h \in \mathcal{H}(\mathbb{R}^N)$ and $r > 0$ we consider

$$M_2(h, r) = \left(\int_{S(r)} |h|^2 \, \mathrm{d}\sigma \right)^{1/2}.$$

Then, for any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$ and $1 \leqslant k \leqslant N$, there exists a harmonic function h on $\mathbb{R}^N$ that is $\partial/\partial x_k$-hypercyclic such that

$$M_2(h, r) \leqslant \varphi(r) \frac{\mathrm{e}^r}{r^{(N-1)/2}} \quad \text{for } r > 0 \text{ sufficiently large},$$

while there can be no $\partial/\partial x_k$-hypercyclic harmonic function h that satisfies [**1**]

$$M_2(h, r) \leqslant C \frac{\mathrm{e}^r}{r^{(N-1)/2}} \quad \text{for } r > 0.$$

Note that Aldred and Armitage phrase their result in terms of a different notion of universality, but their proof gives the result stated above.

We study the situation for frequent hypercyclicity. We shall need the following lemma, where $\mathcal{H}_{j,N}$, $j \geqslant 0$, denotes the space of homogeneous harmonic polynomials on $\mathbb{R}^N$ of degree j.

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Lemma 4.1. *Let $\alpha \in \mathbb{N}_0^N$. For any $H \in \mathcal{H}_{j,N}$, $j \geqslant 0$, there exists a unique polynomial $G \in \mathcal{H}_{j+|\alpha|,N}$ with*

$$D^\alpha G = H.$$

Existence is obtained in [**2**, Lemma 2]. For $|\alpha| = 1$, uniqueness is easily deduced from [**1**, Lemma 3 (ii) and (10)]; the case of general α follows by induction (it is important to note that the uniqueness assertion in [**1**, Lemma 3 (ii)] does not depend on the orthogonality of the stated representation; see [**26**, Theorem 3]).

Theorem 4.2. *Let $1 \leqslant k \leqslant N$.*

(a) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be any function with $\varphi(r) \to \infty$ as $r \to \infty$. Then there is a $\partial/\partial x_k$-frequently hypercyclic harmonic function h on $\mathbb{R}^N$ with*

$$M_2(h, r) \leqslant \varphi(r) \frac{\mathrm{e}^r}{r^{N/2-3/4}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

(b) *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be any function with $\psi(r) \to 0$ as $r \to \infty$. Then there is no $\partial/\partial x_k$-frequently hypercyclic harmonic function h on $\mathbb{R}^N$ with*

$$M_2(h, r) \leqslant \psi(r) \frac{\mathrm{e}^r}{r^{N/2-3/4}} \quad \text{for } r > 0 \text{ sufficiently large.}$$

Proof. (a) We assume without loss of generality that $\inf_{r>0} \varphi(r) > 0$ and define

$$X = \left\{ h \in \mathcal{H}(\mathbb{R}^N) : \|h\|_X := \sup_{r>0} \frac{M_2(h, r) r^{N/2-3/4}}{\varphi(r) \mathrm{e}^r} < \infty \right\}.$$

Then, as in [**1**], X is a Banach space that is continuously embedded in $\mathcal{H}(\mathbb{R}^N)$, endowed with the topology of local uniform convergence.

For $n \in \mathbb{N}$, we consider the continuous mappings

$$T_n : X \to \mathcal{H}(\mathbb{R}^N), \qquad T_n = \frac{\partial^n}{\partial x_k^n}\bigg|_X,$$

and for Y_0 we take the space of harmonic polynomials on $\mathbb{R}^N$.

