Proceedings of the Royal Society of Edinburgh, page 1 of 29
DOI:10.1017/prm.2024.21

Limit cycles in a rotated family of generalized Liénard systems allowing for finitely many switching lines

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(Received 1 August 2023; accepted 2 February 2024)

Analytic rotated vector fields have four significant properties: as the rotated parameter α changes, the amplitude of each stable (or unstable) limit cycle varies monotonically, each semi-stable limit cycle bifurcates at most two limit cycles, the isolated homoclinic loop (if exists) disappears while a unique limit cycle with the same stability arises or no closed orbits arise oppositely, and a unique limit cycle arises near the weak focus (if exists). In this paper, we prove that the four properties remain true for a rotated family of generalized Liénard systems having finitely many switching lines. Furthermore, we discuss variational exponent and use it to formulate multiplicity of limit cycles. Then we apply our results to give exact number of limit cycles to a continuous piecewise linear system with three zones and answer to a question on the maximum number of limit cycles in an SD oscillator.

Keywords: rotated vector field; switching generalized Liénard system; limit cycle; homoclinic loop

2020 *Mathematics Subject Classification:* 34C05; 34C07; 34C23; 34C37

1. Introduction

Consider a planar differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x, y, \alpha), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x, y, \alpha), \tag{1.1}$$

parameterized by $\alpha \in I$ (an interval of $\mathbb{R}$), and use $(P(x, y, \alpha), Q(x, y, \alpha))$ to present its vector field, where functions $P, Q, \partial P/\partial\alpha, \partial Q/\partial\alpha$ are Lipschitzian in $D \times I$ and

$D \subset \mathbb{R}^2$ is a connected open set. The vector field $(P(x, y, \alpha), Q(x, y, \alpha))$ is called a *complete family of rotated vector fields* with a *rotated parameter* α if the following conditions hold:

(D1) Equilibria fixed: The number and location of equilibria are fixed as α varies.

(D2) Direction fixed: $(P(x, y, \alpha), Q(x, y, \alpha))$ rotates counter-clockwise at any regular point (x, y) as the rotated parameter α increases.

(D3) Symmetrically periodic: P, Q are periodic functions in α with minimum period 2π, and $(P(x, y, \alpha + \pi), Q(x, y, \alpha + \pi)) = -(P(x, y, \alpha),\ Q(x, y, \alpha))$.

This concept, originated by Duff ([8]) in 1953, was proved to have the following properties: **(DR1)** the limit cycles $L(\alpha_1)$ and $L(\alpha_2)$ of the vector fields with different α_1 and α_2 respectively in the family do not intersect each other; **(DR2)** every simple limit cycle expands or contracts monotonically as α varies monotonically; **(DR3)** every semi-stable limit cycle splits into a stable cycle and an unstable one when α increases or decreases, but disappears when α varies in the opposite direction; **(DR4)** the outer boundary and the inner one of the annulus $\mathcal{R}$ covered by all limit cycles $L(\alpha)$, $\alpha \in I$, i.e., $\mathcal{R} := \{(x, y) \in \mathbb{R}^2 : (x, y) \in L(\alpha), \alpha \in I\}$, surround an equilibrium each. These properties attracted great attentions (see e.g. [3, 9, 12, 21, 22, 24, 25, 28–30]) to rotated vector fields because they can be used to discuss the non-existence and the uniqueness of limit cycles as well as bifurcations of heteroclinic loops.

Perko ([20, 23]) weakened Duff's complete version to an uncomplete one, not requiring the symmetric periodicity but allowing the vector field not to rotate on an analytic curve $\Omega(x, y) = 0$ not having a branch congruent to a limit cycle of (1.1). He called $(P(x, y, \alpha), Q(x, y, \alpha))$ a *family of rotated vector fields (mod* $\Omega = 0$*)* with the *rotated parameter* α if the following conditions hold:

(P1) Equilibria fixed: the same as **(D1)**.

(P2) Direction fixed: $(P(x, y, \alpha), Q(x, y, \alpha))$ rotates counter-clockwise at any regular point (x, y) as the rotated parameter α increases except on the curve $\Omega(x, y) = 0$.

Assuming that the family of rotated vector fields is analytic in (x, y, α), he proved in [23, 26] the following results:

- **(PR1)** If the rotated vector field with $\alpha = \alpha_0$ exhibits a limit cycle Γ_0 of odd multiplicity then the cycle remains for $\alpha := \alpha_0 + \varepsilon$ with small enough $|\varepsilon|$ and expands or contracts monotonically as ε increases (see [26, Theorem 1 of Section 6 of Chapter IV]).

- **(PR2)** If the rotated vector field with $\alpha = \alpha_0$ exhibits a limit cycle Γ_0 of even multiplicity then, as the parameter α increases or decreases, Γ_0 splits into two simple limit cycles Γ_α^- and Γ_α^+, where the inner one Γ_α^- contracts and the outer one Γ_α^+ expands, but disappears as α varies oppositely (see [26, Theorem 2 of Section 6 of Chapter IV]).

- **(PR3)** If the origin O of the rotated vector field with $\alpha = \alpha_0$ is a weak focus then a unique limit cycle occurs in a small neighbourhood of O as α varies from α_0 with the change of stability at O (see [**26**, Theorem 5 of Section 6 of Chapter IV]) and, moreover, the limit cycle is of the same stability as the weak focus at the origin when $\alpha = \alpha_0$.

- **(PR4)** If the rotated vector field with $\alpha = \alpha_0$ has an isolated homoclinic loop Γ_* then, as the parameter α increases or decreases, the loop disappears while a unique limit cycle Γ_α with the same stability arises near Γ_*, but no closed orbits arise as α varies oppositely (see [**26**, Theorem 3 of Section 6 of Chapter IV]).

However, the above properties **(PR1)**–**(PR4)** may not be true if the rotated vector field is not analytic. The two examples of piecewise-defined families given in § 2, which are smooth but not analytic and satisfy the rotated conditions, show that a limit cycle of odd multiplicity may produce new limit cycles and a limit cycle of even multiplicity may produce more than two limit cycles, which do not match **(PR1)** and **(PR2)** separately.

In this paper, we investigate rotated vector fields of piecewise analytic generalized Liénard form and see which one of results **(PR1)**–**(PR4)** remains true. For this purpose, we consider the family of generalized Liénard equations

$$\ddot{x} + f(x, \dot{x}, \alpha)\dot{x} + g(x) = 0 \tag{1.2}$$

with non-analytic functions f and g in x or $\dot{x}$ such that its corresponding planar vector field is rotated with the parameter $\alpha \in I$. In § 2 we give our main results on the relation between variational exponent and multiplicity of hyperbolic limit cycles (which was not obtained for analytic rotated vector fields), non-hyperbolic limit cycles and semi-stable limit cycles of the vector field of (1.2) as α varies and answer to the questions:

(Q1) Can we find conditions such that the aforementioned results **(PR1)**–**(PR4)** still hold?

(Q2) Can we give an expanding (or contracting) rate for limit cycles in terms of the rotated parameter α?

(Q3) Can we use the rotated rule to determine the number of bifurcated limit cycles for a class of non-analytic systems?

In § 3 we further investigate the number of limit cycles bifurcated from a weak focus or a homoclinic loop in rotated vector fields. In § 4 and 5 we apply our main results obtained in § 2 to an SD oscillator and a continuous piecewise linear differential system with three zones and asymmetry for the number of limit cycles, respectively.

2. Main results

Let us begin this section from the family

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \tilde{X}_n(x, y, \alpha) := X_n(x, y)\cos\alpha - Y_n(x, y)\sin\alpha,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \tilde{Y}_n(x, y, \alpha) := X_n(x, y)\sin\alpha + Y_n(x, y)\cos\alpha\end{aligned} \tag{2.1}$$

with the functions

$$\begin{aligned}X_n(x, y) &:= \begin{cases} -y + x\tan\left((r - r_0)^{2n+1}\left(\sin\dfrac{1}{r - r_0} + 2\right)\right), & \text{as } r \neq r_0,\\ -y, & \text{as } r = r_0,\end{cases}\\ Y_n(x, y) &:= \begin{cases} x + y\tan\left((r - r_0)^{2n+1}\left(\sin\dfrac{1}{r - r_0} + 2\right)\right), & \text{as } r \neq r_0,\\ x, & \text{as } r = r_0,\end{cases}\end{aligned} \tag{2.2}$$

or

$$\begin{aligned}X_n(x, y) &:= \begin{cases} -y + x\tan\left((r - r_0)^{2n}\left(\sin\dfrac{1}{r - r_0} + 2\right)\right), & \text{as } r \neq r_0,\\ -y, & \text{as } r = r_0,\end{cases}\\ Y_n(x, y) &:= \begin{cases} x + y\tan\left((r - r_0)^{2n}\left(\sin\dfrac{1}{r - r_0} + 2\right)\right), & \text{as } r \neq r_0,\\ x, & \text{as } r = r_0,\end{cases}\end{aligned} \tag{2.3}$$

where $r := \sqrt{x^2 + y^2}$ is near $r_0 > 0$ and n is a positive integer. Note that system (2.1) with (2.2) (resp. (2.3)) is continuous and even $\mathcal{C}^n$ (resp. $\mathcal{C}^{n-1}$) but not analytic because its derivative is $\mathcal{C}^{n-1}$ (resp. $\mathcal{C}^{n-2}$) by an inductive proof from $n = 1$.

