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TOPOLOGICAL ENTROPY FOR THE CANONICAL COMPLETELY POSITIVE MAPS ON GRAPH C^*-ALGEBRAS

JA A. JEONG AND GI HYUN PARK

Let $C^*(E) = C^*(s_e, p_v)$ be the graph C^*-algebra of a directed graph $E = (E^0, E^1)$ with the vertices E^0 and the edges E^1. We prove that if E is a finite graph (possibly with sinks) and $\phi_E : C^*(E) \to C^*(E)$ is the canonical completely positive map defined by

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e{}^*,$$

then Voiculescu's topological entropy $\mathrm{ht}(\phi_E)$ of ϕ_E is $\log r(A_E)$, where $r(A_E)$ is the spectral radius of the edge matrix A_E of E. This extends the same result known for finite graphs with no sinks. We also consider the map ϕ_E when E is a locally finite irreducible infinite graph and prove that $\sup_{E'}\{\mathrm{ht}(\phi_{E'})\} \leqslant \mathrm{ht}(\phi_E)$, where the supremum is taken over the set of all finite subgraphs of E.

1. INTRODUCTION

Given a directed graph E with the vertex set E^0 and the edge set E^1 it is well known that there exists a universal C^*-algebra $C^*(E)$ generated by partial isometries $\{s_e \mid e \in E^1\}$ and mutually orthogonal projections $\{p_v \mid v \in E^0\}$ satisfying certain relations determined by the graph E. A classical Cuntz–Krieger algebra $\mathcal{O}_A$ of an $n \times n$ $\{0,1\}$ matrix A is now well understood as a graph C^*-algebra $C^*(E)$ of a finite directed graph E with the vertex matrix A ($\mathcal{O}_A \cong \mathcal{O}_B$ for the edge matrix B of E). If A has no zero rows or columns, the map $\phi_A : \mathcal{O}_A \to \mathcal{O}_A$ defined by

$$\phi_A(x) = \sum_{j=1}^{n} s_j x s_j^*, \quad x \in \mathcal{O}_A$$

is unital and completely positive, where s_j's, $1 \leqslant j \leqslant n$, are the partial isometries that generate $\mathcal{O}_A$. If A is the edge matrix of E, ϕ_A corresponds to the unital completely positive map $\phi_E : C^*(E) \to C^*(E)$ given by

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e{}^*.$$

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Then one can think of Voiculescu's topological entropy of ϕ_E (or ϕ_A), and it turns out that if E is a finite directed graph with no sinks

$$\mathrm{ht}(\phi_E) = \log r(A_E),$$

where $r(A)$ is the spectral radius of the edge matrix A_E of E (see [**15, 4, 7, 3, 5, 14**]). One purpose of the present paper is to extend this result to a finite graph possibly with sinks, and the other is to provide a lower bound for $\mathrm{ht}(\phi_E)$ when E is a locally finite irreducible infinite graph.

In Section 2, we review several definitions and properties of graph C^*-algebras, entropies, and Voiculescu's topological entropy of a completely positive map. Then Section 3 is devoted to obtaining $\mathrm{ht}(\phi_E)$ for an arbitrary finite graph E with the sinks $\mathcal{S}(E)$. To this end we consider another completely positive map ψ_E on $C^*(E)$,

$$\psi_E(x) = \phi_E(x) + \sum_{v \in \mathcal{S}(E)} p_v x p_v,$$

and show that

$$\mathrm{ht}(\phi_E) = \mathrm{ht}(\psi_E) = \log r(A_E).$$

We first prove that $\log r(A_E) \leqslant \mathrm{ht}(\psi_E)$ by considering the topological entropy $h_{\mathrm{top}}(X_{E_S}, \sigma)$ of the (compact) edge shift space (X_{E_S}, σ) of the finite graph E_S which we obtain from E by adding a loop edge to each sink of E. For the reverse inequality $\mathrm{ht}(\psi_E) \leqslant \log r(A_E)$ we shall modify the proof of [**3**, Theorem 1] to cover our general situation. Then $\mathrm{ht}(\phi_E) = \mathrm{ht}(\psi_E)$ is proved.

In Section 4 we consider a locally finite (irreducible) infinite graph E, and prove that the map ϕ_E given by

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e^*, \ x \in C^*(E),$$

is a (well defined) completely positive contraction. But in this case the edge shift space X_E may not be compact, so we shall consider Gureyic's compactification $\overline{X}_E$ of X_E in order to find its topological entropy $h_{\mathrm{top}}(\overline{X}_E)$ as a lower bound for $\mathrm{ht}(\phi_E)$. Note from [**8**] that $h_{\mathrm{top}}(\overline{X}_E) = \sup_{E'} h_{\mathrm{top}}(X_{E'})$, where the supremum is taken over all the finite subgraphs of E. Then it follows that $\mathrm{ht}(\phi_E) = \infty$ for many infinite irreducible graphs E. Nevertheless it would be interesting and important to know the exact value of $\mathrm{ht}(\phi_E)$ when $\mathrm{ht}(\phi_E)$ is finite.

2. Preliminaries

2.1. Graphs and graph C^*-algebras. Let $E = (E^0, E^1, r, s)$ be a directed graph (or simply a graph) with a countable vertex set E^0 and a countable edge set E^1, where $r, s : E^1 \to E^0$ are the range and source maps. If each vertex of E emits and receives only finitely many edges, E is called *locally finite*. By $\mathcal{S}(E)$ we denote the set of all sinks (vertices which emit no edges) of E. A sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of edges satisfying $r(\alpha_i) = s(\alpha_{i+1})$, $i = 1, \dots, n-1$, is called a (finite) *path of length* $|\alpha| = n$. We simply write α as $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and extend the maps r, s to finite paths by $s(\alpha) = s(\alpha_1)$, $r(\alpha) = r(\alpha_n)$. E^n will denote the set of all finite paths of length n (each vertex is regarded as a finite path of length zero), and $E^* = \bigcup_{n=0}^{\infty} E^n$ denotes the set of all finite paths. Similarly an *infinite path* is defined to be an infinite sequence $\alpha = \alpha_1\alpha_2\cdots$ of edges with $r(\alpha_i) = s(\alpha_{i+1})$, $i = 1, 2, \dots$. If a path α $\big(|\alpha| > 0\big)$ satisfies $s(\alpha) = r(\alpha)$ we call α a *loop*. A loop α is called a *loop edge* if $|\alpha| = 1$.

