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A CONVEXITY THEOREM FOR BOUNDARIES OF ORDERED SYMMETRIC SPACES

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ABSTRACT. We consider a class of real flag manifolds which occur as Fürstenberg boundaries of ordered symmetric spaces and study the image of associated momentum maps. The presence of the order structure is responsible for much stronger convexity properties than in the general case.

1. **Introduction.** After Kostant's seminal paper [Ko73], in which he studied convexity properties of Iwasawa projections and adjoint orbits of compact groups, many convexity theorems appeared and proved useful in a wide variety of mathematical disciplines like harmonic analysis, topology or Lie theory (*cf.* [AL92], [At82], [vdB86], [BFR90], [BR91], [CDM88], [Du83], [GuSt82], [Ki84], [LR91], [Ne91], [Ne93], [Pa84], [Wi89]). There is not yet a unified approach to all these results even though many of them can be phrased in terms of symplectic geometry. In that context one has a symplectic manifold $\mathcal{M}$ and a hamiltonian group action of a Lie group G on $\mathcal{M}$ for which one can construct the so called momentum map $\Phi\colon \mathcal{M} \longrightarrow \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. In the case that G is a torus and $\mathcal{M}$ is compact, Atiyah and Guillemin-Sternberg showed that $\Phi(\mathcal{M})$ is a convex polyhedron (*cf.* [At82], [GuSt82]). If the manifold $\mathcal{M}$ in addition admits an involution which in a suitable sense is compatible with the symplectic structure and the hamiltonian action according to Duistermaat (*cf.* [Du83]) one even has $\Phi(\mathcal{M}) = \Phi(\mathcal{M}^\tau)$, where $\mathcal{M}^\tau$ denotes the set of fixed points of τ. The results of Kirwan (*cf.* [Ki84]) show that one cannot expect results of this generality for non-abelian compact groups. In the case of unitary representations viewed as hamiltonian actions on complex projective space Arnal and Ludwig in [AL92] completely determined when the image of the momentum map is convex in terms of the highest weight of the representation. On the other hand one has convexity theorems even for non-compact, non-abelian groups in the presence of order structures (*cf.* [Pa84], [Ne93]). These results, even though formulated in purely Lie theoretic terms, also have a symplectic interpretation. One is lead to the conjecture that there is some underlying principle that connects convexity properties of moment maps and the presence of order structures. The present paper gives some evidence that this is so. We study real flag manifolds that are closely related to the orderable symmetric spaces (*cf.* [Ols82], [Ola90]) and view them as fixed points under complex conjugation on corresponding complex flag manifolds which are

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Kähler, hence symplectic. The group we let act on the complex flag manifold is a compact form for the isotropy group of the symmetric space we started out with. Then the result is that the momentum map restricted to the real flag manifold has a compact convex image. This may be viewed as an extension of Duistermaat's theorem for these flag manifolds to a non-abelian group action. On the other hand it is not true that the image of the complex flag manifold under the momentum map is convex (and hence not equal to the image of the real flag manifold). The cases covered by our theorem include the real Grassmannians sitting inside the complex Grassmannians with the action of the unitary group and Herman's convexity theorem (*cf.* [Wo72]) which asserts that hermitean symmetric spaces may be realized as bounded convex domains. Even though the statement of our theorem makes sense for more general classes of symmetric spaces, counterexamples show that it in general becomes false in the absence of an order structure.

Let $\mathbf{X} = G/H$ be a pseudo-Riemannian symmetric space, *i.e.*, G is a reductive group and H an open subgroup of the group G^τ if fixed points of an involutive automorphism $\tau: G \longrightarrow G$. Then there exists a Cartan involution $\theta: G \longrightarrow G$ which commutes with τ. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$ be the eigenspace decompositions of the Lie algebra $\mathfrak{g}$ of G with respect to τ and θ for the eigenvalues 1 and -1. We assume here that G is contained in a complexification $G_{\mathbb{C}}$. Then the space $\mathbf{X}$ carries a G-invariant infinitesimally generated order structure (*cf.* [Ola90]) if and only if it is of *regular type*, which means that the centralizer of $\{X \in \mathfrak{p} \cap \mathfrak{q} : [X, \mathfrak{p} \cap \mathfrak{q}] = \{0\}\}$ in $\mathfrak{q}$ is equal to $\mathfrak{p} \cap \mathfrak{q}$. In this case any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$ is also maximal abelian in $\mathfrak{p}$ and $\mathfrak{q}$ and one finds an element $X_0 \in \mathfrak{a}$ such that $\operatorname{spec}(\operatorname{ad} X_0) = \{-1, 0, 1\}$ and the centralizer of X_0 in $\mathfrak{g}$ is $Z_{\mathfrak{g}}(X_0) = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p} =: \mathfrak{g}_0$. Consider the system $\Delta := \Delta(\mathfrak{g}, \mathfrak{a})$ of restricted roots for the pair $(\mathfrak{g}, \mathfrak{a})$. It can be split up into *compact* and *non-compact* roots according to $\Delta_0 := \{\alpha \in \Delta : \alpha(X_0) = 0\}$ and $\Delta_\tau := \{\alpha \in \Delta : \alpha(X_0) \neq 0\}$. The condition on the spectrum of $\operatorname{ad} X_0$ ensures that $\Delta_\tau = \Delta_{-1} \dot\cup \Delta_1$, where $\Delta_{\pm 1} := \{\alpha \in \Delta : \alpha(X_0) = \pm 1\}$ and $\dot\cup$ denotes disjoint union. We may and will assume that we have an ordering on Δ such that $\Delta_1 \subseteq \Delta^+$ and any non-compact positive root is larger than all the compact roots. Furthermore we set

$$\mathfrak{n}_i := \sum_{\alpha \in \Delta_i} \mathfrak{g}^\alpha, \quad i = \pm 1,$$

and note that

$$\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha$ is the root space for α and $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathcal{P} := \mathfrak{g}_0 + \mathfrak{n}_1$ is a maximal parabolic subalgebra and the H-orbit in the flag manifold G/P, where P is the subgroup corresponding to $\mathcal{P}$, plays the role of the Fürstenberg boundary for the positive domain in the ordered symmetric space $\mathbf{X}$ (*cf.* [FHO92]).