Next, by Lemma 4.1, for any $H \in \mathcal{H}_{j,N}$ and $n \in \mathbb{N}$, there exists a unique harmonic polynomial $P_n(H) \in \mathcal{H}_{j+n,N}$ such that

$$\frac{\partial^n}{\partial x_k^n} P_n(H) = H.$$

Moreover, by [**1**, Lemma 4] and the homogeneity of $P_n(H)$, we have

$$M_2(P_n(H), r) \leqslant c_{N,j,n} r^{j+n} M_2(H, 1) \tag{4.1}$$

with

$$c_{N,j,n} = \left(\frac{(N + 2j - 2)!}{n!(N + 2j + n - 3)!(N + 2j + 2n - 2)} \right)^{1/2}.$$

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We note that, for fixed j,

$$c_{N,j,n} \sim \frac{1}{(n+j)!(n+j+1)^{N/2-1}}. \tag{4.2}$$

Since any harmonic polynomial h of degree m has a unique representation

$$h = \sum_{j=0}^{m} H_j, \quad H_j \in \mathcal{H}_{j,N},$$

(see [**8**, Proposition 1.26, Theorem 1.27]), we may define, for $n \in \mathbb{N}$, the mappings

$$S_n : Y_0 \to X, \qquad h = \sum_{j=0}^{m} H_j \mapsto \sum_{j=0}^{m} P_n(H_j).$$

With these definitions we have for any harmonic polynomial h and $l \in \mathbb{N}$ that

$$\sum_{n=1}^{l} T_l S_{l-n} h = \sum_{n=1}^{l} T_n h;$$

this converges unconditionally in $\mathcal{H}(\mathbb{R}^N)$, uniformly for $l \in \mathbb{N}$, because $\sum_{n=1}^{\infty} T_n h$ is a finite series. Moreover,

$$T_n S_n h = h \quad \text{for any } n \geqslant 1.$$

In addition, the uniqueness of the $P_n(H)$ implies that

$$T_l S_{l+n} h = S_n h \quad \text{for } n \geqslant 1.$$

Thus, in order for conditions (i)–(iv) in the Frequent Universality Criterion to hold it suffices to show that

$$\sum_{n=1}^{\infty} S_n H = \sum_{n=1}^{\infty} P_n(H)$$

converges unconditionally in X, for any polynomial $H \in \mathcal{H}_{j,N}$, $j \geqslant 0$.

Since the degrees of the $P_n(H)$, $n \geqslant 1$, are different, these functions are orthogonal with respect to the inner product defining $M_2(f,r)$ [**8**, Theorem 5.3]. Hence, for $F \subset \mathbb{N}$ finite, using (4.1), (4.2) and [**1**, Lemma 4], we have

$$\begin{aligned}
\Big\| \sum_{n\in F} P_n(H) \Big\|_X &= \sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)\mathrm{e}^r} M_2\Big(\sum_{n\in F} P_n(H), r\Big) \\
&= \sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)\mathrm{e}^r} \Big(\sum_{n\in F} M_2^2(P_n(H), r)\Big)^{1/2} \\
&\leqslant C \sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)\mathrm{e}^r} \Big(\sum_{n\in F} c_{N,j,n}^2 r^{2(j+n)}\Big)^{1/2} \\
&\leqslant C \sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)\mathrm{e}^r} \Big(\sum_{n\in F} \frac{1}{((n+j)!)^2}(n+j+1)^{N-2} r^{2(j+n)}\Big)^{1/2}.
\end{aligned}$$

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By Lemma 2.2 we have that

$$\biggl(\sum_{n=0}^{\infty}\frac{1}{(n!)^2(n+1)^{N-2}}r^{2n}\biggr)^{1/2}\leqslant C\frac{\mathrm{e}^r}{r^{N/2-3/4}},$$

which, as in the proof of Theorem 2.3, implies the unconditional convergence of $\sum_{n=0}^{\infty}P_n(H)$.