For a graph E, a family $\{s_e, p_v \mid e \in E^1,\ v \in E^0\}$ of partial isometries s_e (with mutually orthogonal ranges) and mutually orthogonal projections p_v is called a *Cuntz–Krieger E-family* if it satisfies the following.

$$\begin{aligned} s_e^* s_e &= p_{r(e)}, \\ s_e s_e^* &\leqslant p_{s(e)}, \text{ and} \\ p_v &= \sum_{s(e)=v} s_e s_e^* \quad \text{if } 0 < \big|s^{-1}(v)\big| < \infty. \end{aligned}$$

It is known (see [**2**, **12**] for example) that there exists a universal C^*-algebra $C^*(E)$ (or $C^*(s_e, p_v)$) generated by a Cuntz–Krieger E-family $\{s_e, p_v\}$. We call $C^*(E)$ the *graph C^*-algebra* associated with E. It is useful to note that $\operatorname{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*\}$ is dense in $C^*(E)$, where $s_\alpha = s_{\alpha_1}\cdots s_{\alpha_k}$ if $\alpha = \alpha_1\cdots\alpha_k \in E^k$, $k \geqslant 1$, and $s_\alpha = p_v$ if $\alpha = v \in E^0$.

2.2. Shift space and entropies. Let $\mathcal{A}$ be a finite set. Then a subset $X \subset \mathcal{A}^{\mathbf{N}}$ is called a (one-sided) *shift space* if there is a collection $\mathcal{F}$ of words over $\mathcal{A}$ such that X is the set of all sequences x in which no word of $\mathcal{F}$ can appear. By σ_X we denote the shift map on X. Since $\mathcal{A}$ is finite (so compact in discrete topology), a shift space $X \subset \mathcal{A}^{\mathbf{N}}$ is a compact space and σ_X is continuous, hence (X, σ_X) carries the entropies which we review below.

(i) ([**13**, Definition 4.1.1] or [**10**, p.23]) The *entropy* $h(X)$ of X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log\big|W_n(X)\big|,$$

where $W_n(X)$ is the set of all words of length n that appear in a sequence of X. If $X \neq \emptyset$ we have $0 \leqslant h(X) < \log|\mathcal{A}| < \infty$ since $1 \leqslant \big|W_n(X)\big| \leqslant |\mathcal{A}|^n$. In particular, the full shift space $X_n = \mathcal{A}^{\mathbf{N}}$ $\big(|\mathcal{A}| = n\big)$ has $h(X_n) = \log n$. If $X = \emptyset$ then $h(X) = -\infty$ by definition.

(ii) ([**16**, Chapter 7]) Let $T : X \to X$ be a continuous map on a compact space X. If $\mathcal{U}$ is an open cover of X then so is $T^{-1}\mathcal{U}$. By $N(\mathcal{U})$ we denote the number of sets in a finite subcover of $\mathcal{U}$ with smallest cardinality. Then the *entropy of T relative* to $\mathcal{U}$ is given by

$$h_{\text{top}}(T, \mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} \log\Big(N\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\Big)\Big),$$

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where $\mathcal{U} \vee \mathcal{V}$ denotes the join of $\mathcal{U}$ and $\mathcal{V}$, and the *topological entropy* of (X, T) is defined to be

$$h_{\text{top}}(X, T) = \sup_{\mathcal{U}} h_{\text{top}}(T, \mathcal{U}),$$

where the supremum is taken over all the open covers (or equivalently, over all the finite open covers) of X.

REMARK 2.1. (a) If E is a finite graph we have the *edge shift space*

$$X_E = \{\alpha = (\alpha_i) \in (E^1)^{\mathbb{N}} \mid r(\alpha_i) = s(\alpha_{i+1}),\ i \in \mathbb{N}\}$$

(or the infinite path space) and the shift map σ_E given by $\sigma_E(\alpha)_i = \alpha_{i+1}$ for each $i \in \mathbb{N}$. For E with no infinite paths, we have $h(X_E) = -\infty$. Otherwise it is known [**16**, Theorem 7.13] that

$$h_{\text{top}}(X_E, \sigma_E) = h(X_E).$$

(b) Let $\Sigma_E (\subset (E^1)^{\mathbb{Z}})$ be the two-sided shift space associated with a finite graph E. Then we know from ([**10**, p.23]) that

$$h(X_E) = h(\Sigma_E).$$

We call a graph E *irreducible* if for any two vertices v, w there exists a finite path α with $s(\alpha) = v$, $r(\alpha) = w$. So a finite graph E is irreducible if and only if its vertex matrix V_E (or edge matrix A_E) is irreducible. Here a real, nonnegative square matrix $A = (A_{ij})_{1 \leqslant i,j \leqslant n}$ is *irreducible* if for each i, j there exists an $m \geqslant 1$ such that $(A^m)_{ij} > 0$.

If E is a finite graph, the vertex matrix V_E has irreducible components $V_1, \dots, V_k$ in the sense that each V_i is an irreducible nonnegative square integer matrix and there exists a permutation matrix P such that PV_EP^{-1} is in a block triangular form with blocks $V_1, \dots, V_k$ on its diagonal. Let λ_{V_i} be the Perron-Frobenius eigenvalue of V_i. Then the *Perron value* $\lambda_E = \max_{1 \leqslant i \leqslant k} \lambda_{V_i}$ is the largest eigenvalue of V_E, hence $\lambda_E = r(V_E)$, the spectral radius of V_E (see [**13**, Section 4.4]). One can write E^0 as the disjoint union of vertices E_i^0 $(1 \leqslant i \leqslant k)$ so that each V_i is a matrix with the index E_i^0. Let E_i be the subgraph of E with the vertex set E_i^0 and edge set $E_i^1 = \{e \in E^1 \mid s(e),\ r(e) \in E_i^0\}$, then E_i is irreducible, and E_i's are called the irreducible components of E. If E_i^0 is a singleton and $|E_i^1| = 1$, then $\log \lambda_{V_i} = 0$, thus the subgraph E_i makes no contribution to the value of $h(X_E)$ because

$$h(X_E) = \log \lambda_E = \max_{1 \leqslant i \leqslant k} \log \lambda_{V_i}$$

([**13**, Theorem 4.4.4]). On the other hand, it is easy to see that $r(A_E) = r(V_E)$. In fact, the rectangular matrices $R = (R_{ev})_{e \in E^1, v \in E^0}$, $S = (S_{ve})_{v \in E^0, e \in E^1}$, where

$$R_{ev} = \begin{cases} 1, & \text{if } r(e) = v, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad S_{ve} = \begin{cases} 1, & \text{if } s(e) = v, \\ 0, & \text{otherwise}, \end{cases}$$

satisfy $RS = A_E$ and $SR = V_E$, which implies that λ is an eigenvalue of V_E if and only if λ is an eigenvalue of A_E. Hence we have the following.

Proposition 2.2. *Let E be a finite graph and X_E be the one-sided shift space associated with E. Then*

$$h(X_E) = \log \lambda_E = \log r(A_E),$$

where λ_E is the Perron value of the edge matrix A_E (or the vertex matrix V_E) of E and $r(A_E)$ is the spectral radius of A_E.