We assume that G is contained in a simply connected group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Then G/P may be viewed as a connected component of the set of real points in the complex flag manifold $G_{\mathbb{C}}/P_{\mathbb{C}}$, where $P_{\mathbb{C}}$ is the parabolic subgroup of $G_{\mathbb{C}}$ corresponding to the complexification $\mathcal{P}_{\mathbb{C}}$ of $\mathcal{P}$. Note that $G_{\mathbb{C}}/P_{\mathbb{C}}$ is a compact Kähler manifold. For a suitable compact form U of $G_{\mathbb{C}}$ the group U acts on $G_{\mathbb{C}}/P_{\mathbb{C}}$ by Kähler isomorphisms. Thus we have a moment map $\Phi\colon G_{\mathbb{C}}/P_{\mathbb{C}} \longrightarrow \mathfrak{u}^*$, where $\mathfrak{u}$ is the Lie algebra of U and $\mathfrak{u}^*$ its dual (*cf.* [At82]). If we restrict the action to some subgroup of U then the corresponding moment map is just Φ followed by the canonical projection onto the dual of the Lie algebra of the subgroup. We will show below that U may be chosen in such a way that $\mathfrak{u}_{\mathfrak{h}} = \mathfrak{u} \cap \mathfrak{h}_{\mathbb{C}}$ is a compact form of $\mathfrak{h}$. We consider the moment map $\Phi_{\mathfrak{h}}\colon G_{\mathbb{C}}/P_{\mathbb{C}} \longrightarrow \mathfrak{u}_{\mathfrak{h}}^*$ for the corresponding subgroup of U. Then our convexity result is

THEOREM 1.1. *(i) $\Phi_{\mathfrak{h}}(G/P)$ is a convex set.*

(ii) $\Phi_{\mathfrak{h}}\colon HP/P \longrightarrow \mathfrak{u}_{\mathfrak{h}}^*$ is a diffeomorphism onto the interior of $\Phi_{\mathfrak{h}}(G/P)$ in the real vector space spanned by $\Phi_{\mathfrak{h}}(G/P)$. ■

The proof of Theorem 1.1 rests on the fact that one can view the real flag manifold G/P as sitting in a real projective space derived naturally from a suitable highest weight representation of G (*cf.* Proposition 2.4).

I would like to thank the referee for the simple argument leading to the proof of Theorem 1.1(ii) which replaces a tedious Sl(2)-reduction from an earlier version.

2. **Flag manifolds and moment maps.** The algebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ is a compact form of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{u}$ containing $i\mathfrak{a}$ then $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Consider the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Similarly to the case of Δ we decompose $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ as

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \Delta_{-1}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \dot{\cup} \Delta_0(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \dot{\cup} \Delta_1(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$$

and set

$$\mathfrak{p}_{\pm} := \sum_{\lambda \in \Delta_{\pm 1}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \mathfrak{g}_{\mathbb{C}}^{\lambda}$$

and

$$\mathfrak{k}_{\mathbb{C}}^c := \mathfrak{t}_{\mathbb{C}} + \sum_{\lambda \in \Delta_0(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \mathfrak{g}_{\mathbb{C}}^{\lambda}.$$

Now we have $\mathcal{P}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^c + \mathfrak{p}_+$ and $\mathcal{P} = (\mathfrak{k}_{\mathbb{C}}^c + \mathfrak{p}_+) \cap \mathfrak{g}$.

We want to realize $G_{\mathbb{C}}/P_{\mathbb{C}}$ as a projective variety (*cf.* [BE89]). To this end we note first that we may find a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ which contains $\Delta_1(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and for which any element of $\Delta_1(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is larger than any element of $\Delta_0(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. In fact, we can even choose $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that Δ^+ consists of the restrictions of the elements in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ to $\mathfrak{a}$. Fix a basis Σ of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and note that Σ contains only one non-compact root, which we denote by β_0. Then $\Sigma_0 := \Sigma \cap \Delta_0(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is a basis for $\Delta_0^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. As usual we define elements $H_\beta \in \mathfrak{t}_{\mathbb{C}}$ for $\beta \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ via

$$B(H, H_\beta) = \beta(H) \quad \forall H \in \mathfrak{t}_{\mathbb{C}},$$

where B is the Killing form. Then we have

$$\mathfrak{t}_{\mathbb{R}} := \sum_{\beta \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \mathbb{R} H_\beta = i\mathfrak{t}$$

and the Killing form induces Euclidean inner products $(\cdot,\cdot)$ on $\mathfrak{t}_\mathbb{R}$ and its dual $\mathfrak{t}_\mathbb{R}^*$ which allows us to identify these to spaces. The co-roots $\check{\beta} \in \mathfrak{t}_\mathbb{R}$ are given by

$$\lambda(\check{\beta}) = 2\frac{(\lambda,\beta)}{(\beta,\beta)} \quad \forall \lambda \in \mathfrak{t}_\mathbb{R}^*.$$

Define an element $\lambda_0 \in \mathfrak{t}_\mathbb{R}^*$ via

$$\lambda_0(\check{\beta}) = \begin{cases} 0 & \text{for } \beta \in \Sigma_0 \\ 1 & \text{for } \beta = \beta_0. \end{cases}$$

REMARK 2.1. $\lambda_0 = \frac{(\beta_0,\beta_0)}{2} X_0 \in \mathfrak{a} \subseteq \mathfrak{t}_\mathbb{R} = \mathfrak{t}_\mathbb{R}^*$.