We first consider family (2.1) with (2.2), which is a Duff's complete family of rotated vector fields with the rotated parameter $\alpha \in [-\pi, \pi)$ and therefore a Perko's family because

$$(\tilde{X}_n(x, y, \alpha + \pi), \tilde{Y}_n(x, y, \alpha + \pi)) = -(\tilde{X}_n(x, y, \alpha), \tilde{Y}_n(x, y, \alpha)) \tag{2.4}$$

and

$$\det\begin{pmatrix} \tilde{X}_n(x, y, \alpha) & \tilde{Y}_n(x, y, \alpha)\\ \dfrac{\partial\tilde{X}_n(x, y, \alpha)}{\partial\alpha} & \dfrac{\partial\tilde{Y}_n(x, y, \alpha)}{\partial\alpha}\end{pmatrix} = \left(\tilde{X}_n(x, y, \alpha)\right)^2 + \left(\tilde{Y}_n(x, y, \alpha)\right)^2 > 0 \tag{2.5}$$

at each regular point. For $\alpha = 0$, $\tilde{X}_n(x, y, 0) = X_n(x, y)$ and $\tilde{Y}_n(x, y, 0) = Y_n(x, y)$. Then system (2.1) with (2.2) as $\alpha = 0$, denoted by **(E1)**, has a unique limit cycle $r =$

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r_0, which is of multiplicity $2n+1$, because the function $V(x,y) := x^2+y^2$ satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}t}|_{(\mathbf{E1})} = 2r\frac{\mathrm{d}r}{\mathrm{d}t}|_{(\mathbf{E1})} = \begin{cases} 2(x^2+y^2)\tan \\ \quad \left((r-r_0)^{2n+1}\left(\sin\dfrac{1}{r-r_0}+2\right)\right) > 0, & \text{as } r > r_0, \\ 0, & \text{as } r = r_0, \\ 2(x^2+y^2)\tan \\ \quad \left((r-r_0)^{2n+1}\left(\sin\dfrac{1}{r-r_0}+2\right)\right) < 0, & \text{as } r < r_0 \end{cases}$$

for r near r_0. On the other hand, for arbitrary $m \in \mathbb{Z}_+$ there exist $\alpha > 0$ and three values $r_1 = r_0 + 2/(5\pi + 4m\pi)$, $r_2 = r_0 + 2/(\pi + 4m\pi)$ and $r_3 = r_0 + 2/(-\pi + 4m\pi)$ such that $3(r_1 - r_0)^{2n+1} < \alpha < 3(r_2 - r_0)^{2n+1} < 3(r_3 - r_0)^{2n+1}$ and $r_1 < r_2 < r_3$. Thus, it is not difficult to check that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left\{ r\cos\alpha \left\{ \tan\left((r-r_0)^{2n+1}\left(\sin\frac{1}{r-r_0}+2\right)\right) - \tan\alpha \right\}, \quad \text{as } r \neq r_0, \right.$$
$$-r\sin\alpha, \quad \text{as } r = r_0$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}t} \begin{cases} < 0, & \text{as } r \leqslant r_0, \\ < 0, & \text{as } r = r_1, \\ > 0, & \text{as } r = r_2, \\ < 0, & \text{as } r = r_3. \end{cases}$$

By the mean value theorem, $\mathrm{d}r/\mathrm{d}t$ has two zeros, one lies in (r_1, r_2) and the other lies in (r_2, r_3), which implies that system (2.1) has at least two limit cycles in a small neighbourhood of $r = r_0$ if the positive integer m is large enough, a result different from **(PR1)**. This implies that for an arbitrary ε-neighbourhood of the circle $r = r_0$ there exists a number $N \in \mathbb{Z}_+$ such that the circle $r = r_i$ for $i = 1, 2, 3$ lies in the ε-neighbourhood of $r = r_0$ if $m > N$.

Family (2.1) with (2.3) is also a Duff's complete family of rotated vector fields with the rotated parameter $\alpha \in [-\pi, \pi)$ and therefore a Perko's family because (2.4) and (2.5) hold for this system at each regular point. Then system (2.1) with (2.3) as $\alpha = 0$, denoted by **(E2)**, has a unique limit cycle $r = r_0$, which is of multiplicity $2n$ and semi-stable, because

$$\frac{\mathrm{d}V}{\mathrm{d}t}|_{(\mathbf{E2})} = 2r\frac{\mathrm{d}r}{\mathrm{d}t}|_{(\mathbf{E2})} = \begin{cases} 2(x^2+y^2)\tan \\ \quad \left((r-r_0)^{2n}\left(\sin\dfrac{1}{r-r_0}+2\right)\right) > 0, & \text{as } r > r_0, \\ 0, & \text{as } r = r_0, \\ 2(x^2+y^2)\tan \\ \quad \left((r-r_0)^{2n}\left(\sin\dfrac{1}{r-r_0}+2\right)\right) > 0, & \text{as } r < r_0 \end{cases}$$

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for r near r_0. On the other hand, for $\alpha \neq 0$ we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \begin{cases} r\cos\alpha\left(\tan\left((r-r_0)^{2n}\left(\sin\dfrac{1}{r-r_0}+2\right)\right)-\tan\alpha\right), & \text{as } r \neq r_0, \\ -r\sin\alpha, & \text{as } r = r_0. \end{cases}$$

Since $\mathrm{d}r/\mathrm{d}t > 0$, there is no limit cycle as $\alpha < 0$. For $\alpha = \varepsilon > 0$ is small, we can obtain two limit cycles by a similar discussion as for system (2.1) with (2.2) in both cases $r > r_0$ and $r < r_0$ near $r = r_0$ respectively, which implies that system (2.1) with (2.3) has at least four limit cycles in a small neighbourhood of $r = r_0$, a result different from **(PR2)**.

The above two examples show that the properties **(PR1)** and **(PR2)** do not always hold for some non-analytic rotated vector fields, which suggest considering some non-analytic rotated vector fields for those properties **(PR1)**–**(PR4)**. In this paper, we discuss the generalized Liénard equation (1.2), which is equivalent to the following planar system

$$\begin{aligned} \dot{x} &= y =: X(x,y,\alpha), \\ \dot{y} &= -g(x) - f(x,y,\alpha)y =: Y(x,y,\alpha), \end{aligned} \tag{2.6}$$

where $(x,y) \in D$, a connected open set in $\mathbb{R}^2$. The system defines the corresponding vector field $\mathcal{L}_\alpha := (y, -g(x) - f(x,y,\alpha)y)$.

Let a_1(resp. a_2) denote the minimum (resp. maximum) for abscissas of points in D, which may be $-\infty$ (resp. $+\infty$). We need the following hypotheses:

(H_1) Piecewise analytic : The function $g(x)$ is piecewise analytic on $(a_1,x_1)\cup(x_1,x_2)\cup(x_2,x_3)\cup\ldots\cup(x_n,a_2)$, and functions $f(x,y,\alpha)$, $\partial f(x,y,\alpha)/\partial y$ are piecewise analytic on $D_1\cup D_2\cup\ldots\cup D_{n+1}$, where $D_1=\{(x,y)\in D : a_1<x<x_1\}$, $D_i=\{(x,y)\in D : x_{i-1}<x<x_i\}$ for $i=2,\ldots,n$ and $D_{n+1}=\{(x,y)\in D : x_n<x<a_2\}$.

(H_2) (H_2) Rotated

$$\frac{\partial f(x,y,\alpha)}{\partial\alpha} \geqslant 0 \ (\text{or } \leqslant 0) \tag{2.7}$$

in D and the equality in (2.7) does not hold on an entire closed orbit of (2.6).

Note that $D = D_1\cup\bar{D}_2\cup\ldots\cup\bar{D}_n\cup D_{n+1}$, where $\bar{D}_i$ denotes the closure of D_i. Without loss of generality, in this paper we only consider the case '$\geqslant$' in (2.7). Otherwise, we can make the transformation $(y,t)\to(-y,-t)$. Inequality (2.7) is not very restrictive because, for example, one can easily check that the function $f(x,y,\alpha) := \alpha x^{2m}y^{2n} + \hat{f}(x,y)$ with non-negative integers m,n and a piecewise analytic function $\hat{f}(x,y)$ satisfies $\frac{\partial f(x,y,\alpha)}{\partial\alpha} = x^{2m}y^{2n} \geqslant 0$, i.e. inequality (2.7). The following proposition indicates that the generalized Liénard system (2.6) is rotated if inequality (2.7) is satisfied.