2.3. Topological entropy of a completely positive map. We briefly review the definition of topological entropy for a completely positive map of a C^*-algebra which was first defined for automorphisms of unital nuclear C^*-algebras by Voiculescu [**15**] and then extended to automorphisms of exact C^*-algebras by Brown [**4**]. See also [**7**] and [**3**] for the following definition of topological entropy for a completely positive map.

Let $\pi : A \to B(H)$ be a faithful representation of a C^*-algebra A and $Pf(A)$ be the set of all finite subsets of A. For $\omega \in Pf(A)$ and $\delta > 0$, we put

$$\begin{aligned} \mathrm{CPA}(\pi, A) &:= \Big\{(\phi, \psi, B) \;\Big|\; \phi : A \to B, \psi : B \to B(H) \\ &\qquad\qquad \text{contractive completely positive maps, } \dim B < \infty\Big\}, \\ \mathrm{rcp}(\pi, \omega, \delta) &:= \inf\Big\{ \mathrm{rank}(B) \;\Big|\; (\phi, \psi, B) \in \mathrm{CPA}(\pi, A), \big\|\psi \circ \phi(x) - \pi(x)\big\| < \delta, \\ &\qquad\qquad \text{for all } x \in \omega\Big\}, \end{aligned}$$

where $\mathrm{rank}(B) :=$ the dimension of a maximal Abelian subalgebra of B.

It is well known [**9**] that every exact C^*-algebra A is nuclearly embeddable, that is, there exists a faithful representation $\pi : A \to B(H)$ such that for each finite subset $\omega \subset A$ and $\delta > 0$ there is $(\phi, \psi, B) \in \mathrm{CPA}(\pi, A)$ with $\psi \circ \phi$ close to π within δ on ω. Moreover the value $\mathrm{rcp}(\pi, \omega, \delta)$ is independent of the choice of π (see [**4, 3**]). Since graph C^*-algebras $C^*(E)$ are nuclear (see [**11**, p. 193]) we may write $\mathrm{rcp}(\omega, \delta)$ for $\mathrm{rcp}(\pi, \omega, \delta)$ assuming $C^*(E) \subset B(H)$ for a Hilbert space H.

Definition 2.3: ([**4, 3**]) Let $A \subset B(H)$ be a C^*-algebra and $\Phi : A \to A$ be a completely positive map. Then we define

$$\begin{aligned} \mathrm{ht}(\Phi, \omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log\big(\mathrm{rcp}(\omega \cup \Phi(\omega) \cup \cdots \cup \Phi^{n-1}(\omega), \delta)\big), \\ \mathrm{ht}(\Phi, \omega) &= \sup_{\delta>0} \mathrm{ht}(\Phi, \omega, \delta), \\ \mathrm{ht}(\Phi) &= \sup_{\omega\in Pf(A)} \mathrm{ht}(\Phi, \omega). \end{aligned}$$

$\mathrm{ht}(\Phi)$ is called the *topological entropy* of Φ.

Remark 2.4. We refer the reader to [**3, 4**], and [**7**] for the following useful properties. Let $\Phi : A \to A$ be a completely positive map on an exact C^*-algebra A.

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(a) If $\theta : A \to B$ is a C^*-isomorphism then

$$\mathrm{ht}(\Phi) = \mathrm{ht}(\theta\Phi\theta^{-1}).$$

(b) Let $\widetilde{A}$ be the unital C^*-algebra obtained by adjoining a unit. Let $\widetilde{\Phi} : \widetilde{A} \to \widetilde{A}$ be the extension of Φ. Then

$$\mathrm{ht}(\widetilde{\Phi}) = \mathrm{ht}(\Phi).$$

(c) If A_0 is a Φ-invariant C^*-subalgebra of A, then

$$\mathrm{ht}\big(\Phi|_{A_0}\big) \leqslant \mathrm{ht}(\Phi).$$

(d) If $\{\omega_k\}$ is an increasing sequence of finite subsets in A such that the linear span of the set $\bigcup\limits_{k,l \in \mathbb{Z}^+} \Phi^l(\omega_k)$ is dense in A, then

$$\mathrm{ht}(\Phi) = \sup_k \mathrm{ht}(\Phi, \omega_k).$$

(e) Let $T : X \to X$ be a continuous map on a compact metric space X. Then $\mathrm{ht}(T^*) = h_{\mathrm{top}}(X, T)$, where $T^* : C(X) \to C(X)$ is the completely positive map given by $T^*(f) = f \circ T,\ f \in C(X)$.

3. Finite graphs

In this section we consider the following two completely positive maps ϕ_E, ψ_E on the graph C^*-algebra $C^*(E)$ associated with a finite graph E,

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e{}^*,$$
$$\psi_E(x) = \sum_{e \in E^1} s_e x s_e{}^* + \sum_{v \in S(E)} p_v x p_v.$$

We call ϕ_E the *canonical completely positive map* of $C^*(E)$ which is not unital if E contains a sink while ψ_E is always. A computation shows that

(1) $$\psi_E^n(x) = \sum_{|\mu|=n} s_\mu x s_\mu{}^* + \sum_{\substack{0<|\eta|<n \\ r(\eta) \in S(E)}} s_\eta x s_\eta{}^* + \sum_{v \in S(E)} p_v x p_v.$$

Hence if E has no infinite paths then there exists an N such that the first term $\sum\limits_{|\mu|=n} s_\mu x s_\mu{}^*$ vanishes and $\psi_E^n(x) = \psi_E^N(x)$ whenever $n > N$. Thus it follows that $\mathrm{ht}(\psi_E) = 0$. But the edge matrix A_E has no nonzero irreducible components and so its Perron value is 0. Hence we see from Proposition 2.2 that $\log r(A_E) = -\infty$.

We now compute $\mathrm{ht}(\psi_E)$ (and $\mathrm{ht}(\phi_E)$) for E which contains an infinite path.

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THEOREM 3.1. *Let E be a finite graph with the edge matrix A_E. If E contains an infinite path then*

$$\mathrm{ht}(\psi_E) = \log r(A_E),$$

where $r(A_E)$ is the spectral radius of A_E.