(b) We may assume that ψ is monotonically decreasing. Let $h\in\mathcal{H}(\mathbb{R}^N)$ satisfy the given growth condition. We can write h as

$$h=\sum_{n=0}^{\infty}H_n,\quad H_n\in\mathcal{H}_{n,N}.$$

By orthogonality of this sum we have

$$M_2(h,r)^2=\sum_{n=0}^{\infty}M_2(H_n,r)^2.$$

In addition,

$$\frac{\partial^n}{\partial x_k^n}h(0)=\sum_{j=0}^{\infty}\frac{\partial^n}{\partial x_k^n}H_j(0)=\frac{\partial^n}{\partial x_k^n}H_n(0);$$

in view of [**1**, Lemma 1] we deduce that

$$\biggl|\frac{\partial^n}{\partial x_k^n}h(0)\biggr|\leqslant n!\sqrt{d_{n,N}}\,r^{-n}M_2(H_n,r),$$

where $d_{n,N}=\dim\mathcal{H}_{n,N}$. We note that $d_{n,N}=O(n^{N-2})$ (see [**1**] or [**8**, p. 94]). Combining these results, we find that

$$\sum_{n=0}^{\infty}\biggl|\frac{\partial^n}{\partial x_k^n}h(0)\biggr|^2\frac{1}{(n!)^2n^{N-2}}r^{2n}\leqslant C\psi(r)^2\frac{\mathrm{e}^{2r}}{r^{N-3/2}},$$

that is,

$$\sum_{n=0}^{\infty}\biggl|\frac{\partial^n}{\partial x_k^n}h(0)\biggr|^2\frac{r^{2n+N-3/2}\mathrm{e}^{-2r}}{\psi(r)^2(n!)^2n^{N-2}}\leqslant C.$$

We can now argue exactly as in the proof of Theorem 2.4; note that the function

$$g(r)=\frac{r^{2n+N-3/2}\mathrm{e}^{-2r}}{(n!)^2n^{N-2}}$$

has its maximum at $a_n=n+\frac{1}{2}N-\frac{3}{4}$ with $g(a_n)\sim 1/\sqrt{n}$ and an inflection point at

$$n+\tfrac{1}{2}N-\tfrac{3}{4}+\sqrt{\tfrac{1}{2}n+\tfrac{1}{4}N-\tfrac{3}{8}}.$$

We then obtain that

$$\frac{1}{m}\sum_{n=0}^{m}\biggl|\frac{\partial^n}{\partial x_k^n}h(0)\biggr|^2\to 0,$$

so that h cannot be frequently hypercyclic. $\square$

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In [**2**], Aldred and Armitage studied rates of growth in terms of sup-norms on spheres, and they did this for arbitrary differentiation operators D^α.

In order to formulate their results we need to introduce the constants c_N given by

$$c_2 = 1, \qquad c_N = N\biggl(\prod_{j=1}^{N-1} \frac{(2j)^{2j}}{(2j+1)^{2j+1}}\biggr)^{1/(2N)}, \quad N \geqslant 3.$$

Then [**2**]

$$c_N > \sqrt{\tfrac{1}{2}N} \quad \text{for } N \geqslant 3 \quad \text{and} \quad c_N = \sqrt{\tfrac{1}{2}N} + o(1) \quad \text{as } N \to \infty.$$

Aldred and Armitage then show that, for any $\alpha \in \mathbb{N}_0^N$, $\alpha \neq 0$, there exists a harmonic function h on $\mathbb{R}^N$ that is hypercyclic for D^α such that, for any $\varepsilon > 0$, there is some $C_\varepsilon > 0$ with

$$|h(x)| \leqslant C_\varepsilon \mathrm{e}^{(c_N+\varepsilon)r} \quad \text{for } \|x\| = r > 0,$$

while, for $\alpha = (1, 1, \dots, 1)$, there can be no D^α-hypercyclic harmonic function h that satisfies

$$|h(x)| \leqslant C\mathrm{e}^{cr} \quad \text{for } \|x\| = r > 0$$

for any $c < \sqrt{\frac{1}{2}N}$. Here, $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^N$.