PROPOSITION 2.1. *$\mathcal{L}_\alpha$ with the hypotheses (H_1) and (H_2) satisfies conditions* **(P1)** *and* **(P2)** *in Perko's definition.*

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Proof. One can check that **(P1)** is true because all equilibria lie at the x-axis and the abscissas of all equilibria are the roots of $g(x) = 0$ which are independent of α. In order to check **(P2)**, let θ denote the angle from the x-axis to the vector $(X(x, y, \alpha), Y(x, y, \alpha))$ of system (1.1) in counter-clockwise direction. Then

$$\frac{\partial\theta}{\partial\alpha} = \frac{\partial}{\partial\alpha}(\arctan(Y/X)) = \frac{1}{X^2 + Y^2}\begin{vmatrix} X & Y \\ \dfrac{\partial X}{\partial\alpha} & \dfrac{\partial Y}{\partial\alpha} \end{vmatrix}.$$

For $\mathcal{L}_\alpha$, we have $X(x, y, \alpha) = y$ and $Y(x, y, \alpha) = -g(x) - f(x, y, \alpha)y$. Then, it implies that the vector $(X(x, y, \alpha), Y(x, y, \alpha))$ rotates counter-clockwise at a regular point $P_0(x, y)$ as α varies since

$$\begin{vmatrix} X(x, y, \alpha) & Y(x, y, \alpha) \\ \dfrac{\partial X}{\partial\alpha}(x, y, \alpha) & \dfrac{\partial Y}{\partial\alpha}(x, y, \alpha) \end{vmatrix} = \begin{vmatrix} y & -g(x) - f(x, y, \alpha)y \\ 0 & -\dfrac{\partial f(x, y, \alpha)}{\partial\alpha}y \end{vmatrix} = -\frac{\partial f(x, y, \alpha)}{\partial\alpha}y^2 < 0$$

at the regular point $P_0(x, y)$ for all $\alpha \in I$ except in the curves $\Omega(x, y) := \frac{\partial f(x,y,\alpha)}{\partial\alpha}y^2 = 0$. Thus, condition **(P2)** holds. □

As indicated in the theory of non-analytic dynamical systems ([**1**, **15**]), limit cycles of piecewise analytic vector field $\mathcal{L}_\alpha$ are referred to the three types: crossing limit cycle (limit cycle intersecting a switching manifold transversely and only containing isolated crossing points of the switching manifold), grazing limit cycle (limit cycle tangent to a switching manifold) and sliding limit cycle (a curve segment of the limit cycle lies on a switching manifold). System (2.6) has no sliding limit cycles because only one point $\{(x_i, 0)\}$ lies in the sliding set of each switching line $x = x_i$. Thus, we only discuss crossing limit cycles and grazing limit cycles.

The first equation of (2.6) shows that the x-axis is a unique vertical isocline of the vector field $\mathcal{L}_\alpha$. Then each limit cycle Γ_α of system (2.6) has exactly two intersection points with the x-axis, denoted by $Q_0 : (x_l, 0)$ and $P_0 : (x_r, 0)$ with $x_l < x_r$. Without loss of generality, in what follows we assume that Γ_α is a crossing limit cycle; otherwise, we can research similarly because any small vicinity of the grazing limit cycle Γ_α (where Γ_α is not included) only possibly contains the crossing points (see the definition in [**1**, **16**]) of the switching line $x = x_i$ for $i = 1, \ldots, n$. Since any small neighbourhood of the crossing limit cycle Γ_α only possibly contains the crossing points of the included switching lines, the solution of system (2.6) is Lipschitzian ([**2**]). Then, the well-known existence and uniqueness theorem and the continuous dependence theorem of solutions remain true for system (2.6) in the small neighbourhood of Γ_α. Obviously, the x-axis is the normal of Γ_α at P_0 and Q_0. In this sense, we call the right-hand intersection point $P_0 : (x_r(\alpha), 0)$ the *right-endpoint* of the limit cycle Γ_α and regard x_r, denoted by $x_r(\alpha)$ for the dependence on the rotated parameter α, as the *x-radius* of the cycle. By a translation, we can assume that $x_r(\alpha) = 0$ without loss of generality. Thus, $g(0) > 0$ and $g(x_l) < 0$. Consider variation ε of α and suppose that the vector field $\mathcal{L}_{\alpha+\varepsilon}$ has a limit cycle $\Gamma_{\alpha+\varepsilon}$ with right-endpoint $\tilde{P}_0 : (\tilde{x}_r(\alpha + \varepsilon), 0)$. Then the difference between x-radii

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$\tilde{x}_r(\alpha+\varepsilon)$ and $x_r(\alpha)$ depends on ε, i.e. the function

$$\Delta_\alpha(\varepsilon) := \tilde{x}_r(\alpha+\varepsilon) - x_r(\alpha)$$

has the same smoothness as $x_r(\alpha)$ or $\tilde{x}_r(\alpha)$. By the continuity, $\Delta_\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. In the following theorem 2.3, we prove that

$$\Delta_\alpha(\varepsilon) = \zeta\varepsilon^\ell + o(\varepsilon^\ell) \text{ as } \varepsilon \to 0+ \tag{2.8}$$

for a certain $\ell \in \mathbb{R}^+$. Clearly, the amplitude of the limit cycle Γ_α expands outwards if $\zeta > 0$ or inwards (or say contracts) if $\zeta < 0$ with the *variational exponent* ℓ.

On the other hand, we also consider those spiral orbits near limit cycles. Let P denote $(x_0, 0)$, where $|x_0| < \delta$ and $\delta > 0$ is small. By the continuous dependence theorem, the orbit $\varphi(P, I^+)$ starting from P intersects the x-axis again for the first time at a point $P_1 = (x_1, 0)$, as shown in figure 1. Here, without loss of generality, we only show the case of the external neighbourhood of Γ in figure 1. Thus, we can define the map $\mathcal{P} : P \mapsto P_1$ (or $x_0 \mapsto x_1$ equivalently) on the x-axis, called a Poincaré map, and the successive function ϱ between P and $\mathcal{P}(P)$, i.e.

$$\varrho(x_0) := x_1 - x_0. \tag{2.9}$$

As indicated in Definition 2 of [**26**, p.216] or Section 2 of [**30**, Chapter 4], Γ_α is externally stable (resp. unstable) when there is a sufficiently small $\delta > 0$ such that $\varrho(x) < 0$ (resp. > 0) for all $x \in (0, \delta)$. Similarly, Γ_α is internally stable (resp. unstable) if $\varrho(x) > 0$ (resp. < 0) for all $x \in (-\delta, 0)$. Γ_α is *stable* (resp. *unstable*) if it is both externally stable (resp. unstable) and internally stable (resp. unstable). Naturally, we have $\varrho(0) = 0$ and $\varrho(x) = x^k h(x)$, where $k > 0$ is a real number and h is a continuous function such that $h(0) \neq 0$. Γ_α is called a *limit cycle of multiplicity k (which may not be an integer) if*

$$\varrho(x_0) = c_k x_0^k + o(|x_0|^k), \quad c_k \neq 0. \tag{2.10}$$

In particular, if system (2.6) *is smooth then $k \in \mathbb{Z}_+$ and condition* (2.10) *is equivalent to the following* $\varrho(0) = \varrho'(0) = \cdots = \varrho^{(k-1)}(0) = 0$ and $\varrho^{(k)}(0) \neq 0$ *by* [**30**, *Chapter 4.2, Definition 2.1*]. *Then, Γ_α is called a simple or hyperbolic limit cycle for* $k = 1$; *for odd k, Γ_α is stable (resp. unstable) if $\varrho^{(k)}(0) < 0$ or $c_k < 0$ (resp.$\varrho^{(k)}(0) > 0$ or $c_k > 0$);* for even k, Γ_α is semi-stable. The multiplicity k is an integer if the system is smooth (see Theorem 7.19 in [**11**, p.196]), but may not be integer for piecewise analytic systems. For example, the following system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - x(x^2 + y^2 - 1)^{p/q}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - y(x^2 + y^2 - 1)^{p/q}$$

with an integer p and an odd number q has the limit cycle $x^2 + y^2 = 1$, which is of multiplicity p/q because one can reduce the system to the equation $\frac{\mathrm{d}r}{\mathrm{d}t} = -r(r^2 - 1)^{p/q}$, where $r = \sqrt{x^2 + y^2}$. For simplicity, let $p/q = 5/3$. Then, for an initial value $(r, \theta) = (1 + \varepsilon, 0)$ with arbitrarily small $|\varepsilon|$, near the cycle $r = 1$ we can use the method of indeterminate coefficients to give the successive function

$$\varrho(\varepsilon) = c_k\varepsilon^k + o(|\varepsilon|^k) = \varepsilon^k(c_k + o(1)),$$

where $k = 5/3$ and $c_k = -2^{8/3}\pi$.