PROOF. Note first that $\mathbb{R}X_0 \subseteq \mathfrak{a} \subseteq \mathfrak{t}_\mathbb{R}$ is orthogonal to $\sum_{\beta\in\Sigma_0} \mathbb{R}\check{\beta} \subseteq \mathfrak{t}_\mathbb{R} \cap [\mathfrak{k}_\mathbb{C}^c, \mathfrak{k}_\mathbb{C}^c]$. But these two sets span $\mathfrak{t}_\mathbb{R}$ so the definition of λ_0 says that $\mathbb{R}X_0 = \mathbb{R}\lambda_0$. Since $\beta_0 \in \Delta_1(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ we now have

$$(\beta_0, X_0) = \beta_0(X_0) = 1 = \lambda_0(\check{\beta}) = 2\frac{(\beta_0,\lambda_0)}{(\beta_0,\beta_0)}. \qquad \blacksquare$$

Let $\pi\colon \mathfrak{g}_\mathbb{C} \longrightarrow \mathrm{gl}(V)$ be the holomorphic representation with highest weight λ_0 and v_0 a highest weight vector. The representation π can be integrated to a holomorphic representation of $G_\mathbb{C}$ which we also denote by π. If $\mathbb{P}(V)$ is the complex projective space associated with V and $[v]$ the line through $v \in V \setminus \{0\}$, then, using Remark 2.1, we see $P_\mathbb{C} = \{g \in G_\mathbb{C} : g \cdot [v_0] = [v_0]\}$ where $g \cdot [v]$ denotes the action induced on $\mathbb{P}(V)$ by π. Therefore $G_\mathbb{C}/P_\mathbb{C}$ gets identified with the orbit $G_\mathbb{C} \cdot [v_0]$ which is a projective variety. Therefore we may realize G/P as the G-orbit of $[v_0]$ in $\mathbb{P}(V)$.

LEMMA 2.2. *There exists a real form $V_\mathbb{R}$ of V containing v_0 which is $\mathfrak{g}$-invariant.*

PROOF. We have $\mathfrak{g} = \mathfrak{n}_{-1}+\mathfrak{g}_0+\mathfrak{n}_1 = \mathfrak{n}_{-1}+\theta(\mathfrak{n}_0)+\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$ and $\mathcal{P} = \theta(\mathfrak{n}_0)+\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$, where $\mathfrak{n}_0$ is the sum of the root spaces for the $\alpha \in \Delta_0^+$ and $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_1$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and set $V_\mathbb{R} := U(\mathfrak{g})v_0$. Note that $\mathbb{R}v_0$ is a weight space for $\mathfrak{a}$ so that $\mathcal{P}\mathbb{C}v_0 \subseteq \mathbb{C}v_0$ is possible only if $\pi\big(\theta(\mathfrak{n}_0)\big)v_0 = \{0\}$ and $\pi(\mathfrak{n})v_0 = \{0\}$. Note that $\mathfrak{m} \subseteq \mathfrak{k} \cap \mathfrak{g}_0 \subseteq \mathfrak{k}_\mathbb{C}^c$ is orthogonal to X_0 with respect to the Killing form. Therefore we even have $\mathfrak{m} \subseteq [\mathfrak{k}_\mathbb{C}^c, \mathfrak{k}_\mathbb{C}^c]$ which shows that $\pi(\mathfrak{m})$ annihilates v_0 since λ_0 vanishes on all the $\check{\beta}$ with $\beta \in \Delta_0(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. But then the Poincaré-Birkhoff-Witt Theorem shows that $U(\mathfrak{g})v_0 \cap \mathbb{C}v_0 = \mathbb{R}v_0$ and hence $V_\mathbb{R}$ is a proper subspace of V. It obviously is $\mathfrak{g}$-invariant and since $V_\mathbb{R}+iV_\mathbb{R}$ and $V_\mathbb{R}\cap iV_\mathbb{R}$ are $\mathfrak{g}_\mathbb{C}$-invariant we have $V_\mathbb{R}+iV_\mathbb{R} = V$ and $V_\mathbb{R}\cap iV_\mathbb{R} = \{0\}$ which proves our claim. $\blacksquare$

From Lemma 2.2 we see that the orbit $G\cdot[v_0]$ is contained in the *real* projective space $\mathbb{P}(V_\mathbb{R})$ which is the set of real points with respect to the complex conjugation on $\mathbb{P}(V)$ induced by the real form $V_\mathbb{R}$ of V.

REMARK 2.3. $\pi(\sigma X)v = \overline{\pi(X)\bar{v}}$, where $v \longmapsto \bar{v}$ denotes the complex conjugation on V with respect to $V_\mathbb{R}$ and σ is the complex conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$. $\blacksquare$

Let σ also denote the involutive automorphism of $U(\mathfrak{g}_\mathbb{C})$ induced by $\sigma\colon \mathfrak{g}_\mathbb{C} \longrightarrow \mathfrak{g}_\mathbb{C}$. Then Remark 2.3 shows that σ induces $v \longrightarrow \bar{v}$ via $\overline{a \cdot v_0} = \sigma(a) \cdot v_0$ for all $a \in U(\mathfrak{g}_\mathbb{C})$. Moreover we find $\overline{g \cdot v_0} = \sigma(g) \cdot v_0$ for all $g \in G_\mathbb{C}$. Thus $G_\mathbb{C} \cdot [v_0] \cap \mathbb{P}(V_\mathbb{R})$ corresponds to the fixed point set of the involution (still denoted by σ) induced on $G_\mathbb{C}/P_\mathbb{C}$ by σ.