We improve the positive part of this result in two directions: we strengthen the growth condition and we extend the result to frequent hypercyclicity. Note, however, that the result of Aldred and Armitage even covers a more general notion of universality.

Theorem 4.3. *Let $\alpha \in \mathbb{N}_0^N$, $\alpha \neq 0$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\varphi(r)/r^p \to \infty$ as $r \to \infty$ for any $p \geqslant 0$. Then there exists a harmonic function h on $\mathbb{R}^N$ with*

$$|h(x)| \leqslant \varphi(r)\mathrm{e}^{c_N r} \quad \textit{for } \|x\| = r \textit{ sufficiently large}$$

that is frequently hypercyclic for D^α.

Proof. The proof follows the same lines as that of Theorem 4.2. Assuming, as usual, that $\inf_{r>0} \varphi(r) > 0$, we consider the Banach space

$$X = \biggl\{h \in \mathcal{H}(\mathbb{R}^N) : \|h\|_X := \sup_{r>0} \sup_{\|x\|=r} \frac{|h(x)|}{\varphi(r)\mathrm{e}^{c_N r}} < \infty\biggr\},$$

which is continuously embedded in $\mathcal{H}(\mathbb{R}^N)$, and we define continuous mappings

$$T_n : X \to \mathcal{H}(\mathbb{R}^N), \qquad T^n = D^{n\alpha}|_X.$$

For Y_0 we consider the space of harmonic polynomials on $\mathbb{R}^N$. The mappings $S_n : Y_0 \to X$ are defined by

$$h = \sum_{j=0}^{m} H_j \mapsto \sum_{j=0}^{m} P_n(H_j),$$

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where $H_j \in \mathcal{H}_{j,N}$ and $P_n(H_j)$ denotes the unique polynomial $G_j \in \mathcal{H}_{j+n|\alpha|,N}$ with $D^{n\alpha}G_j = H_j$ (see Lemma 4.1). It follows from [**2**, Lemma 4] that

$$\sup_{\|x\|=r} |P_n(H_j)| \leqslant C\frac{n^A|\alpha|^A(j+1)^{(N-1)/2}(c_N r)^{n|\alpha|}}{(n|\alpha|)!}\sup_{\|x\|=r}|H_j(x)|,$$

where $A, C > 0$ are constants depending only on N; we may assume that $A \in \mathbb{N}$.

Now, as in the proof of Theorem 4.2, the conditions of the Frequent Universality Criterion are satisfied in this setting; we need only note that here we have

$$\begin{aligned}\sup_{r>0}\frac{1}{\varphi(r)\mathrm{e}^{c_N r}}\sum_{n=L+1}^{\infty} C\frac{n^A|\alpha|^A(j+1)^{(N-1)/2}(c_N r)^{n|\alpha|}}{(n|\alpha|)!}\sup_{\|x\|=r}|H_j(x)| \\ \leqslant C\sup_{r>0}\frac{1}{\varphi(r)\mathrm{e}^{c_N r}}\sum_{n=L+1}^{\infty}\frac{n^A(c_N r)^{n|\alpha|}}{(n|\alpha|)!}r^j \\ \leqslant C\sup_{r>0}\frac{r^{j+A}}{\varphi(r)\mathrm{e}^{c_N r}}\sum_{n=L+1}^{\infty}\frac{(c_N r)^{n|\alpha|-A}}{(n|\alpha|-A)!},\end{aligned}$$

where we have used the homogeneity of H_j. □

Our previous result contains, in particular, the following.

Corollary 4.4. *For every $\alpha \in \mathbb{N}_0^N$, $\alpha \neq 0$, the operator D^α is frequently hypercyclic on $\mathcal{H}(\mathbb{R}^N)$.*

We complement Theorem 4.3 by giving lower estimates for possible growth rates, where we obtain rates in terms of L^2-norms. Note, however, that since

$$M_2(h,r) \leqslant \sup_{\|x\|=r}|h(x)|,$$

the result is also true for M_∞. For the same reason, Theorem 4.3 also gives a result in terms of M_2.