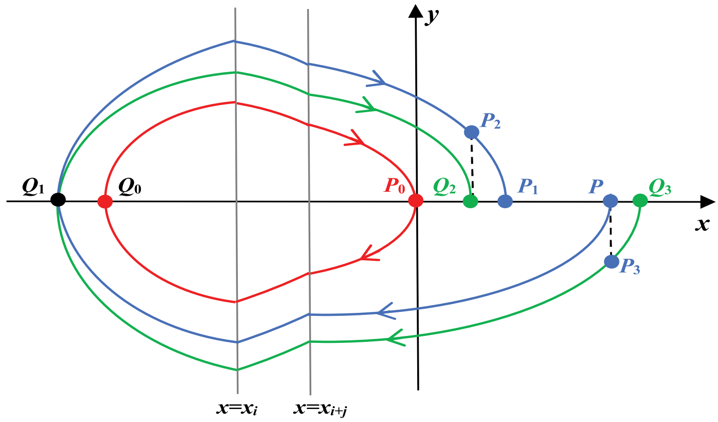

Figure 1. The orbits close to limit cycle Γ for $\varepsilon > 0$.

Unlike (2.10), Γ_α is called an externally (or internally) *compound limit cycle* (see [**30**, Chapter 4.2]) if for arbitrarily given small $\delta > 0$ there are two points $x_0, \tilde{x}_0 \in (0, \delta)$ (or $\in (-\delta, 0)$) such that

$$\mathcal{P}(x_0) = x_0 \text{ and } \mathcal{P}(\tilde{x}_0) \neq \tilde{x}_0,$$

i.e. near a side of Γ_α there is not a periodic annulus but there is a sequence of periodic solutions approaching the cycle Γ_α. Although any analytic system does not have a compound limit cycle ([**30**, Theorem 2.1 of Chapter 4]), a piecewise analytic Liénard system may have. For example, consider system (2.6) with $g(x) = x$ and

$$f(x,y) = \begin{cases} (x^2+y^2-1)^k \sin \dfrac{2\pi}{x^2+y^2-1}, & \text{as } x^2+y^2 \neq 1, \\ 0, & \text{as } x^2+y^2 = 1, \end{cases} \tag{2.11}$$

where k is a positive integer. In the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, we obtain

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= -r\sin^2\theta f(r\cos\theta, r\sin\theta) \\ &= \begin{cases} -r\sin^2\theta(r^2-1)^k \sin \dfrac{2\pi}{r^2-1}, & \text{as } r \neq 1, \\ 0, & \text{as } r = 1. \end{cases} \end{aligned}$$

It implies that system (2.6) with $g(x) = x$ and $f(x)$ exhibited in (2.11) has a compound limit cycle $x^2 + y^2 = 1$ because it has closed orbits $x^2 + y^2 = 1 \pm 2/n$ for all integers $n \geqslant 3$, but for each $n \geqslant 3$ in the annular regions $\sqrt{1 + 2/(n+1)} < r < \sqrt{1 + 2/n}$ and $\sqrt{1 - 2/n} < r < \sqrt{1 - 2/(n+1)}$ system (2.6) with $g(x) = x$ and $f(x)$ exhibited in (2.11) has no closed orbits. Otherwise, integrating along a closed

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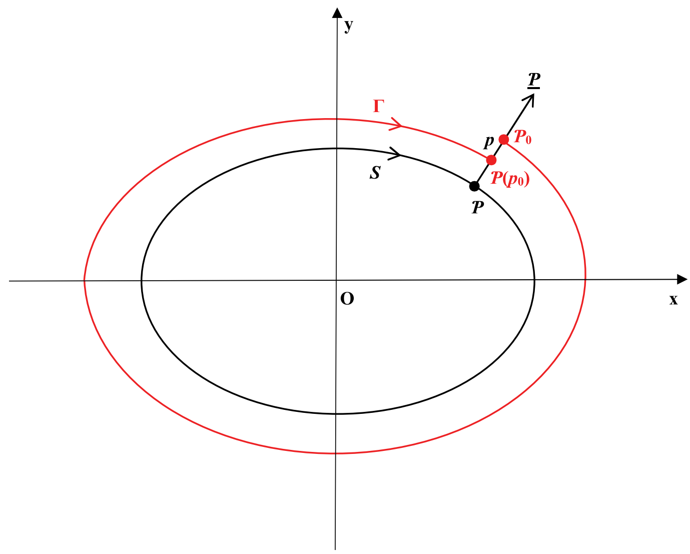

Figure 2. The curvilinear coordinates in the annulus $S(\Gamma, \varepsilon)$.

orbit γ (if exists in one of the annular region), we obtain a contradiction

$$0 = \oint_\gamma \mathrm{d}r = r(2\pi) - r(0) = \int_0^{2\pi} \frac{\mathrm{d}r}{\mathrm{d}\theta}\mathrm{d}\theta$$
$$= \int_0^{2\pi} \frac{r\sin^2\theta(r^2-1)^k \sin\frac{2\pi}{r^2-1}}{1+\sin\theta\cos\theta(r^2-1)^k\sin\frac{2\pi}{r^2-1}}\mathrm{d}\theta \neq 0$$

because $\sin\frac{2\pi}{r^2-1} \neq 0$ and $1+\sin\theta\cos\theta(r^2-1)^k\sin\frac{2\pi}{r^2-1} > 0$.

The above two examples show that compound limit cycle and limit cycle of fractional multiplicity are both possible for piecewise analytic systems, but neither of them happens in a Liénard system (2.6) with the specific piecewise analyticity (H_1).

LEMMA 2.2. *Any limit cycle of the generalized Liénard system (2.6) with hypothesis (H_1) is neither compound nor of fractional multiplicity.*

Proof. Assume that system (2.6) with hypothesis (H_1) has a limit cycle Γ, as shown in figure 2. For compound structure and fractional multiplicity, we need to consider a small annulus surrounding Γ. Since the function g is piecewise analytic on $(a_1, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup \ldots \cup (x_n, a_2)$, and functions $f(x, y, \alpha)$, $\partial f(x, y, \alpha)/\partial y$ are piecewise analytic on $D_1 \cup D_2 \cup \ldots \cup D_{n+1}$, there is a small $\varepsilon > 0$ such that each normal at any $P \in \Gamma$ restricted to the open annular neighbourhood $S(\Gamma, \varepsilon)$ of Γ with the radius ε is non-contact. Thus, we can set up curvilinear coordinates in the annulus $S(\Gamma, \varepsilon)$. Each point $\underline{P}$ in $S(\Gamma, \varepsilon)$ has a corresponding point $P \in \Gamma$ such that $\underline{P}$ lies on the normal at P.

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Note that a point on limit cycle Γ can be written as

$$(x, y) = (\varphi(s), \psi(s)),$$

where s is the arclength (parameter) from P to a specified point in the clockwise direction. Let p be the length from P to $\underline{P}$ positively in the outward direction. Thus, as shown in [**30**, Chapter 4.2], we can represent the point $\underline{P} : (x, y)$ in the curvilinear coordinates s and p in each region D_j $(j = 1, 2, \ldots, n+1)$ as

$$x = \varphi(s) - p\psi'(s), \quad y = \psi(s) + p\varphi'(s),$$

where

$$\varphi'(s) = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{X_0}{\sqrt{{X_0}^2 + {Y_0}^2}}, \quad \psi'(s) = \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{Y_0}{\sqrt{{X_0}^2 + {Y_0}^2}},$$
$$X_0 = X(\varphi(s), \psi(s), \alpha), \quad Y_0 = Y(\varphi(s), \psi(s), \alpha).$$

Therefore, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\psi'(s) + \frac{dp}{\mathrm{d}s}\varphi'(s) + p\varphi''(s)}{\varphi'(s) - \frac{dp}{\mathrm{d}s}\psi'(s) - p\psi''(s)} = \frac{Y(\varphi(s) - p\psi'(s), \psi(s) + p\varphi'(s), \alpha)}{X(\varphi(s) - p\psi'(s), \psi(s) + p\varphi'(s), \alpha)},$$

implying that

$$\frac{dp}{\mathrm{d}s} = \frac{Y\varphi' - X\psi' - p(X\varphi'' + Y\psi''(s))}{X\varphi' + Y\psi'} = F(s, p, \alpha), \tag{2.12}$$

where

$$\varphi''(s) = \frac{-Y_0}{\sqrt{{X_0}^2 + {Y_0}^2}} \left({X_0}^2 \frac{\partial Y}{\partial x}|_{p=0} + X_0 Y_0 \left(\frac{\partial Y}{\partial y}|_{p=0} - \frac{\partial X}{\partial x}|_{p=0} \right) - {Y_0}^2 \frac{\partial X}{\partial y}|_{p=0} \right),$$
$$\psi''(s) = \frac{X_0}{\sqrt{{X_0}^2 + {Y_0}^2}} \left({X_0}^2 \frac{\partial Y}{\partial x}|_{p=0} + X_0 Y_0 \left(\frac{\partial Y}{\partial y}|_{p=0} - \frac{\partial X}{\partial x}|_{p=0} \right) - {Y_0}^2 \frac{\partial X}{\partial y}|_{p=0} \right).$$

Having the above curvilinear coordinates, we first consider the case that Γ does not intersect any switching line. Let $P_0 : (0, p_0) \in \overline{P\underline{P}}$ and p_0 be the length from P to P_0 positively in the outward direction, as shown in figure 2. It follows from (2.12) that the Poincaré map $\mathcal{P}$ satisfies

$$\mathcal{P}(p_0) = p(l, p_0) = p_0 + \int_0^l F(s, p(s, p_0), \alpha)\mathrm{d}s,$$

where l is the total arc length of Γ.