PROPOSITION 2.4. *The real flag manifold G/P may be viewed as a connected component of the set of fixed points $(G_{\mathbb{C}}/P_{\mathbb{C}})^{\sigma} := G_{\mathbb{C}} \cdot [v_0] \cap \mathbb{P}(V_{\mathbb{R}})$ in $G_{\mathbb{C}}/P_{\mathbb{C}}$ for the complex conjugation σ on $G_{\mathbb{C}}/P_{\mathbb{C}}$ induced from $\sigma\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$.*

PROOF. We have seen above that $G \cdot [v_0]$ is contained in $(G_{\mathbb{C}}/P_{\mathbb{C}})^{\sigma}$. But

$$T_{[v_0]}(G_{\mathbb{C}} \cdot [v_0]) = (\mathfrak{p}_- + \mathfrak{k}^c_{\mathbb{C}} + \mathfrak{p}_+) \cdot v_0 = \mathfrak{p}_- \cdot v_0$$

so that the real dimensions of $(G_{\mathbb{C}}/P_{\mathbb{C}})^{\sigma}$ and G/P agree. Since G is connected and $(G_{\mathbb{C}}/P_{\mathbb{C}})^{\sigma}$ is G-invariant, this implies the claim. ∎

REMARK 2.5. We can choose a U-invariant Hermitian inner product $(\cdot|\cdot)$ on V which is real on $V_{\mathbb{R}}$ and satisfies

$$(v|w) = (\operatorname{Re} v|\operatorname{Re} w) + (\operatorname{Im} v|\operatorname{Im} w) - i(\operatorname{Re} v|\operatorname{Im} w) + i(\operatorname{Im} v|\operatorname{Re} w),$$

where Re and Im denote the real and imaginary parts in V with respect to the complex conjugation. In particular we have

$$(\bar{v}|\bar{w}) = \overline{(v|w)}.$$ ∎

The inner product $(\cdot|\cdot)$ induces a Fubini-Study metric on $\mathbb{P}(V)$ and then U acts on $\mathbb{P}(V)$ preserving the Kähler structure. Restrict the Kähler metric to $G_{\mathbb{C}} \cdot [v_0]$ then U also preserves the Kähler structure of $G_{\mathbb{C}}/P_{\mathbb{C}}$. Using the fact that the embedding $G_{\mathbb{C}}/P_{\mathbb{C}} \to \mathbb{P}(V)$ is U-equivariant, it is easy to see that the moment map for the action of U on $G_{\mathbb{C}}/P_{\mathbb{C}}$ is just the restriction of the moment map for the action of U on $\mathbb{P}(V)$ (*cf.* [GeSe87,§3]). But the latter is given by $\Phi\colon \mathbb{P}(V) \to \mathfrak{u}^*$

$$\langle \Phi([v]), X\rangle = i\frac{\big(\pi(X)v|v\big)}{(v|v)}.$$

REMARK 2.6. $\Phi([\bar{v}]) = -\sigma\Phi([v])$. ∎

Remark 2.6 shows that the image of G/P under Φ and $\Phi_{\mathfrak{h}}$ consists of purely imaginary elements of $\mathfrak{u}^*$ and $\mathfrak{u}^*_{\mathfrak{h}}$ respectively. If we identify $\mathfrak{u}^*$ with $\mathfrak{u}$ then $\mathfrak{u}^*_{\mathfrak{h}}$ gets identified with $\mathfrak{u}_{\mathfrak{h}} = \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{p} \cap \mathfrak{h})$ so that $\Phi_{\mathfrak{h}}$ restricts to a map $\Phi_{\mathfrak{h}}\colon G/P \to i(\mathfrak{p} \cap \mathfrak{h})$.

3. **Strongly orthogonal roots.** Consider the c-dual $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ of $\mathfrak{g}$. Then $(\mathfrak{g}^c, \tau)$ is a symmetric pair of *Hermitian type* (*cf.* [Ola91]). The complex conjugation η of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}^c$ is given by $\sigma\tau = \tau\sigma$ and a Cartan decomposition of $\mathfrak{g}^c$ compatible with τ is $\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$, where

$$\mathfrak{k}^c := \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{q} \cap \mathfrak{p})$$

and

$$\mathfrak{p}^c := \mathfrak{h} \cap \mathfrak{p} + i(\mathfrak{q} \cap \mathfrak{k}).$$

We denote the corresponding Cartan involution by θ^c. Note here that the complexifications of $\mathfrak{k}^c$ and $\mathfrak{g}_0$ agree so that this notation is compatible with the one from Section 2.

Let Ψ be a maximal system of strongly orthogonal roots in $\Delta_1(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ which is invariant under $-\tau$ (*cf.* [Ola91, §3]). We can choose root vectors E_μ for $\mu \in \Delta_1(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ satisfying

$$\tau E_\mu = E_{\tau\mu}, \quad \eta E_\mu = E_{-\mu}, \quad [E_\mu, E_{-\mu}] = \check{\mu}.$$

Set

$$\hat{E}_\mu = \begin{cases} E_\mu & \text{if } -\tau\mu = \mu \\ E_\mu + \tau E_{-\mu} & \text{if } -\tau\mu \neq \mu, \end{cases}$$

$X_\mu = E_\mu + \eta E_\mu$ and $\hat{X}_\mu = \hat{E}_\mu + \eta \hat{E}_\mu$. Then according to [Ola91, Lemma 3.3], the set

$$\mathfrak{b}^c := \sum_{\mu \in \Psi} \mathbb{R} X_\mu$$

is a maximal abelian subspace in $\mathfrak{p}^c$ and

$$\mathfrak{b}_\mathfrak{h} := \mathfrak{b}^c \cap \mathfrak{h} = \sum_{\mu \in \Psi} \mathbb{R} \hat{X}_\mu$$

is maximal abelian in $\mathfrak{h} \cap \mathfrak{p}$.

LEMMA 3.1. *Let* $\mu \in \Delta_1(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ *then*

$$\pi(\hat{X}_\mu)v_0 = \begin{cases} \pi(E_{-\mu})v_0 & \textit{if } -\tau\mu = \mu \\ \pi(E_{-\mu} + E_{\tau\mu})v_0 & \textit{if } -\tau\mu \neq \mu. \end{cases}$$

PROOF. This is an immediate calculation using $\eta E_\mu = E_{-\mu}$ and $\pi(\mathfrak{p}_+)v_0 = \{0\}$. ■

LEMMA 3.2. *Let R be the group generated by Ψ in $i\mathfrak{t}^*$ then* $R \cap \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) = \Psi$.