Let $\mathcal{D}_E$ be the commutative C^*-subalgebra of $C^*(E)$ generated by projections of the form $p_\mu = s_\mu s_\mu^*$, $\mu \in E^*$. Then $\mathcal{D}_E$ is ψ_E-invariant and

$$\mathcal{D}_E = \overline{\mathrm{span}}\{p_\mu = s_\mu s_\mu^* \in C^*(E) \mid \mu \in E^*\}.$$

Now we seek a shift space (X, σ_X) such that there exists an isomorphism $w : \mathcal{D}_E \to C(X)$ satisfying $w(\psi_E|_{\mathcal{D}_E})w^{-1} = \sigma_X^*$ from which we deduce that $h(X) \leqslant \mathrm{ht}(\psi_E)$. Let $E_{\mathcal{S}}$ be the graph obtained from E by adding a loop edge e_v to each sink $v \in \mathcal{S}(E)$, that is,

$$E_{\mathcal{S}}^0 = E^0, \quad E_{\mathcal{S}}^1 = E^1 \cup \{e_v \mid s(e_v) = r(e_v) = v, \; v \in \mathcal{S}(E)\}$$

and consider the shift space $X_{E_{\mathcal{S}}}$ of infinite paths. Then the cylinder sets $[\mu] = \{\mu\alpha \mid \mu\alpha \in X_{E_{\mathcal{S}}}\}$, $\mu \in E_{\mathcal{S}}^*$, are both open and compact, and form a basis for the subspace topology of the compact space $X_{E_{\mathcal{S}}} \subset (E_{\mathcal{S}}^1)^{\mathbb{N}}$. Hence the characteristic functions $\chi_{[\mu]}$, $\mu \in E_{\mathcal{S}}^*$, are continuous on $X_{E_{\mathcal{S}}}$. Moreover applying the Stone–Weierstrass theorem one sees that the linear span of the characteristic functions $\{\chi_{[\mu]} \mid \mu \in E_{\mathcal{S}}^n, \; n \in \mathbb{N}\}$ is dense in $C(X_{E_{\mathcal{S}}})$. Then as in [**6**, Proposition 2.5] and [**14**, Corollary 7.2], one obtains the following.

LEMMA 3.2. *The linear map $w : \mathcal{D}_E \to C(X_{E_{\mathcal{S}}})$ given by*

$$w(p_\mu) = \begin{cases} \chi_{[\mu]}, & \text{if } |\mu| \geqslant 1, \\ \chi_{[e_v]}, & \text{if } \mu = v \in \mathcal{S}(E) \end{cases}$$

is a $*$-isomorphism such that $w(\psi_E|_{\mathcal{D}(E)})w^{-1} = (\sigma_{X_{E_{\mathcal{S}}}})^*$.

PROPOSITION 3.3. $h_{\mathrm{top}}(X_{E_{\mathcal{S}}}, \sigma_{X_{E_{\mathcal{S}}}}) = \mathrm{ht}(\psi_E|_{\mathcal{D}_E}) \leqslant \mathrm{ht}(\psi_E)$.

PROOF: By Remark 2.4(e), we have $h_{\mathrm{top}}(X_{E_{\mathcal{S}}}, \sigma_{X_{E_{\mathcal{S}}}}) = \mathrm{ht}((\sigma_{X_{E_{\mathcal{S}}}})^*)$. Also Remark 2.4(a) and Lemma 3.2 imply that $\mathrm{ht}((\sigma_{X_{E_{\mathcal{S}}}})^*) = \mathrm{ht}(\psi_E|_{\mathcal{D}_E})$. The last inequality follows from Remark 2.4(c). ∎

PROPOSITION 3.4.

(a) $h(X_E) = h(X_{E_{\mathcal{S}}})$.

(b) *Let G be the graph obtained from E by removing vertices v with $s^{-1}(v)$ consisting of a loop edge and all edges in $r^{-1}(v)$ and then adding a loop edge to each newly formed sink, if any. Then $h(X_E) = h(X_G)$.*

PROOF: (a) immediately follows from Proposition 2.2 and the arguments before it. For (b), apply (a) and the arguments before Proposition 2.2 repeatedly. ∎

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PROPOSITION 3.5. $\log r(A_E) = h(X_E) \leqslant \mathrm{ht}(\psi_E)$.

PROOF: $h(X_E) = h(X_{E_S})$ by Proposition 3.4(a), and $h(X_{E_S}) = h_{\mathrm{top}}(X_{E_S}, \sigma_{X_{E_S}})$ by Remark 2.1.(a). Then Proposition 3.3 proves the assertion. ∎

For the proof of the reverse inequality $\mathrm{ht}(\psi_E) \leqslant \log r(A_E)$, we modify the proof in [**3**] according to our general situation. But we have to deal with more complicated situation due to the existence of sinks which do not appear in case of [**3**], so we present a proof here. Put

$$W(n) := E^n \cup \left\{ \mu \in \bigcup_{k=0}^{n-1} E^k \,\middle|\, r(\mu) \in \mathcal{S}(E) \right\}.$$

Then there is a one to one correspondence between $W(n)$ and the set $(E_S)^n$ of finite paths of length n in E_S, and so the following lemma is an immediate consequence of Proposition 3.4(a).

LEMMA 3.6. $\lim_{n\to\infty} (1/n) \log\left|W(n)\right| = \log r(A_E)$.

As in [**3**] we define a map $\rho_m : C^*(E) \to M_{|W(m)|} \otimes C^*(E)$ by

$$\rho_m(x) := \sum_{\mu,\nu \in W(m)} e_{\mu\nu} \otimes s_\mu^* x s_\nu.$$

LEMMA 3.7. *ρ_m is an injective $*$-homomorphism.*

PROOF: Since $\sum_{\mu\in W(m)} s_\mu s_\mu^* = I$, the unit of $C^*(E)$, it easily follows that ρ_m is a $*$-homomorphism. To see that ρ_m is injective, suppose $\rho_m(x) = 0$ (in [**3**], $C^*(E)$ was simple). Then $s_\mu^* x s_\nu = 0$ for all $\mu, \nu \in W(m)$. Thus for each pair of vertices $v, w \in E^0$,

$$\sum_{\substack{\mu\in W(m), s(\mu)=v \\ \nu\in W(m), s(\nu)=w}} s_\mu s_\mu^* x s_\nu s_\nu^* = 0,$$

which implies that $p_v x p_w = 0$ since

$$p_v = \sum_{\mu\in W(m), s(\mu)=v} s_\mu s_\mu^*.$$

Therefore $x = 0$ and ρ_m is injective. ∎

LEMMA 3.8. *Let $n \in \mathbb{N}$, $|\beta| \leqslant |\alpha| \leqslant n_0$, and $m \geqslant n + n_0$. Then for each $0 \leqslant l \leqslant n-1$,*

$$\rho_m\left(\psi_E^l(s_\alpha s_\beta^*)\right) = \sum_{\mu\in W(|\alpha|-|\beta|)} X(\mu,\alpha,\beta,l,m) \otimes s_\mu$$

for some partial isometries $X(\mu,\alpha,\beta,l,m)$.