Theorem 4.5. *Let $\alpha \in \mathbb{N}_0^N, \alpha \neq 0$. Let ν denote the number of non-zero α_k and let*

$$\mu = \sqrt{\frac{|\alpha|}{2(\prod_{k=1}^N \alpha_k^{\alpha_k})^{1/|\alpha|}}}.$$

If $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is any function with $\psi(r) \to 0$ as $r \to \infty$, then there is no D^α-frequently hypercyclic harmonic function h on $\mathbb{R}^N$ with

$$M_2(h,r) \leqslant \psi(r)\frac{\mathrm{e}^{\mu r}}{r^{(N+\nu-2)/4}} \quad \textit{for } r > 0 \textit{ sufficiently large.}$$

In particular, if $\alpha = (1, 1, \dots, 1)$, then there exists no D^α-frequently hypercyclic function h with

$$M_2(h,r) \leqslant \psi(r)\frac{\exp(\sqrt{N/2}r)}{r^{(N-1)/2}}.$$

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Proof. By [**22**, Lemma 2.4] we have that

$$M_2(h,r)^2 = \Gamma(\tfrac{1}{2}N)\sum_{m=0}^{\infty}\frac{|\nabla_m h(0)|^2}{2^m m!\,\Gamma(m+N/2)}r^{2m}$$

with

$$|\nabla_m h|^2 = m!\sum_{|\alpha|=m}\frac{|D^\alpha h|^2}{\alpha_1!\cdots\alpha_N!}.$$

Hence, we obtain that

$$\sum_{n=0}^{\infty}\frac{|D^{n\alpha}h(0)|^2}{2^{n|\alpha|}\Gamma(n|\alpha|+\frac{1}{2}N)(n\alpha_1)!\cdots(n\alpha_N)!}r^{2|n|\alpha} \leqslant CM_2(h,r)^2.$$

Using the definition of μ and ν we deduce that

$$\sum_{n=0}^{\infty}\frac{|D^{n\alpha}h(0)|^2}{n^{(N+\nu-1)/2}}\left(\frac{\mathrm{e}\mu}{n|\alpha|}\right)^{2n|\alpha|}r^{2n|\alpha|} \leqslant CM_2(h,r)^2.$$

Now the proof can be completed as for Theorem 2.4. □

Remark 4.6. It is instructive to compare Theorem 4.5 with Theorem 4.2 (b).

(a) First, Theorem 4.2 (b) immediately extends to any operator D^α with $\alpha = (0,\dots,0,\nu,0,\dots,0)$, $\nu \geqslant 1$, because any function that is frequently hypercyclic for such a D^α is also frequently hypercyclic for some $\partial/\partial x_k$. Even more is true; one can show that Theorem 4.2 (b) holds with the same growth rate for an arbitrary operator D^α, $\alpha \neq 0$. One need only repeat the proof, noting that

$$D^{n\alpha}\sum_{j=0}^{\infty}H_j(0) = \sum_{j=0}^{\infty}D^{n\alpha}H_j(0) = D^{n\alpha}H_{n|\alpha|}(0)$$

and that [**1**, Lemma 1] is valid for arbitrary D^α.

(b) Together with Theorem 4.5 we now have two lower estimates for growth rates for arbitrary operators D^α. In some cases, Theorem 4.2 (b) gives the better growth estimate (for example, when α has exactly one non-zero entry); in others Theorem 4.5 gives the better estimate (for example, if $\alpha = (1,1,\dots,1)$, $N \geqslant 3$).

5. Frequently hypercyclic harmonic functions for translation operators

We finally consider translation operators

$$T_a f(x) = f(x+a), \quad a \neq 0,$$

on spaces of harmonic functions. The hypercyclicity of these operators was obtained by Dzagnidze [**20**]. An alternative proof is due to Armitage and Gauthier [**6**] (see also [**4**, § 11]); using their approach, we can show that the translation operators are even frequently hypercyclic.