PROOF. This is clear, since the elements of Ψ are long roots and orthogonal. ■

PROPOSITION 3.3. $\Phi\Big(\mathbb{P}\big(U(\mathfrak{b}_\mathfrak{h})v_0\big)\Big) \subseteq i\mathfrak{b}_\mathfrak{h}$.

PROOF. Let

$$V_\mathfrak{h} := \sum_{\mu \in R} V^{\lambda_0 + \mu},$$

where the $V^{\mu'}$ are the weight spaces, then $U(\mathfrak{b}_\mathfrak{h})v_0 \subseteq V_\mathfrak{h}$ since $\mathfrak{b}_\mathfrak{h} \subseteq \sum_{\mu\in\Psi}(\mathfrak{g}_\mathbb{C}^\mu + \mathfrak{g}_\mathbb{C}^{-\mu})$. Lemma 3.2 implies $\pi(X)V_\mathfrak{h} \perp V_\mathfrak{h}$ for all $X \in \mathfrak{g}_\mathbb{C}^\mu, \mu \notin \Psi \cup -\Psi$ because weight vectors for different weights are orthogonal. This in turn shows that

$$\Phi\big(\mathbb{P}(V_\mathfrak{h})\big) \subseteq \Big(\mathfrak{u} \cap \sum_{\mu \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \setminus \Psi} (\mathfrak{g}_\mathbb{C}^\mu + \mathfrak{g}_\mathbb{C}^{-\mu})\Big)^\perp = i\mathfrak{b}^c + \mathfrak{t}.$$

On the other hand we have $\Phi\Big(\mathbb{P}\big(U(\mathfrak{b}_\mathfrak{h})v_0\big)\Big) \subseteq \Phi\big(\mathbb{P}(V_\mathbb{R})\big) \subseteq i\mathfrak{p}$, which proves the claim. ■

LEMMA 3.4. *Let* $\mathrm{p}\colon \mathfrak{u}^* \longrightarrow i\mathfrak{b}_{\mathfrak{h}}^*$ *be the canonical projection then*
(i) $\mathrm{p}\circ\Phi([v_0]) = 0.$
(ii) $\mathrm{p}\circ d\Phi([v_0])$ *is surjective.*

PROOF. (i)

$$\langle \mathrm{p}\circ\Phi([v_0]), \hat{X}_\mu\rangle = i\frac{\big(\pi(\hat{X}_\mu)v_0|v_0\big)}{(v_0|v_0)} = 0.$$

(ii)

$$\begin{aligned}\Big\langle \mathrm{p}\circ d\Phi([v_0])\big(\pi(\hat{X}_\mu)v_0\big), \hat{X}_{\mu'}\Big\rangle &= i\frac{d}{dt}\Big|_{t=0}\frac{\big(\pi(\hat{X}_{\mu'})(v_0+t\pi(\hat{X}_\mu)v_0)|v_0+t\pi(\hat{X}_\mu)v_0\big)}{\big(v_0+t\pi(\hat{X}_\mu)v_0|v_0+t\pi(\hat{X}_\mu)v_0\big)}\\ &= 2i\big(\pi(\hat{X}_{\mu'})v_0|\pi(\hat{X}_\mu)v_0\big)\\ &= \begin{cases}0 & \text{if } \mu\neq\mu'\\ 2i\|\pi(\hat{X}_\mu)v_0\|^2 & \text{if } \mu=\mu'.\end{cases}\end{aligned}$$

This proves the claim since $\mathfrak{p}_-\cap\mathcal{P}_{\mathbb{C}} = \{0\}$ and hence $\pi(\hat{X}_\mu)v_0$ does not vanish. ■

4. **Torus actions and projections.** Let $\mathfrak{t}_\sharp$ be a maximal abelian subalgebra of $\mathfrak{u}$ containing $i\mathfrak{b}^c$. Consider the torus $T_\sharp$ corresponding to $\mathfrak{t}_\sharp$ and the associated moment map $\Phi_\sharp\colon G_{\mathbb{C}}/P_{\mathbb{C}} \longrightarrow \mathfrak{t}_\sharp^*$. Let $W_\sharp$ be the Weyl group of the root system $\Delta_\sharp := \Delta\big(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{t}_\sharp)_{\mathbb{C}}\big)$ and $\lambda_\sharp$ an extremal weight of π with respect to $(\mathfrak{t}_\sharp)_{\mathbb{C}}$ then [GeSe87, §5], implies that $\Phi_\sharp(G_{\mathbb{C}}/P_{\mathbb{C}}) = \mathrm{conv}(W_\sharp\cdot\lambda_\sharp)$, where $\mathrm{conv}(W_\sharp\cdot\lambda_\sharp)$ is the (closed) convex hull of the Weyl group orbit $W_\sharp\cdot\lambda_\sharp$. Let now $T_\mathfrak{h}$ be the torus belonging to $i\mathfrak{b}_\mathfrak{h}$ and $\Phi_b\colon G_{\mathbb{C}}/P_{\mathbb{C}} \longrightarrow i\mathfrak{b}_\mathfrak{h}^*$ the corresponding moment map, then $\Phi_b = \mathrm{p}\circ\Phi_\sharp = \mathrm{p}\circ\Phi$.