Second, consider the case that Γ intersects only one switching line $x = x_j$. It is clear that Γ is divided by $x = x_j$ into two parts: left part and right part. Consider $P_0 : (0, p_0)$ to be the initial point of the Poincaré map on $x = x_j$, and let (l_1, p_1) (resp. (l_2, p_2)) denote the first intersection point of the positive-(resp. negative-) half orbit with $x = x_j$. Without loss of generality, we can assume that the segment $\overline{PP_0}$ on the switching line $x = x_j$ is vertical to Γ. Otherwise, a rotation can achieve

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this because the switching line $x = x_j$ is transversal to Γ. Then, we can obtain the two half Poincaré maps

$$\mathcal{P}_1(p_0) = p_1(l, p_0) = p_0 + \int_0^{l_1} F(s, p(s, p_0), \alpha)\mathrm{d}s$$

and

$$\mathcal{P}_2(p_0) = p_2(l, p_0) = p_0 - \int_{l_1}^{l} F(s, p(s, p_0), \alpha)\mathrm{d}s,$$

as shown in figure 3. Notice that the denominator of the right-hand side of (2.12) does not equal zero since $X^2(\varphi(s), \phi(s), \alpha) + Y^2(\varphi(s), \phi(s), \alpha) \neq 0$. Moreover, the vector field of system (2.6) is analytic for $x < x_j$ and $x > x_j$ in $S(\Gamma, \varepsilon)$. Therefore,

$$p_1(l, p_0) = p_0 + \sum_{i=1}^{\infty} a_i {p_0}^i \text{ and } p_2(l, p_0) = p_0 + \sum_{i=1}^{\infty} b_i {p_0}^i.$$

Thus, we obtain the following successive function

$$\varrho(p_0) = \mathcal{P}_1(p_0) - \mathcal{P}_2(p_0) = \sum_{i=1}^{\infty} (a_i - b_i){p_0}^i. \tag{2.13}$$

The coefficients $a_i - b_i$ in the above series have the two cases:

(i) $a_i - b_i = 0$ for each positive integer $i \in \mathbb{Z}_+$;

(ii) there exists a positive integer i_1 such that $a_{i_1} - b_{i_1} \neq 0$ and $a_j - b_j = 0$ for all $1 \leqslant j < i_1$.

In case **(i)** we have a periodic annulus and no limit cycles exist. In case **(ii)**, Γ is a limit cycle of multiplicity $i_1 \in \mathbb{Z}_+$ and there are at most i_1 zeros of $\varrho(p_0)$ near the origin by the Malgrange preparation theorem ([**6**]). Consequently, limit cycle Γ is not compound and has a multiplicity $i_1 \in \mathbb{Z}_+$ if Γ intersects only one switching line.

In addition, if the crossing limit cycle Γ intersects two or more switching lines, we can prove our result similarly as we do for the case of only one switching line. Actually, if Γ intersects n ($n > 1$) switching lines, a small neighbourhood of the crossing limit cycle Γ only possibly contains the crossing points of the switching lines. Thus, we can present the Poincaré map in $2n$ pieces, each of which is a map from a witching line to the next similar to (2.13). Since Γ intersects transversally the switching lines, each piece of the Poincaré map is well defined and close to the corresponding segment of the orbit Γ in the corresponding zone D_i between the executive switching lines. In contrast, if Γ is a grazing limit cycle, there are at most two grazing points, at which Γ intersects two switching lines at the x-axis. Therefore, any small vicinity of Γ (where Γ is not included) only possibly contains the crossing points of the switching lines and thus the successive function can be constructed similarly as we did above for the case that Γ intersects only one switching line. □

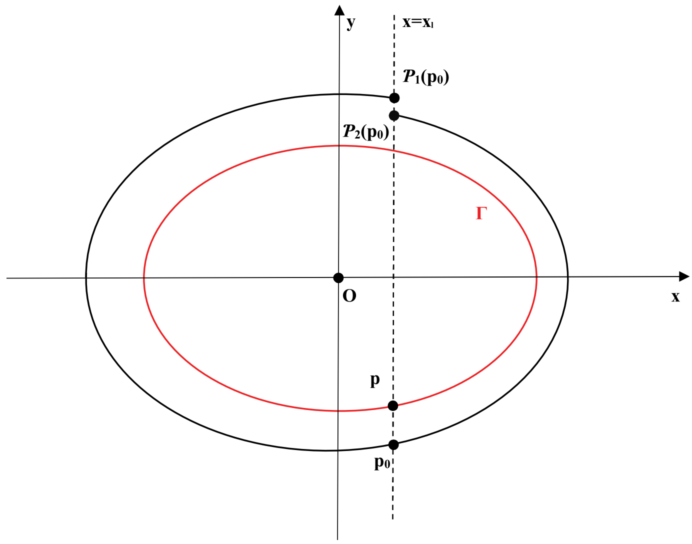

Figure 3. The successive function $\varrho(p_0) = \mathcal{P}_1(p_0) - \mathcal{P}_2(p_0)$.

Having the above lemma, we are ready to prove the following theorem, which shows that properties **(PR1)** and **(PR2)**, obtained by Perko ([**23**, **26**]) for analytic families of rotated vector field, remain true for Liénard system (2.6) with hypotheses (H_1) and (H_2). Note that system (2.6) with hypotheses (H_1) and (H_2) is rotated in Perko's definition by proposition 2.1. Additionally, we give the relation between variational exponent and multiplicity of limit cycles.

THEOREM 2.3. *Assume that system (2.6) satisfies hypotheses (H_1) and (H_2) and has a limit cycle Γ of multiplicity k for $\alpha = \alpha_0$. Then*

(a) *When k is odd, the cycle Γ still exists, denoted by Γ_α, and expands inwards (resp. outwards) monotonically as $\alpha := \alpha_0 + \varepsilon$ increases if Γ is stable (resp. unstable), where $\varepsilon > 0$ is sufficiently small. Moreover, the variational exponent of the cycle Γ_α is $\varepsilon^{2/k}$.*

(b) *When k is even, Γ splits into exact two simple limit cycles $\Gamma_\alpha^\pm$, one of which is stable but the other is unstable as the parameter α varies in one direction and Γ disappears as α varies in the opposite direction. Moreover, the outer limit cycle Γ_α^+ expands outwards and the inner one Γ_α^- expands inwards as α varies. Additionally, the variational exponent of the two cycles $\Gamma_\alpha^\pm$ is $\varepsilon^{2/k}$, where $\alpha = \alpha_0 + \varepsilon$.*

Proof. By lemma 2.2, k is a positive integer. In the following, we discuss the distance between an orbit without perturbation and the orbit under perturbation near a limit cycle. From the distance we can further investigate zeros of successive function and the relation between variational exponent and multiplicity of limit cycles.

Without loss of generality, we only discuss the case that the limit cycle Γ is externally stable as $\alpha = \alpha_0$, i.e. those points P_0, P_1 and P defined by the Poincaré

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map and the successive function ϱ (see (2.9) and figure 1) are ranked by $0 < x_1 < x_0$. Otherwise, Γ is externally unstable, for which we make a time-rescaling $t \to -t$, so that the limit cycle of the rescaled system is externally stable.