PROOF: Note first that if $\mu, \nu \in W(m)$ and $|\mu| \neq |\nu|$ then $s_\mu^* s_\nu = 0$. Also if $\mu, \nu \in E^*$, $r(\mu) \in \mathcal{S}(E)$, and $|\mu| < |\nu|$ then $s_\mu^* s_\nu = 0$. Then from the formula (1)

$$\begin{aligned}\rho_m\big(\psi_E^l(s_\alpha s_\beta^*)\big) &= \sum_{\eta \in W(l)} \rho_m(s_\eta s_\alpha s_\beta^* s_\eta^*) \\ &= \sum_{\eta \in W(l)} \sum_{\mu,\nu \in W(m)} e_{\mu\nu} \otimes s_\mu^* s_\eta s_\alpha s_\beta^* s_\eta^* s_\nu \\ &= \sum_{\mu \in W(|\alpha|-|\beta|)} \sum_{\substack{\eta\alpha\mu', \eta\beta\mu'\mu \in W(m) \\ \eta \in W(l)}} e_{\eta\alpha\mu', \eta\beta\mu'\mu} \otimes s_\mu,\end{aligned}$$

and $X(\mu,\alpha,\beta,l,m) := \sum_{\substack{\eta\alpha\mu', \eta\beta\mu'\mu \in W(m) \\ \eta \in W(l)}} e_{\eta\alpha\mu', \eta\beta\mu'\mu}$ is a partial isometry with the range projection $X(\mu,\alpha,\beta,l,m)X(\mu,\alpha,\beta,l,m)^* = \sum_{\substack{\eta\alpha\mu' \in W(m) \\ \eta \in W(l)}} e_{\eta\alpha\mu', \eta\alpha\mu'}$. ∎

For each $n_0 \geqslant 1$, put

$$\omega(n_0) := \big\{ s_\alpha s_\beta^* \mid |\beta| \leqslant |\alpha| \leqslant n_0 \big\}.$$

Then the following proposition implies that

$$\mathrm{ht}(\psi_E) \leqslant \log r(A_E),$$

since the linear span of the set $\bigcup_{k \geqslant 1} \big(\omega(k) \cup \omega(k)^*\big)$ is dense in $C^*(E)$ (Remark 2.4.(d)).

PROPOSITION 3.9. *Let $n_0 \geqslant 1$ and $\delta > 0$. Then*

$$\mathrm{ht}\big(\psi_E, \omega(n_0), \delta\big) = \limsup_n \frac{1}{n} \log \mathrm{rcp}\left(\bigcup_{i=0}^{n-1} \psi_E^i\big(\omega(n_0)\big), \delta \right) \leqslant \log r(A_E).$$

PROOF: Let H be a Hilbert space on which $C^*(E)$ acts faithfully. Since $C^*(E)$ is nuclear, there exists $(\phi_0, \psi_0, M_{m_0}) \in \mathrm{CPA}\big(id_{C^*(E)}, C^*(E)\big)$ such that

$$\big\|\psi_0\phi_0(s_\gamma) - s_\gamma\big\| < \frac{\delta}{|W(n_0)|}, \qquad \gamma \in W(n_0). \tag{2}$$

Now for $n \geqslant 1$, let $m = m(n) = n + n_0$ and $B = M_{|W(m)|} \otimes M_{m_0}$. Then by Arveson's extension theorem (see [**4**, p. 349]) the $*$-isomorphism $\rho_m^{-1} : \rho_m\big(C^*(E)\big) \to C^*(E)$ extends to a unital completely positive map

$$\Psi_m : M_{|W(m)|} \otimes C^*(E) \to B(H).$$

Now consider the completely positive maps ϕ and ψ given by

$$\phi = (id \otimes \phi_0)\rho_m : C^*(E) \to B \quad \text{and} \quad \psi = \Psi_m(id \otimes \psi_0) : B \to B(H).$$

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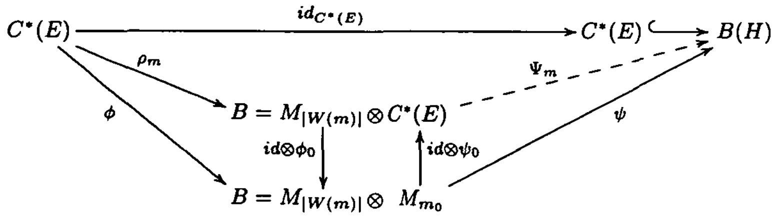

Let $a = s_\alpha s_\beta^* \in \omega(n_0)$. Then by Lemma 3.8 there exist partial isometries $X(\mu) = X(\mu, \alpha, \beta, l, m)$ such that

$$\rho_m \psi_E^l(a) = \sum_{\mu \in W(|\alpha|-|\beta|)} X(\mu) \otimes s_\mu. \tag{3}$$

Then as in [3] it follows from (2) and (3) that

$$\left\| \psi\phi\big(\psi_E^l(a)\big) - \psi_E^l(a) \right\| < \left|W(n_0)\right| \cdot \frac{\delta}{|W(n_0)|} = \delta.$$

Therefore

$$\operatorname{rcp}\left(\bigcup_{i=0}^{n-1} \psi_E^i\big(\omega(n_0)\big),\, \delta \right) \leqslant m_0 \left|W(m)\right| = m_0 \left|W(n+n_0)\right|,$$

and so $\limsup_n (1/n) \log \operatorname{rcp}\left(\bigcup_{i=0}^{n-1} \psi_E^i\big(\omega(n_0)\big),\, \delta \right) \leqslant \log r(A_E)$ (by Lemma 3.6). ∎

Corollary 3.10. *Let E be a finite directed graph and G be a subgraph of E obtained by removing sinks and edges going into them. Then*

$$\operatorname{ht}(\psi_E) = \operatorname{ht}(\psi_G).$$

In the rest of the section we show that $\operatorname{ht}(\phi_E) = \operatorname{ht}(\psi_E)$.

Lemma 3.11. $\psi_E^l(x) = \phi_E^l(x) + \psi_E^{l-1}\Big(\sum_{v \in S(E)} p_v x p_v \Big), \quad l \in \mathbb{N}.$

Proof: Since $\psi_E(x) = \phi_E(x) + \sum_{v \in S(E)} p_v x p_v$, we have

$$\begin{aligned}
\psi_E^2(x) &= \phi_E\Big(\phi_E(x) + \sum_{v \in S(E)} p_v x p_v \Big) + \sum_{w \in S(E)} p_w \Big(\phi_E(x) + \sum_{v \in S(E)} p_v x p_v \Big) p_w \\
&= \phi_E^2(x) + \phi_E\Big(\sum_{v \in S(E)} p_v x p_v \Big) + \sum_{w \in S(E)} p_w \Big(\sum_{v \in S(E)} p_v x p_v \Big) p_w \\
&= \phi_E^2(x) + \psi_E\Big(\sum_{v \in S(E)} p_v x p_v \Big).
\end{aligned}$$

For $l \geqslant 3$, use induction on l. ∎

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Let $\phi_L : C^*(E) \to C^*(E)$ be the completely positive map given by $\phi_L(x) = \sum_{\substack{e\in E^1 \\ r(e)\notin \mathcal{S}(E)}} s_e x s_e{}^*$.

Proposition 3.12. $\mathrm{ht}(\phi_L) = \mathrm{ht}(\phi_E) \leqslant \mathrm{ht}(\psi_E)$.