Note that $\sigma|_{i\mathfrak{b}_\mathfrak{h}} = -\mathrm{id}$. Moreover σ replaces the Kähler form on $\mathbb{P}(V)$ by its conjugate. Therefore σ is antisymplectic and we may apply [Du83, Theorem 2.5], to G/P with the torus action $T_b\times G_{\mathbb{C}}/P_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}/P_{\mathbb{C}}$. The result is that $\Phi_b(G_{\mathbb{C}}/P_{\mathbb{C}}) = \Phi_b(G/P)$. Let $B_\mathfrak{h}$ be the analytic subgroup of G associated to $\mathfrak{b}_\mathfrak{h}$ then [Du83, Proposition 4.2], implies

$$\Phi_b(G/P) = \Phi_b(\overline{B_\mathfrak{h}\cdot[v_0]})$$

since Lemma 3.4(ii) says that $[v_0]$ is a regular point of Φ_b (Note that $\overline{B_\mathfrak{h}\cdot[v_0]}$ is the *closure* of $B_\mathfrak{h}\cdot[v_0]$). Combining these facts with Proposition 3.3 we obtain

PROPOSITION 4.1. $\Phi(\overline{B_\mathfrak{h}\cdot[v_0]}) = \Phi_\mathfrak{h}(\overline{B_\mathfrak{h}\cdot[v_0]}) == \Phi_b(\overline{B_\mathfrak{h}\cdot[v_0]}) = \Phi_b(G/P) = \Phi_b(G_{\mathbb{C}}/P_{\mathbb{C}})$ *and this set is the convex hull of the images under* Φ_b *of the* T_b*-fixed points in* G/P.

PROOF. All that remains to be noted is $B_\mathfrak{h}\cdot[v_0]\subseteq\mathbb{P}\big(U(\mathfrak{b}_\mathfrak{h})v_0\big)$. ■

Note that determining the T_b-fixed points in $\mathbb{P}(V)$ is the same as determining the weight decomposition of π with respect to T_b.

Consider the action of η on $\Delta_\sharp\subseteq i\mathfrak{t}_\sharp^*\cong i\mathfrak{t}_\sharp$ given by

$$\eta\gamma(X) = \overline{\gamma(\eta X)}\quad \forall X\in(\mathfrak{t}_\sharp)_{\mathbb{C}}$$

(*cf.* [Wa72, p.25]). Then $(\Delta_\sharp, \eta)$ is a *normal* σ- system of roots (*cf.* [Wa72, Lemma 1.1.3.6]).

REMARK 4.2. Let $\gamma \in \Delta_\sharp$. Then the following statements are equivalent

(1) $\eta\gamma = -\gamma$.

(2) $\mathfrak{b}^c \subseteq \ker \gamma$. ∎

For $\gamma \in \Delta_\sharp$ with $\eta\gamma \neq -\gamma$ set $\tilde{\gamma} = \frac{1}{2}(\gamma + \eta\gamma)$. Then, according to Araki's Theorem (*cf.* [Wa72, Proposition 1.1.3.1]), the set

$$\tilde{\Delta}_\sharp := \{\tilde{\gamma} : \gamma \in \Delta_\sharp, \eta\gamma \neq -\gamma\}$$

is a root system (in fact a system of restricted roots) in $(\mathfrak{b}^c)^*$, whose Weyl group $\tilde{W}_\sharp$ is generated by the reflections at the hyperplanes $\ker \gamma \cap \mathfrak{b}^c = \ker \tilde{\gamma} \cap \mathfrak{b}^c$. Moreover Satake's Theorem (*cf.* [Wa72, Proposition 1.1.3.3]) shows that

$$\tilde{W}_\sharp = \{w|_{(\mathfrak{b}^c)^*} : w \in W_\sharp, w(\mathfrak{b}^c) \subseteq \mathfrak{b}^c\}.$$

If now $\mathrm{p}_\sharp \colon i\mathfrak{t}_\sharp^* \longrightarrow (\mathfrak{b}^c)^*$ is the canonical projection then [Ne91, Theorem II.15], shows

$$\mathrm{p}_\sharp\big(\mathrm{conv}(W_\sharp \cdot \mu)\big) = \mathrm{conv}(\tilde{W}_\sharp \cdot \mu)$$

if $\mu \in (\mathfrak{b}^c)^*$. In fact, more is true

PROPOSITION 4.3. *Let* $\mu \in i\mathfrak{t}_\sharp^*$ *then* $\mathrm{p}_\sharp\big(\mathrm{conv}(W_\sharp \cdot \mu)\big) = \mathrm{conv}\big(\tilde{W}_\sharp \cdot \mathrm{p}_\sharp(\mu)\big)$.

PROOF. Let $\tilde{w} \in \tilde{W}_\sharp$ and $w \in W_\sharp$ be such that $w\big((\mathfrak{b}^c)^*\big) = (\mathfrak{b}^c)^*$ and $w|_{(\mathfrak{b}^c)^*} = \tilde{w}$. Then the orthogonality of w shows that also $i(\mathfrak{k}^c \cap \mathfrak{t}_\sharp)^*$ is stable under w and hence w commutes with η. But this shows

$$\mathrm{p}_\sharp\big(w(\mu)\big) = w\Big(\frac{1}{2}(\mu + \eta\mu)\Big) = \tilde{w}(\tilde{\mu}).$$

Now the proof of [Ne91, Theorem II.15] yields the desired formula. ∎

We will use [Ne91, Theorem II.15], once more. Note first that $\mathfrak{b}^c$ is $\mathfrak{h}$-*maximal* in the sense of [Sch84, §7]. But then [Sch84, Proposition 7.2.1], shows that $\Delta_b := \{\mu|_{\mathfrak{b}_\mathfrak{h}} : \mu \in \tilde{\Delta}_\sharp, \mu|_{\mathfrak{b}_\mathfrak{h}} \neq 0\}$ is a root system in $\mathfrak{b}_\mathfrak{h}^*$ whose Weyl group W_b is naturally identified with $N_{K^c}(\mathfrak{b}_\mathfrak{h})/Z_{K^c}(\mathfrak{b}_\mathfrak{h})$. Moreover it is shown in loc. cit. that each element of W_b can be obtained from an element of $\tilde{W}_\sharp$ via restriction. This means that we may apply [Ne91, Theorem II.15], to obtain

$$\mathrm{p}\big(\mathrm{conv}(\tilde{W}_\sharp \cdot \mu)\big) = \mathrm{conv}(W_b \cdot \mu)$$

for $\mu \in \mathfrak{b}_\mathfrak{h}^*$. Note that on $(\mathfrak{b}^c)^*$ the projection p is given by $\mu \mapsto \frac{1}{2}(\mu + \tau\mu)$. But then the same argument as in the proof of Proposition 4.3 shows that $\mathrm{p}\big(\mathrm{conv}(\tilde{W}_\sharp \cdot \mu)\big) = \mathrm{conv}\big(W_b \cdot \mathrm{p}(\mu)\big)$.