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Theorem 5.1. *Every translation operator T_a, $a \neq 0$, is frequently hypercyclic on $\mathcal{H}(\mathbb{R}^N)$.*

Proof. By [**11**, Lemma 2.2] (see also [**15**, Lemma 2.5]) there are pairwise disjoint sets $A(l,\nu) \subset \mathbb{N}$, $l,\nu \geqslant 1$, of positive lower density such that

$$|n-m| \geqslant \nu + \mu \quad \text{for } n \in A(l,\nu),\ m \in A(k,\mu),\ n \neq m.$$

Let $(P_l)_{l\geqslant 1}$ be a dense sequence of harmonic polynomials in $\mathbb{R}^N$. We define a function g by

$$g(x) = P_l(x-na) \quad \text{for } x \in B(na, \tfrac{1}{2}\nu\|a\|),\ n \in A(l,\nu),\ l,\nu \geqslant 1,$$

where $B(b,\rho) = \{x \in \mathbb{R}^N : \|x-b\| \leqslant \rho\}$.

Then g and

$$F = \bigcup_{n=1}^{\infty} B(na, \tfrac{1}{2}\nu\|a\|)$$

satisfy the conditions of [**7**, Theorem 1.1]. Thus, there exists a harmonic function f on $\mathbb{R}^N$ such that

$$|f(x) - g(x)| < \frac{1}{1+\|x\|} \quad \text{for } x \in F.$$

This implies that, for $n \in A(l,\nu)$, $l,\nu \geqslant 1$,

$$\sup_{\|x\|\leqslant \nu\|a\|/2} |f(x+na) - P_l(x)| \to 0 \quad \text{as } n \to \infty.$$

Since each set $A(l,\nu)$ has positive lower density, f is frequently hypercyclic for T_a. $\square$

Armitage [**5**] proved that if $\phi : [0,\infty) \to (0,\infty)$ is a continuous increasing function such that $\log\phi(r)/(\log r)^2 \to \infty$ as $r \to \infty$, then there exists a harmonic function f on $\mathbb{R}^N$ that is hypercyclic for T_a such that

$$|f(x)| \leqslant \phi(\|x\|) \quad \text{for } x \in \mathbb{R}^N \quad \text{and} \quad \limsup_{x\to\infty} \frac{f(x)}{\phi(\|x\|)} = 1.$$

In particular, there are hypercyclic harmonic functions of order 0. However, the methods of [**5**] do not seem to be adaptable to the study of frequent hypercyclicity of T_a.

6. Problems

To end this paper we formulate several open problems.

- Complete the diagram in Figure 1.
- Even for $p = 2$ there remains a problem: does there exist a D-frequently hypercyclic entire function f that satisfies

$$M_2(f,r) \leqslant C\frac{\mathrm{e}^r}{r^{1/4}} \quad \text{for } r > 0?$$

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- Obtain the analogue of Theorem 4.2 for rates of growth in terms of L^p-norms $M_p(h, r)$.

- Does there exist a $\partial/\partial x_k$-frequently hypercyclic harmonic function h on $\mathbb{R}^N$ such that
$$M_2(h, r) \leqslant C \frac{\mathrm{e}^r}{r^{N/2-3/4}} \quad \text{for } r > 0?$$

- Find an optimal result on the possible rates of growth (in the ordinary sense, or in terms of M_p, $1 \leqslant p < \infty$) of harmonic functions that are hypercyclic or frequently hypercyclic for D^α. Aldred and Armitage [**2**] conjecture that, for hypercyclicity and sup-norms, exponential type $\sqrt{\frac{1}{2}N}$ (instead of the larger c_N) is possible.

- Obtain (optimal) results on the possible rates of growth of harmonic functions that are frequently hypercyclic for T_a, $a \neq 0$.

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