Proof: Let $\delta > 0$, $n \in \mathbb{N}$, and let $\omega \subset C^*(E)$ be a finite set of the elements of the form $s_\alpha s_\beta^*$, $\alpha, \beta \in E^*$ such that $\{p_v \mid v \in \mathcal{S}(E)\} \subset \omega$. Then choose an element $(\psi_1, \psi_2, B) \in \mathrm{CPA}(id_{C^*(E)}, C^*(E))$ satisfying $\mathrm{rank}(B) = \mathrm{rcp}\left(\bigcup_{j=0}^{n-1} \psi_E^j(\omega),\, \delta\right)$. If $x \in \omega$, $1 \leqslant l \leqslant n-1$, then by the above lemma

$$\begin{aligned}\left\|\psi_2\psi_1(\phi_E^l(x)) - \phi_E^l(x)\right\| &\leqslant \left\|\psi_2\psi_1(\psi_E^l(x)) - \psi_E^l(x)\right\| \\ &+ \left\|\psi_2\psi_1\left(\psi_E^{l-1}\Big(\sum_{v\in\mathcal{S}(E)} p_v x p_v\Big)\right) - \psi_E^{l-1}\Big(\sum_{v\in\mathcal{S}(E)} p_v x p_v\Big)\right\| \leqslant 2\delta,\end{aligned}$$

and $\mathrm{rcp}\left(\bigcup_{j=0}^{n-1} \phi_E^j(\omega), 2\delta\right) \leqslant \mathrm{rcp}\left(\bigcup_{j=0}^{n-1} \psi_E^j(\omega), \delta\right)$. Thus we have $\mathrm{ht}(\phi_E) \leqslant \mathrm{ht}(\psi_E)$.

To prove the first equality, note that $\phi_E^l(x) = \phi_L^l(x)$ if $x = s_\alpha s_\beta{}^* \in \omega$ with $|\alpha| + |\beta| > 0$, and $\phi_L^l(x) = 0$ if $x = p_v$, $v \in \mathcal{S}(E)$. Thus

$$\bigcup_{i=0}^{n-1} \phi_L^i(\omega) \subseteq \bigcup_{i=0}^{n-1} \phi_E^i(\omega) \cup \{0\},$$

and hence $\mathrm{ht}(\phi_L) \leqslant \mathrm{ht}(\phi_E)$. Put $\overline{\omega} := \omega \cup \phi_E(\omega)$. From definitions of ϕ_E and ϕ_L it is easily seen that $\phi_E^l(x) = \phi_L^{l-1}(\phi_E(x))$, $l \geqslant 1$. Thus

$$\bigcup_{i=0}^{n-1} \phi_E^i(\omega) \subseteq \bigcup_{i=0}^{n-1} \phi_L^i(\overline{\omega}),$$

which also shows that $\mathrm{ht}(\phi_E) \leqslant \mathrm{ht}(\phi_L)$. ∎

Note that the commutative C^*-subalgebra

$$\mathcal{D}'_E := \overline{\mathrm{span}}\{\, p_\mu = s_\mu s_\mu{}^* \mid \mu \in E^*,\ r(\mu) \notin \mathcal{S}(E)\}.$$

of $C^*(E)$ is ϕ_L-invariant and so $\mathrm{ht}(\phi_L|_{\mathcal{D}'_E}) \leqslant \mathrm{ht}(\phi_L)$.

Proposition 3.13. $\mathrm{ht}(\phi_E) = \mathrm{ht}(\psi_E)$.

Proof: Let G be the graph obtained from E by removing the sinks $\mathcal{S}(E)$ and the edges going into them. Then as in Lemma 3.2, one can show that there is an isomorphism $w' : \mathcal{D}'_E \to C(X_G)$ such that

$$\sigma_G^* = w'(\phi_L|_{\mathcal{D}'_E})(w')^{-1},$$

where σ_G is the shift map on X_G. Thus $\mathrm{ht}(\phi_L|_{\mathcal{D}'_E}) = h(X_G)$. Consequently,

$$\mathrm{ht}(\psi_E) = \log r(A_E) = h(X_E) = h(X_G) = \mathrm{ht}(\phi_L|_{\mathcal{D}'_E}) \leqslant \mathrm{ht}(\phi_E)$$

by Theorem 3.1, Corollary 3.10, and Proposition 3.12. ∎

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EXAMPLE 3.14. The Toeplitz algebra $\mathcal{T}$ can be viewed as the graph C^*-algebra $C^*(E)$ of $E = (E^0 = \{v, w\}, E^1 = \{e, f\})$, where $s(e) = r(e) = s(f) = v$, $r(f) = w$. In fact, if $\{s_e, s_f, p_v, p_w\}$ is a Cuntz–Krieger E-family generating $C^*(E)$ the element $U := s_e + s_f$ satisfies that $U^*U = I = p_v + p_w$, $UU^* = p_v$, $U^*U - UU^* = p_w$, $U^2U^* = s_e$, and $U - U^2U^* = s_f$. Thus $C^*(E) = C^*\{U\}$ and so by Coburn's theorem $\mathcal{T} = C^*\{U\} = C^*(E)$. Since $U^*U = I$, the linear span of the set $\{U^m(U^*)^n \mid m, n \geqslant 0\}$ is dense in $C^*(E)$, and one can show that $\phi_E(x) = UxU^*$ for each x of the form $U^m(U^*)^n$. Thus ϕ_E is the endomorphism $\mathrm{Ad}(U)$ on $\mathcal{T}$. Since $r(A_E) = 1$, it follows from Theorem 3.1 and Proposition 3.13 that $\mathrm{ht}(\phi_E) = \log r(A_E) = 0$. Thus $\mathrm{ht}(\mathrm{Ad}(U)) = 0$.

4. INFINITE GRAPHS

In this section we consider the topological entropy of ϕ_E for an infinite graph E.

PROPOSITION 4.1. *Let E be a locally finite infinite graph and let $C^*(E) = C^*(s_e, p_v)$ be its associated C^*-algebra. Then the sum $\sum_{e \in E^1} s_e x s_e^*$ exists for each $x \in C^*(E)$ and the map $\phi_E : C^*(E) \to C^*(E)$ given by*

$$\phi_E(x) = \sum_{e \in E^1} s_e x s_e^*, \ x \in C^*(E)$$

is a completely positive contraction.