THEOREM 4.4. *Let* $\lambda_\sharp \in it_\sharp^*$ *be an extremal weight for* π. *Then* $\Phi_b(G/P) = \operatorname{conv}\big(W_b \cdot \mathrm{p}(\lambda_\sharp)\big)$.

PROOF.

$$\begin{aligned}\Phi_b(G/P) &= \Phi_b(G_{\mathbb{C}}/P_{\mathbb{C}})\\ &= \mathrm{p}\circ\Phi_\sharp(G_{\mathbb{C}}/P_{\mathbb{C}}) = \mathrm{p}\big(\operatorname{conv}(W_\sharp\cdot\lambda_\sharp)\big)\\ &= \mathrm{p}\Big(\operatorname{conv}\big(\tilde{W}_\sharp\cdot \mathrm{p}_\sharp(\lambda_\sharp)\big)\Big)\\ &= \operatorname{conv}\big(W_b\cdot\mathrm{p}(\lambda_\sharp)\big),\end{aligned}$$

since $\mathrm{p}\circ\mathrm{p}_\sharp = \mathrm{p}$. ■

Note that any element of Δ_b may be viewed as a restricted root in $\Delta(\mathfrak{h}, \mathfrak{b}_{\mathfrak{h}})$ since $\mathfrak{h}$, being ad $\mathfrak{b}_{\mathfrak{h}}$-invariant, consists of a sum of joint ad $\mathfrak{b}_{\mathfrak{h}}$-eigenspaces. Thus $W_{\mathfrak{b}}$ is contained in the Weyl group $W_{\mathfrak{h}}$ of $\Delta(\mathfrak{h}, \mathfrak{b}_{\mathfrak{h}})$ which in turn can be identified with $N_{(K_H)_0}(\mathfrak{b}_{\mathfrak{h}})/Z_{(K_H)_0}(\mathfrak{b}_{\mathfrak{h}})$, where $K_H := K\cap H$ is the maximal compact subgroup of H contained in U and $(K_H)_0$ is its connected component. This proves

LEMMA 4.5. $W_b = W_{\mathfrak{h}} = N_{(K_H)_0}(\mathfrak{b}_{\mathfrak{h}})/Z_{(K_H)_0}(\mathfrak{b}_{\mathfrak{h}})$. ■

5. **The convexity theorem.** Consider the Cartan decomposition $H = K_H B_{\mathfrak{h}}(K_H)_0$ of H and note that $(K_H)_0 \subseteq P$ so that the U-equivariance of Φ implies

$$\Phi(H\cdot[v_0]) = \Phi(K_H B_{\mathfrak{h}}\cdot[v_0]) = \mathrm{Ad}^*(K_H)\Phi(B_{\mathfrak{h}}\cdot[v_0]).$$

Analogously, we have

$$\Phi_{\mathfrak{h}}(H\cdot[v_0]) = \mathrm{Ad}^*(K_H)\Phi_{\mathfrak{h}}(B_{\mathfrak{h}}\cdot[v_0]) = \mathrm{Ad}^*(K_H)\Phi_b(B_{\mathfrak{h}}\cdot[v_0]).$$

We set

$$D := \mathrm{Ad}^*(K_H)\Phi_b(G/P) \subseteq i(\mathfrak{h}\cap\mathfrak{p})^*.$$

LEMMA 5.1. *Let* $\mu\in D$ *then* $\operatorname{conv}\big(\mathrm{Ad}^*(K_H)\mu\big)\subseteq D$.

PROOF. Let $\mu'\in\operatorname{conv}\big(\mathrm{Ad}^*(K_H)\mu\big)$ then μ' corresponds to an element of $i(\mathfrak{h}\cap\mathfrak{p})$ so that there exists an element $k\in K_H$ such that $\mathrm{Ad}^*(k)\mu'\in i\mathfrak{b}_{\mathfrak{h}}^*$. Let $[v]\in G/P$ be such that $\mu = \Phi([v])$ then

$$\begin{aligned}\mathrm{p}\Big(\operatorname{conv}\big(\mathrm{Ad}^*(K_H)\mu\big)\Big) &= \operatorname{conv}\Big(\mathrm{p}\big(\Phi(K_H\cdot[v_0])\big)\Big)\\ &\subseteq \operatorname{conv}\big(\Phi_b(G/P)\big)\\ &= \Phi_b(G/P)\subseteq D\end{aligned}$$

and hence $\mathrm{Ad}^*(k)\mu' = \mathrm{p}\big(\mathrm{Ad}^*(k)\mu'\big)\in D$. Thus we have $\mu'\in\mathrm{Ad}^*(K_H)D = D$. ■

PROPOSITION 5.2. $\Phi_{\mathfrak{h}}(G/P) = \Phi_{\mathfrak{h}}(\overline{H\cdot[v_0]}) = D$.