We always let $Q : (x_Q, y_Q)$ be a general point Q. Notice that the closed orbit Γ has two intersection points P_0 and Q_0 with the x-axis. Moreover, the orbit $\widehat{PQ_1P_1}$ starts from $P : (x_0, 0)$, passes through $Q_1 : (x_{Q_1}, 0)$ and returns the positive x-axis at $P_1 : (x_1, 0)$ such that $x_{Q_1} < 0 < x_{P_1} < x_P$, as shown in figure 1. We consider the perturbed system $(2.6)|_{\alpha=\alpha_0+\varepsilon}$ and let $\widehat{Q_3Q_1Q_2}$ be the orbit of system $(2.6)|_{\alpha=\alpha_0+\varepsilon}$ crossing Q_1 and having two intersection points Q_2 and Q_3 with the positive x-axis. For simplicity, let

$$x_{12} := \min\{x_1, x_{Q_2}\} \quad \text{and} \quad x_{03} := \min\{x_0, x_{Q_3}\}.$$

For $x \in (x_{Q_1}, x_{12})$, let

$$z_1 := \tilde{y}_\varepsilon(x) - \tilde{y}_0(x), \tag{2.14}$$

where $y = \tilde{y}_0(x)$ and $y = \tilde{y}_\varepsilon(x)$ represent the orbit segments $\widehat{Q_1P_1}$ and $\widehat{Q_1Q_2}$, respectively. For $x \in (x_{Q_1}, x_{03})$, let

$$z_2 := \hat{y}_\varepsilon(x) - \hat{y}_0(x), \tag{2.15}$$

where $y = \hat{y}_0(x)$ and $y = \hat{y}_\varepsilon(x)$ represent the orbit segments $\widehat{PQ_1}$ and $\widehat{Q_3Q_1}$, respectively. Moreover, let

$$P_2 = \begin{cases} (x_{12}, \tilde{y}_0(x_{12}) & \text{if } x_1 \geqslant x_{Q_2}, \\ (x_{12}, \tilde{y}_\varepsilon(x_{12}) & \text{if } x_1 < x_{Q_2}, \end{cases} \qquad P_3 = \begin{cases} (x_{03}, \hat{y}_\varepsilon(x_{12}) & \text{if } x_0 \leqslant x_{Q_3}, \\ (x_{03}, \hat{y}_0(x_{12}) & \text{if } x_0 > x_{Q_3}. \end{cases}$$

By the mean value theorem, we see from the equations $\tilde{y}_\varepsilon(x_{Q_1}) = \tilde{y}_0(x_{Q_1}) = 0$ that

$$\begin{aligned} z_1(x) &= z_1(x) - z_1(x_{Q_1}) \\ &= \{\tilde{y}_\varepsilon(\tau) - \tilde{y}_0(\tau)\} \,|_{\tau=x_{Q_1}}^{\tau=x} \\ &= \int_{x_{Q_1}}^{x} \left(\frac{-g(\tau) - f(\tau, \tilde{y}_\varepsilon(\tau), \alpha+\varepsilon)\tilde{y}_\varepsilon(\tau)}{\tilde{y}_\varepsilon(\tau)} - \right. \\ &\quad \left. \frac{-g(\tau) - f(\tau, \tilde{y}_0(\tau), \alpha)\tilde{y}_0(\tau)}{\tilde{y}_0(\tau)} \right) \mathrm{d}\tau \\ &= \int_{x_{Q_1}}^{x} \left(\frac{-g(\tau)}{\tilde{y}_\varepsilon(\tau)} + \frac{g(\tau)}{\tilde{y}_0(\tau)} - f(\tau, \tilde{y}_\varepsilon(\tau), \alpha+\varepsilon) + f(\tau, \tilde{y}_0(\tau), \alpha+\varepsilon) \right. \\ &\quad \left. - f(\tau, \tilde{y}_0(\tau), \alpha+\varepsilon) + f(\tau, \tilde{y}_0(\tau), \alpha) \right) \mathrm{d}\tau \end{aligned}$$

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$$
\begin{aligned}
&= \int_{x_{Q_1}}^{x} \left(\frac{-g(\tau)}{\tilde{y}_\varepsilon(\tau)} + \frac{g(\tau)}{\tilde{y}_0(\tau)} - \frac{\partial f(\tau, \tilde{y}^*(\tau), \alpha+\varepsilon)}{\partial y} z_1(\tau) \right. \\
&\qquad \left. - \frac{\partial f(\tau, \tilde{y}_0(\tau), \alpha+\varepsilon_1)}{\partial \alpha}\varepsilon \right) \mathrm{d}\tau \\
&= h_1(x) + \int_{x_{Q_1}}^{x} z_1(\tau) h_2(\tau) \mathrm{d}\tau,
\end{aligned} \tag{2.16}
$$

where $\tilde{y}^*$ lies between $\tilde{y}_0$ and $\tilde{y}_\varepsilon$, ε_1 lies between 0 and ε, and

$$
h_1(x) := -\varepsilon \int_{x_{Q_1}}^{x} \frac{\partial f(\tau, \tilde{y}_0(\tau), \alpha+\varepsilon_1)}{\partial \alpha} \mathrm{d}\tau, \quad h_2(x) := \frac{g(x)}{\tilde{y}_0(x)\tilde{y}_\varepsilon(x)} - \frac{\partial f(\tau, \tilde{y}^*(\tau), \alpha+\varepsilon)}{\partial y}.
$$

By (2.16),

$$
h_2(x) z_1(x) = h_2(x) h_1(x) + h_2(x) \int_{x_{Q_1}}^{x} z_1(\tau) h_2(\tau) \mathrm{d}\tau,
$$

indicating that the function $h_3(x) := \int_{x_{Q_1}}^{x} z_1(\tau)h_2(\tau)\mathrm{d}\tau$ satisfies

$$
\frac{dh_3(x)}{\mathrm{d}x} = z_1(x) h_2(x) = h_2(x) h_3(x) + h_1(x) h_2(x). \tag{2.17}
$$

By the variation of constant formula, we get from (2.17) that

$$
h_3(x) = \int_{x_{Q_1}}^{x} h_1(\tau) h_2(\tau) \exp\left\{ \int_{\tau}^{x} h_2(\eta) \mathrm{d}\eta \right\} \mathrm{d}\tau. \tag{2.18}
$$

Since $h_1(x_{Q_1}) = 0$, we see from (2.7), (2.16) and (2.18) that for all $x \in (x_{Q_1}, x_{12})$

$$
\begin{aligned}
z_1(x) &= h_1(x) + \int_{x_{Q_1}}^{x} h_1(\tau) h_2(\tau) \exp\left\{ \int_{\tau}^{x} h_2(\eta) \mathrm{d}\eta \right\} \mathrm{d}\tau \\
&= h_1(x) - \int_{x_{Q_1}}^{x} h_1(\tau)\, d\left(\exp\left\{ \int_{\tau}^{x} h_2(\eta) \mathrm{d}\eta \right\} \right) \\
&= h_1(x_{Q_1}) \exp\left\{ \int_{x_{Q_1}}^{x} h_2(\eta) \mathrm{d}\eta \right\} + \int_{x_{Q_1}}^{x} h_1'(\tau) \exp\left\{ \int_{\tau}^{x} h_2(\eta) \mathrm{d}\eta \right\} \mathrm{d}\tau \\
&= -\varepsilon \int_{x_{Q_1}}^{x} \frac{\partial f(\tau, \tilde{y}_0(\tau), \alpha+\varepsilon_1)}{\partial \alpha} \exp\left\{ \int_{\tau}^{x} h_2(\eta) \mathrm{d}\eta \right\} \mathrm{d}\tau < 0.
\end{aligned} \tag{2.19}
$$

Similarly, for all $x \in (x_{Q_1}, x_{03})$ we can obtain

$$
z_2(x) = -\varepsilon \int_{x_{Q_1}}^{x} \frac{\partial f(\tau, \hat{y}_0(\tau), \alpha+\varepsilon_2)}{\partial \alpha} \exp\left\{ \int_{\tau}^{x} h_4(\eta) \mathrm{d}\eta \right\} \mathrm{d}\tau < 0, \tag{2.20}
$$

where $\varepsilon_2 \in (0, \varepsilon)$ and

$$h_4(x) := \frac{g(x)}{\hat{y}_0(x)\hat{y}_\varepsilon(x)} - \frac{\partial f(\tau, \hat{y}^*(\tau), \alpha+\varepsilon)}{\partial y}$$

for $\hat{y}^*$ lying between $\hat{y}_0$ and $\hat{y}_\varepsilon$.

Construct an energy function

$$E(x, y) = \int_0^x g(s)\mathrm{d}s + \frac{y^2}{2}. \tag{2.21}$$

Then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -f(x, y, \alpha)y^2 \tag{2.22}$$

restricted on system (2.6). From (2.19) and (2.20) we see that the two points P_2 and P_3 lie on the two orbit segments $\widehat{Q_1P_1}$ and $\widehat{Q_3Q_1}$ respectively, where $x_{P_2} = x_{Q_2}$, $x_{P_3} = x_P$, $y_{P_2} > 0$ and $y_{P_3} < 0$, as shown in figure 1. From (2.19)–(2.22), and $\int_{\widehat{P_2P_1}} \mathrm{d}E = E(P_1) - E(P_2)$, we obtain

$$\begin{aligned}\int_{-z_1(x_{Q_2})}^{0} \frac{f(x,y,\alpha)y^2}{g(x)+f(x,y,\alpha)y}\mathrm{d}y &= \frac{y_{P_1}^2}{2} + \int_0^{x_1} g(s)\mathrm{d}s - \frac{y_{P_2}^2}{2} - \int_0^{x_{Q_2}} g(s)\mathrm{d}s \\ &= \int_{x_{Q_2}}^{x_1} g(s)\mathrm{d}s - \frac{z_1^2(x_{Q_2})}{2},\end{aligned}$$

where $z_1(\cdot)$ is defined in (2.14). Thus, there exist $x^* \in [x_{Q_2}, x_1]$ and $y^* \in [0, -z_1(x_{Q_2})]$ such that