PROOF: For an $x \in C^*(E)$ and $\varepsilon > 0$, choose a finite subgraph F of E and an element $z = \sum_{\alpha,\beta \in F^*} \lambda_{\alpha\beta} s_\alpha s_\beta^*$ $(\lambda_{\alpha\beta} \in \mathbb{C})$ such that $\|x - z\| < \varepsilon$. Put $E^1 = \{e_1, e_2, \dots\}$. Then by the local finiteness of E there is a number N such that

$$F^1 \cup \{e \in E^1 \mid r(e) \in F^0\} \subset E^1_N := \{e_1, e_2, \dots, e_N\},$$

so that $zp_{r(e_k)} = 0$ for $k \geqslant N + 1$. For any finite set E' of edges, let $V_{E'} := \{r(e) \mid e \in E' \setminus E^1_N\}$ and $P := \sum_{v \in V_{E'}} p_v$. Then $\|xP\| = \|(x - z)P\| < \varepsilon$, and

$$\Big\| \sum_{e \in E' \setminus E^1_N} s_e x s_e^* \Big\| = \Big\| \sum_{e \in E' \setminus E^1_N} s_e (xP)^*(xP) s_e^* \Big\|^{1/2} \leqslant \|xP\| < \varepsilon.$$

Thus if E', E'' are two finite sets of edges with $E^1_N \subset E' \cap E''$, then

$$\Big\| \sum_{e \in E'} s_e x s_e^* - \sum_{e \in E''} s_e x s_e^* \Big\| \leqslant \Big\| \sum_{e \in E' \setminus E^1_N} s_e x s_e^* \Big\| + \Big\| \sum_{e \in E'' \setminus E^1_N} s_e x s_e^* \Big\| < 2\varepsilon,$$

which shows that the sum $\sum_{e \in E^1} s_e x s_e^*$ exists and the map ϕ_E is well defined. To see that ϕ_E is a contractive completely positive map, consider a sequence of completely positive

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maps $\phi_n : C^*(E) \to C^*(E)$ given by $\phi_n(x) = \sum_{i=1}^{n} s_{e_i} x s_{e_i}{}^*$. If $x \geqslant 0$ then $\phi_E(x) \geqslant 0$ as the limit of positive elements $\phi_n(x)$ in norm. The same argument also proves that ϕ_E is completely positive. Since each ϕ_n is contractive we have $\|\phi_E\| \leqslant 1$. ∎

The (one-sided) shift space X_E may not be compact for an infinite graph E, which makes the definition $h_{\mathrm{top}}(X_E)$ meaningless. This leads Gurevic [8] to consider a compactification of X_E: Identify the edge set $E^1 = \{e_n\}_{n\in\mathbf{N}}$ with the metric space $\{1, (1/2), (1/3), \dots\} \subset [0,1]$ by $e_n \mapsto (1/n)$, and let $\overline{E}^1 := E^1 \cup \{0\} = \{0, 1, (1/2), (1/3), \dots\}$ be the one-point compactification. Then X_E becomes the subspace of the product space $(\overline{E}^1)^{\mathbf{N}}$ with the closure $\overline{X}_E$, where the metric

$$d\big((x_n),(y_n)\big) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n|, \quad x_n, y_n \in \overline{E}^1$$

is compatible with the product topology. The shift map $\overline{\sigma}_E := \sigma_{\overline{X}_E}$ on the compact metric space $\overline{X}_E$ now has a well-defined topological entropy. Similarly we have the compact metric space $\overline{\Sigma}_E \subset (\overline{E}^1)^{\mathbf{Z}}$ and the shift map $\overline{\sigma}_E := \sigma_{\overline{\Sigma}_E}$. We use the same notation for two shift maps.

Lemma 4.2. *If E is a locally finite irreducible infinite graph then*

$$h_{\mathrm{top}}(\overline{X}_E, \overline{\sigma}_E) = h_{\mathrm{top}}(\overline{\Sigma}_E, \overline{\sigma}_E).$$

Proof: Consider the open cover $\mathcal{P}_n := \{[1], \dots, [1/n], [\overline{1/n}]\}$ of $\overline{\Sigma}_E$, where

$$[1/k] = \{(x_i) \in \overline{\Sigma}_E \mid x_1 = 1/k\}, \ k = 1, \dots, n,$$
$$[\overline{1/n}] = \{(x_i) \in \overline{\Sigma}_E \mid x_1 < 1/n\}.$$

Put $\mathcal{V}_n := \overline{\sigma}_E^n \mathcal{P}_n \vee \overline{\sigma}_E^{n-1}\mathcal{P}_n \vee \dots \vee \overline{\sigma}_E \mathcal{P}_n \vee \mathcal{P}_n \vee \overline{\sigma}_E^{-1}\mathcal{P}_n \vee \dots \vee \overline{\sigma}_E^{-n}\mathcal{P}_n$. Then

$$\begin{aligned} &h_{\mathrm{top}}(\overline{\sigma}_E, \mathcal{V}_n) \\ &\quad= \lim_{k\to\infty} \frac{1}{k} \log N\Big(\bigvee_{l=0}^{k-1} \overline{\sigma}_E^{-l}(\mathcal{V}_n)\Big) \\ &\quad= \lim_{k\to\infty} \frac{1}{k} \log N\big(\overline{\sigma}_E^n \mathcal{P}_n \vee \dots \vee \overline{\sigma}_E^1 \mathcal{P}_n \vee \dots \vee \mathcal{P}_n \vee \dots \vee \overline{\sigma}_E^{n-k+1}\mathcal{P}_n\big) \\ (4) &\quad\leqslant \lim_{k\to\infty} \Big(\frac{1}{k}\log N(\overline{\sigma}_E^n \mathcal{P}_n \vee \dots \vee \overline{\sigma}_E^1 \mathcal{P}_n) + \frac{1}{k}\log N(\mathcal{P}_n \vee \dots \vee \overline{\sigma}_E^{n-k+1}\mathcal{P}_n)\Big) \\ &\quad= \lim_{k\to\infty} \frac{1}{k} \log N(\mathcal{P}_n \vee \overline{\sigma}_E^{-1}\mathcal{P}_n \vee \dots \vee \overline{\sigma}_E^{n-k+1}\mathcal{P}_n). \end{aligned}$$

Similarly for the finite open cover $\mathcal{Q}_n := \{[1], \dots, [1/n], [\overline{1/n}]\}$ of $\overline{X}_E$, where

$$[1/k] = \{(x_i) \in \overline{X}_E \mid x_1 = 1/k\}, \ k = 1, \dots, n,$$
$$[\overline{1/n}] = \{(x_i) \in \overline{X}_E \mid x_1 < 1/n\},$$

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and for $\mathcal{U}_n := \mathcal{Q}_n \vee \overline{\sigma}_E^{-1}\mathcal{Q}_n \vee \cdots \vee \overline{\sigma}_E^{-n}\mathcal{Q}_n$, one has

$$(5) \qquad h_{\text{top}}(\overline{X}_E, \mathcal{U}_n) = \lim_{k\to\infty} \log N\big(\mathcal{Q}_n \vee \overline{\sigma}_E^{-1}\mathcal{Q}_n \vee \cdots \vee \overline{\sigma}_E^{n-k+1}\mathcal{Q}_n\big).$$