PROOF. We have seen already that $D = \Phi_{\mathfrak{h}}(\overline{H\cdot[v_0]})\subseteq\Phi_{\mathfrak{h}}(G/P)$. Conversely let $\mu\in\Phi_{\mathfrak{h}}(G/P)\subseteq i(\mathfrak{h}\cap\mathfrak{p})^*$, then there exists a $k\in K_H$ such that $\mathrm{Ad}^*(k)\mu\in i\mathfrak{b}_{\mathfrak{h}}^*$. But then

$$\mathrm{Ad}^*(k)\mu\in i\mathfrak{b}_{\mathfrak{h}}^*\cap\Phi_{\mathfrak{h}}(G/P)\subseteq\Phi_b(G/P)\subseteq D$$

and hence $\mu\in\mathrm{Ad}^*(k)D\subseteq D$. ■

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THEOREM 5.3. *D is convex.*

PROOF. We know from Theorem 4.4 and Lemma 4.5 that $\Phi_b(G/P) = \operatorname{conv}(W_{\mathfrak{h}} \cdot \mu_0)$, where $\mu_0 = \mathrm{p}(\lambda_\sharp)$ for a suitable $\lambda_\sharp \in i\mathfrak{t}_\sharp^*$. But then Lemma 5.1 implies

$$\begin{aligned} D &= \mathrm{Ad}^*(K_H)\Phi_b(G/P) \\ &= \mathrm{Ad}^*(K_H)\operatorname{conv}(W_{\mathfrak{h}} \cdot \mu_0) \\ &\subseteq \operatorname{conv}\big(\mathrm{Ad}^*(K_H) \cdot \mu_0\big) \\ &\subseteq D. \end{aligned}$$

■

THEOREM 5.4. *The map $\Phi_{\mathfrak{h}}\colon HP/P \longrightarrow i(\mathfrak{h} \cap \mathfrak{p})^*$ is a diffeomorphism onto the image which is open and coincides with the interior of $\Phi_{\mathfrak{h}}(G/P)$ in $i(\mathfrak{h} \cap \mathfrak{p})^*$.*

PROOF. A simple computation like that in Lemma 3.4(ii) shows that the map $\Phi_h\colon B_{\mathfrak{h}} \cdot [v_0] \longrightarrow i\mathfrak{b}_{\mathfrak{h}}^*$ has a bijective differential everywhere. The image of this map is convex hence simply connected. Therefore the map has to be surjective and thus a diffeomorphism. ■

We note at this point that one can now use the arguments of [Wo72, Section 4], to write $\Phi_{\mathfrak{h}}(G/P)$ as the unit ball with respect to a suitable operator norm.

6. **Examples.** We illustrate our results with the simplest family of examples, the ones which have the Graßmannians $G_p(\mathbb{R}^n)$ as flag varieties G/P. In this case the group G is $\mathrm{Sl}(n, \mathbb{R})$. We assume without loss of generality that $n \leq 2p$ and view the elements of G and its Lie algebra as blockmatrices with blocksize according to the partition $(p, n-p)$ of n. Let B_0 be the $(p) \times (n-p)$-matrix with entries

$$b_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Then the image of Φ_b is a cube which one obtains from $\begin{pmatrix} 0 & B_0 \\ B_0^\top & 0 \end{pmatrix}$ upon the action of the respective Weyl group and consequently

$$\Phi_{\mathfrak{h}}(G/P) = \left\{ \begin{pmatrix} o & gBh \\ h^{-1}B^\top g^{-1} & 0 \end{pmatrix} : g \in O(p), h \in O(q), B \in \Phi_b(G/P) \right\}$$

which is linearly isomorphic to the set of linear contractive mappings from $\mathbb{C}^q$ to $\mathbb{C}^p$ with the usual norm. For more details see the seminar report [Hi92].

A prominent class of examples for spaces of regular type is the class of spaces $H_{\mathbb{C}}/H$, where H is the group of symmetries of a Hermitian symmetric domain and $H_{\mathbb{C}}$ a complexification of H. In this case our results can be viewed as a symplectic interpretation (and proof) of Herman's convexity theorem which says that Hermitian symmetric spaces of non-compact type can be realized as convex domains. We leave the details to the reader (*cf.* [Wo72]).

We conclude with a counterexample which indicates the role of the ordering for the convexity of the moment map. Consider again spaces of type $H_{\mathbb{C}}/H$ but this time we don't insist on H being the group of symmetries of a Hermitian symmetric space. In fact, we restrict our attention to the case $H = \mathrm{Gl}(n+1,\mathbb{R})$, and $H_{\mathbb{C}} = \mathrm{Gl}(n+1,\mathbb{C})$ embedded as the diagonal in $G_{\mathbb{C}} = \mathrm{Gl}(n+1,\mathbb{C}) \times \overline{\mathrm{Gl}(n+1,\mathbb{C})}$. As maximal parabolic we choose

$$\mathcal{P} = \left\{ \begin{pmatrix} a & v^{\top} \\ 0 & D \end{pmatrix} \in \mathrm{gl}(n+1,\mathbb{C}) : a \in \mathbb{C}, v \in \mathbb{C}^n \right\}.$$

Then $G/P = \mathbb{P}(\mathbb{C}^n)$ and $G_{\mathbb{C}}/P_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^n) \times \overline{\mathbb{P}(\mathbb{C}^n)}$ and a compact form of $H_{\mathbb{C}}$ is $\mathrm{U}(n+1)$. The moment map $\Phi_{\mathfrak{h}} \colon G_{\mathbb{C}}/P_{\mathbb{C}} \longrightarrow \mathfrak{u}(n+1)$ associated with this action is

$$\Phi_{\mathfrak{h}}\big((z_0 : \cdots : z_n), (w_0 : \cdots : w_n)\big) = \frac{i}{2}\left(\frac{z_j \bar{z}_k}{\sum_{k=0}^n |z_k|^2} + \frac{\bar{w}_j \bar{w}_k}{\sum_{k=0}^n |w_k|^2}\right)_{j,k=0,\ldots,n}$$

and when restricted to G/P

$$\Phi_{\mathfrak{h}}\big((z_0 : \cdots : z_n)\big) = \frac{i}{\sum_{k=0}^n |z_k|^2}(\operatorname{Re} z_j \bar{z}_k).$$

Whereas the image is a disk for $n = 1$ (Hermitian!), it is non-convex for $n = 2$. This can be seen from a somewhat tedious computation using affine coordinates on the projective plane.

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