$$\begin{aligned}\int_{-z_1(x_{Q_2})}^{0} \frac{f(x,y,\alpha)y^2}{g(x)+f(x,y,\alpha)y}\mathrm{d}y &= \frac{f(x,y^*,\alpha)(y^*)^2}{g(x)+f(x,y^*,\alpha)y^*} z_1(x_{Q_2}) \\ &= (x_1 - x_{Q_2})g(x^*) - \frac{z_1^2(x_{Q_2})}{2}\end{aligned}$$

by the mean value theorem for integrals. Thus,

$$x_1 - x_{Q_2} = \frac{z_1^2(x_{Q_2})}{2g(x_0)} + o(\varepsilon^2) \tag{2.23}$$

because $g(x^*) = g(x_0) + O(\varepsilon)$. From (2.19)–(2.22) and the equality

$$\int_{\widehat{Q_3P_3}} \mathrm{d}E = E(P_3) - E(Q_3),$$

we obtain

$$\begin{aligned}-\int_{z_2(x_0)}^{0} \frac{f(x,y,\alpha)y^2}{g(x)+f(x,y,\alpha)y}\mathrm{d}y &= \frac{y_{P_3}^2}{2} + \int_0^{x_0} g(s)\mathrm{d}s - \frac{y_{Q_3}^2}{2} - \int_0^{x_{Q_3}} g(s)\mathrm{d}s \\ &= \int_{x_{Q_3}}^{x_0} g(s)\mathrm{d}s + \frac{{z_2}^2(x_0)}{2},\end{aligned} \tag{2.24}$$

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where $z_2(\cdot)$ is defined in (2.15). It follows from equality (2.24) that

$$x_{Q_3} - x_0 = \frac{z_2^2(x_0)}{2g(x_0)} + o(\varepsilon^2). \tag{2.25}$$

On the other hand, Γ is a limit cycle of multiple k, i.e.

$$x_0 - x_1 = -a_k {x_1}^k + o({x_1}^k), \tag{2.26}$$

as seen in the definition of a limit cycle of multiple k in [**30**, Chapter 4.2]. Moreover, Γ is externally stable, i.e. $a_k < 0$. It follows from (2.23)–(2.26) that

$$x_{Q_3} - x_{Q_2} = \frac{z_1^2(x_{Q_2})}{2g(x_0)} + \frac{z_2^2(x_0)}{2g(x_0)} + o(\varepsilon^2) - a_k {x_1}^k + o({x_1}^k). \tag{2.27}$$

Since $\dot{y} = -g(x) < 0$ on both the positive x-axis and the negative x-axis near the origin, we see that $g(x_0) > 0$.

Consider the case that k is odd, i.e. Γ is stable in both internal and external neighbourhoods of Γ. In the external neighbourhood of Γ, we can obtain that $x_{Q_3} - x_{Q_2} > 0$ by (2.27) and the inequalities $a_k < 0$, $g(x_0) > 0$ and $x_1 > 0$, implying that no limit cycles exist in the external neighbourhood of Γ when α increases. In the internal neighbourhood of Γ, we can also obtain equality (2.27) and the inequalities $a_k < 0$, $g(x_0) > 0$ but $x_1 < 0$ from a similar discussion to the external neighbourhood of Γ. Since the solution of system (2.6) is Lipschitzian, the implicit function theorem is applicable. Thus, from (2.27) with $X_1 := {x_1}^k$, we see that equality $x_{Q_3} - x_{Q_2} = 0$ has a unique root $X_1 = (z_1^2(x_{Q_2}) + z_2^2(x_0))/(2a_k g(x_0)) + o(\varepsilon^2) < 0$, which is equivalent to

$$x_1 = \left(\frac{z_1^2(x_{Q_2}) + z_2^2(x_0)}{2a_k g(x_0)}\right)^{1/k} + o(\varepsilon^{2/k}) < 0. \tag{2.28}$$

It implies that system $(2.6)|_{\alpha=\alpha_0+\varepsilon}$ produces a stable limit cycle Γ_α in the internal neighbourhood of Γ, where the bifurcated limit cycle Γ_α passes through the point $(x_1, 0)$. Moreover, the stable limit cycle expands inwards (or contracts) monotonically as α increases. From (2.8) and (2.28), we can obtain

$$\Delta_\alpha(\varepsilon) := x_1 - 0 = x_1,$$

implying that the variational exponent is $2/k$. This proves the results of statement **(a)**.

In the case that k is even, i.e. Γ is semi-stable (externally stable but internally unstable), we obtain equality (2.27), $a_k < 0$ and $g(x_0) > 0$ in the neighbourhood of Γ, implying that $x_{Q_3} - x_{Q_2} > 0$ near the origin and then system (2.6) has no limit cycles in a neighbourhood of Γ as α increases.

In order to completely investigate limit cycles bifurcating from semi-stable Γ for even k, we consider the case that $\alpha := \alpha_0 - \varepsilon$ decreases. By a similar calculation to

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(2.27), we obtain

$$x_{Q_3} - x_{Q_2} = -\frac{z_1^2(x_{P_2})}{2g(x_0)} - \frac{z_2^2(x_{Q_3})}{2g(x_0)} + o(\varepsilon^2) - a_k {x_1}^k + o({x_1}^k), \tag{2.29}$$

where $g(x_0) > 0$ and $a_k < 0$. Let $k := 2n$ for $n \in \mathbb{Z}_+$ and $X_2 := {x_1}^n$. By the implicit function theorem and (2.29), the equality $x_{Q_3} - x_{Q_2} = 0$ has exactly two roots $X_2 = \pm\sqrt{-(z_1^2(x_{P_2}) + z_2^2(x_{Q_3}))/(2a_k g(x_0))} + o(\varepsilon^2)$, which is equivalent to

$$x_1 = \pm\left(-\frac{z_1^2(x_{P_2}) + z_2^2(x_{Q_3})}{2a_k g(x_0)}\right)^{1/k} + o(\varepsilon^{2/k}). \tag{2.30}$$

It follows that two simple limit cycles $\Gamma_\alpha^\pm$ of system $(2.6)|_{\alpha=\alpha_0-\varepsilon}$ exist in a neighbourhood of Γ, and the outer limit cycle Γ_α^+ is stable but the inner one Γ_α^- is unstable. Moreover, Γ_α^+ expands outwards monotonically and Γ_α^- expands inwards monotonically as α decreases. From (2.8) and (2.30), we can obtain $\Delta_\alpha(\varepsilon) := x_1 - 0 = x_1$, implying that the variational exponent is $2/k$. This proves the results of statement **(b)**.

In conclusion, for odd k system (2.6) produces a stable (resp. unstable) limit cycle Γ_α when Γ is stable (resp. unstable) in an internal (resp. external) neighbourhood of Γ_α as α increases. On the other hand, for even k the externally stable and internally unstable (resp. externally stable and internally unstable) Γ splits into exact two simple limit cycles $\Gamma_\alpha^\pm$. Moreover, the outer limit cycle Γ_α^+ is stable (resp. unstable) and expands outwards, but the inner one Γ_α^- is unstable (resp. stable) and expands inwards as α decreases (resp. increases). However, the limit cycle Γ disappears as α varies in the opposite direction. Furthermore, the variational exponent of the new limit cycle is $\varepsilon^{2/k}$. □

Theorem 2.3 shows how the limit cycle expands or bifurcates as the rotated parameter α varies, where the stable (or unstable) limit cycle may be hyperbolic or non-hyperbolic. Clearly, the results of theorem 2.3 are true for analytic rotated Liénard systems. Additionally, theorem 2.3 gives variational exponents to show the expanding rates of the limit cycles depending on the rotated parameter α, which was not discussed yet even for analytic rotated vector fields.

3. Rotated Hopf bifurcation and rotated homoclinic bifurcation

In this section, we introduce results on homoclinic loops and Hopf bifurcation of the one-parameter family of rotated vector fields (2.6) with hypotheses (H_1) and (H_2).