But $N(\mathcal{P}_n \vee \overline{\sigma}_E^{-1}\mathcal{P}_n \vee \cdots \vee \overline{\sigma}_E^{n-k+1}\mathcal{P}_n) = N(\mathcal{Q}_n \vee \overline{\sigma}_E^{-1}\mathcal{Q}_n \vee \cdots \vee \overline{\sigma}_E^{n-k+1}\mathcal{Q}_n)$ follows easily, thus from (4) and (5), we have

$$h_{\text{top}}(\overline{\Sigma}_E, \mathcal{V}_n) = h_{\text{top}}(\overline{X}_E, \mathcal{U}_n).$$

On the other hand, the sequence $\{\mathcal{U}_n\}$ ($\{\mathcal{V}_n\}$, respectively) is refining (see [**1**]), that is, $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ and for every (finite) open cover $\mathcal{B}$ there exists an n such that $\mathcal{U}_n$ is a refinement of $\mathcal{B}$, which implies that

$$\begin{aligned} h_{\text{top}}(\overline{X}_E, \overline{\sigma}_E) &= \lim_{n\to\infty} h_{\text{top}}(\overline{X}_E, \mathcal{U}_n) \\ h_{\text{top}}(\overline{\Sigma}_E, \overline{\sigma}_E) &= \lim_{n\to\infty} h_{\text{top}}(\overline{\Sigma}_E, \mathcal{V}_n). \end{aligned}$$

∎

REMARK 4.3. For an infinite graph E, Gurevic [**8**] introduced an entropy

$$\sup\{h(\Sigma_{E'}) \mid E' \subset E \text{ finite subgraph}\},$$

and proved that $h_{\text{top}}(\overline{\Sigma}_E) = \sup_{E'} h(\Sigma_{E'})$ holds if E is irreducible. Moreover the supremum can be taken over all the irreducible finite subgraphs by [**8**, Lemma 2].

THEOREM 4.4. *Let E be a locally finite irreducible infinite graph. Then*

$$h_{\text{top}}(\overline{X}_E) = \sup_{E'} h(X_{E'}) \leqslant \operatorname{ht}(\phi_E),$$

where the supremum is taken over all the finite subgraphs of E.

PROOF: Recall that $h(\Sigma_{E'}) = h(X_{E'})$ for any finite subgraph E' of E (see Remark 2.1(b)). Then the first equality follows from Lemma 4.2 and Remark 4.3.

Note that for the locally compact shift space $X_E(\subset (\overline{E}^1)^{\mathbb{N}})$ the cylinder sets

$$[\alpha] = \{x = (x_1, x_2, \dots) \in X_E \mid x_i = \alpha_i, 1 \leqslant i \leqslant |\alpha|\},\ \alpha \in E^*$$

are both compact and open and form a basis for the topology. Also one can easily show that the closure $\overline{X}_E$ is nothing but the one point compactification of X_E. As in a finite graph case, let

$$\mathcal{D}_E := C^*\{p_\alpha \mid \alpha \in E^*\}$$

be the commutative C^*-subalgebra of $C^*(E)$ generated by projections $p_\alpha = s_\alpha s_\alpha^*$. Then clearly $\phi_E(\mathcal{D}_E) \subset \mathcal{D}_E$, hence $\operatorname{ht}(\phi_E|_{\mathcal{D}_E}) \leqslant \operatorname{ht}(\phi_E)$. Thus it suffices to see that

$$\operatorname{ht}(\phi_E|_{\mathcal{D}_E}) = h_{\text{top}}(\overline{X}_E).$$

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We prove that the map $w : \mathcal{D}_E \to C_0(X_E)$, $w(p_\alpha) = \chi_{[\alpha]}$, is a $*$-isomorphism such that

$$w(\phi_E|_{\mathcal{D}_E})w^{-1} = \sigma_E^*, \tag{6}$$

which then implies that $\mathrm{ht}(\phi_E|_{\mathcal{D}_E}) = \mathrm{ht}({\sigma_E}^*)$, and thus by Remark 2.4(b) we have

$$\mathrm{ht}(\phi_E|_{\mathcal{D}_E}) = \mathrm{ht}(\widetilde{\phi}_E|_{\widetilde{\mathcal{D}}_E}) = h_{\mathrm{top}}(\overline{X}_E).$$

Since it is tedious to show that w is an injective $*$-homomorphism satisfying (6), here we only prove that w is surjective. It is enough to see that the linear span of the characteristic functions $\chi_{[\alpha]}$ is dense in $C_0(X_E)$. Let $f \in C_0(X_E)$ and $\varepsilon > 0$. Then there is a compact subset $K \subset X_E$ such that $\|f|_{X_E \setminus K}\| < \varepsilon$. For each $x = (x_n) \in K$, consider the cylinder set

$$[x]_n := \{\, y = (y_n) \in X_E \mid x_k = y_k,\ 1 \leqslant k \leqslant n \,\}.$$

Since f is continuous at x there is a neighborhood U_x of x such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad y \in U_x.$$

Moreover we can choose $U_x = [x]_N$ for some $N \in \mathbb{N}$. Then there exists a finite subcover of $\{U_x \mid x \in K\}$ consisting of disjoint open sets, say $\{[x^1]_{N_1}, \dots, [x^m]_{N_m}\}$. Put $g := \sum_{j=1}^{m} f(x^j)\chi_{[x^j]_{N_j}}$. Then $g(y) = 0$ for $y \notin \bigcup_{j=1}^{m} [x^j]_{N_j}$. If $y \in [x^j]_{N_j}$ for some j then

$$|f(y) - g(y)| \leqslant |f(y) - f(x^j)| + |f(x^j) - g(y)| < \varepsilon.$$

Therefore $|g(y) - f(y)| < \varepsilon$ for each $y \in X_E$. ∎

REMARK 4.5. It would be nice to obtain an upper bound for the topological entropy $\mathrm{ht}(\phi_E)$ for E in Theorem 4.4. Let E be a locally finite irreducible infinite graph and let $\mathcal{A}_E$ be the AF subalgebra of $C^*(E) = C^*\{p_v, s_e\}$ generated by the partial isometries of the form $s_\alpha s_\beta^*$ with $|\alpha| = |\beta|$. Then $\mathcal{A}_E$ is ϕ_E-invariant and contains the commutative subalgebra $\mathcal{D}_E$, so that $\mathrm{ht}(\phi_E|_{\mathcal{D}_E}) \leqslant \mathrm{ht}(\phi_E|_{\mathcal{A}_E})$. We shall give an upper bound for $\mathrm{ht}(\phi_E|_{\mathcal{A}_E})$ elsewhere.

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Mathematical Sciences Division
BK21, Seoul National University
Seoul, 151–742
Korea
e-mail: jajeong@math.snu.ac.kr

Department of Mathematics
Hanshin University
Osan, 447–791
Korea
e-mail: ghpark@hanshin.ac.kr