THEOREM 3.1. *Properties* **(PR3)** *and* **(PR4)** *in §1 are still true for system (2.6) with hypotheses (H_1) and (H_2).*

Proof. First, we prove property **(PR3)**. Without loss of generality, assume that the origin of system (2.6) is a weak focus as $\alpha = \alpha_0$ and the weak focus is stable, as shown in figure 4. When the vector field of system (2.6) is analytic in a small

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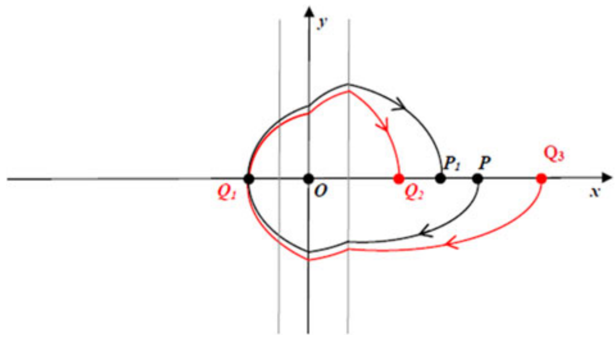

Figure 4. Poincaré map near O in system (2.6).

neighbourhood of the origin, the conclusion holds directly by [**23**]. Since the origin is a weak focus, we obtain that

$$\mathcal{P}(x) - x = a_k x^k + o(x^k)$$

on the x-axis, where $\mathcal{P}(x)$ is the Poincaré map, k is a positive integer and $x > 0$ is small. As proven in theorem 2.3, (2.27) still holds, where P, P_1, Q_1, Q_2, Q_3, $x_0, x_1, x_{Q_1}, x_{Q_2}, x_{Q_3}$ are defined similarly in figure 1 and the proof of theorem 2.3, as shown in figure 4. Then, $a_k < 0$ since the weak focus is stable. For positive integer k and $\alpha = \alpha_0 - \varepsilon$, the equality $x_{Q_3} - x_{Q_2} = 0$ has a unique positive zero

$$x_1 = \left(-\frac{z_1^2(x_{P_2}) + z_2^2(x_{Q_3})}{2a_k g(x_0)} \right)^{1/k} + o(\varepsilon^{2/k}) > 0,$$

where $\varepsilon > 0$ is small and $g(x_0) > 0$. When $\alpha = \alpha_0 + \varepsilon$, for even k the equality $x_{Q_3} - x_{Q_2} = 0$ has no positive zeros. Thus, system $(2.6)|_{\alpha=\alpha_0-\varepsilon}$ produces a new stable limit cycle in a small neighbourhood of the origin. Thus, property **(PR3)** still holds.

Next, we prove property **(PR4)**. Assume that system (2.6) exhibits a homoclinic loop Γ_0 as $\alpha = \alpha_0$. Without loss of generality, we consider that the saddle in the homoclinic loop is the origin and the homoclinic loop is stable. By the sign of the vector field $(y, -g(x) - f(x, y, \alpha)y)$ near the saddle, in a small neighbourhood of the origin one side of the stable manifold and one side of the unstable manifold of the saddle lie in the left-half plane, but the other sides of the two manifolds lie in the right-half plane. Assume that Γ_0 intersects the positive (or negative) y-axis and surrounds neither one stable manifold nor one unstable manifold of the origin other than those in Γ_0, as shown in figures 5a, b. Notice that $\dot{x} = y > 0$ in the positive y-axis and $\dot{x} = y < 0$ in the negative y-axis. However, in figures 5a, b the sign of $\dot{x}$ at the intersection point Γ_0 and the y-axis is opposite by the location of the stable and unstable manifolds of Γ_0. Thus, system (2.6) has no homoclinic loops which intersect the positive (or negative) y-axis and do not surround the other stable and unstable manifolds. In other words, if system (2.6) has a homoclinic loop Γ_0 which does not surround one stable and one unstable manifolds of O other than those in Γ_0, then Γ_0 lies in the left-half (or right-half) plane. It is similar to prove

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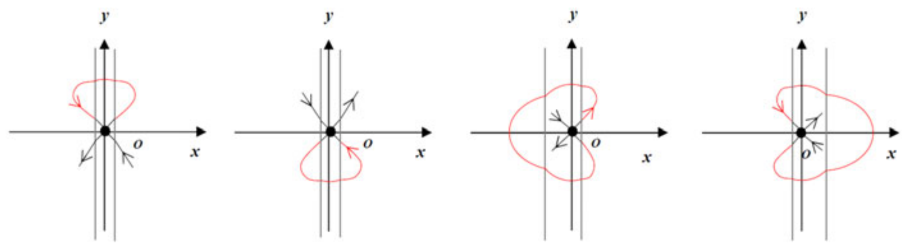

Figure 5. Impossible cases of homoclinic loops for system (2.6).

that system (2.6) has no homoclinic loops which intersect the positive and negative y-axes, and surround the other two manifolds, as shown in figures 5c, d. Therefore, it is sufficient to prove that the homoclinic loop has one of the configurations shown in figure 6. Otherwise, we apply the transformation $(x, y) \to (-x, -y)$. The case offigure 6b can be treated by a similar mean to the case of figure 6a, where the only difference is that the Poincaré map need to be considered in the outer neighbourhood of the homoclinic loop for the case of figure 6b. Therefore, we only need to discuss the case of figure 6a. Now, consider $\delta_0 := -f(0, 0, \alpha_0) \neq 0$. In other words, the sum of the two eigenvalues of the hyperbolic saddle is not equal to zero. Let

$$\delta := -\frac{\partial f(\varphi(t), \phi(t), \alpha_0)}{\partial y}\phi(t) - f(\varphi(t), \phi(t), \alpha_0),$$

where $(x, y) = (\varphi(t), \phi(t))$ represents the homoclinic loop. Consider the perturbed system of (2.6) for $\alpha = \alpha_0 + \varepsilon$ with sufficiently small $|\varepsilon|$. By Theorem 3.7 of [**7**, Chapter 3], we have the following conclusions:

(a) There is exactly one limit cycle bifurcating from the homoclinic loop of system (2.6) when $\delta_0\varepsilon\Delta > 0$ (resp. < 0), which is stable for $\delta_0 < 0$ and unstable for $\delta_0 > 0$, where

$$\Delta := -\int_{-\infty}^{+\infty} e^{-\int_0^t \delta(s)\mathrm{d}s}\frac{\partial f(\varphi(t), \phi(t), \alpha)}{\partial \alpha}|_{\alpha=\alpha_0}\phi^2(t)\mathrm{d}t.$$

(b) There are no limit cycles in a small neighbourhood of the homoclinic loop of system (2.6) when $\delta_0\varepsilon\Delta < 0$ (resp. > 0).

We consider the case $\delta_0 = 0$. In this case we cannot apply [**7**, Theorem 3.7 of Chapter 3] directly, which only considers the case $\delta_0 \neq 0$. We can obtain the Poincaré map again

$$\mathcal{P}(x) - x = a_k x^k + o(x^k)$$

on the x-axis, defined in an interior neighbourhood of any homoclinic loop, where k is a positive integer. As proven in theorem 2.3, (2.27) still holds, where P, P_0, P_1, Q_1, Q_2, Q_3, $x_0, x_1, x_{Q_2}, x_{Q_3}$ are defined similarly in the proof of theorem 2.3, as

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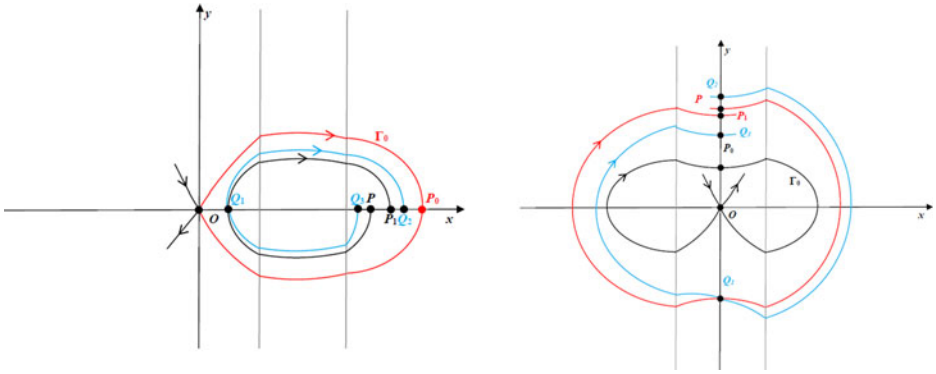

(a) (b)

Figure 6. Classification of homoclinic loops for system (2.6).

shown in figure 6. Then, $a_k > 0$ since the homoclinic loop is stable. For $\alpha = \alpha_0 + \varepsilon$ and positive integer k, the equality $x_{Q_3} - x_{Q_2} = 0$ has a unique zero

$$x_1 = \tilde{x}_0 - \left(\frac{z_1^2(x_{Q_2}) + z_2^2(x_0)}{2a_k g(x_0)} \right)^{1/k} + o(\varepsilon^{2/k}) < \tilde{x}_0,$$

where $\tilde{x}_0$ is the abscissa of P_0 and $\varepsilon > 0$ is small. When $\alpha = \alpha_0 - \varepsilon$, for even k and small $\varepsilon_1 > 0$ the equality $x_{Q_3} - x_{Q_2} = 0$ has no zeros in $(\tilde{x}_0 - \varepsilon_1, \tilde{x}_0)$. Thus, the proof is completed. □

Note that theorems 2.3 and 3.1 give positive answers to questions **(Q1)** and **(Q2)** mentioned in the Introduction.

The following assumptions are needed for studying singular closed orbits (homoclinic loops or heteroclinic loops).

(H_{a}) Assume that system (1.1) defines a family of analytic rotated vector fields. Let system (1.1) have a singular closed orbit $L_0 \subset \bar{D}$ for some $\lambda_0 \in I$ such that the Poincaré map is well defined on one side of L_0, where $\bar{D}$ is the closure of D.

For singular closed orbits in a family of analytic rotated vector fields, it was proved by Han in [**14**, Theorem 2.4] that

(i) if L_0 is non-isolated, system (1.1) has no closed orbits in a small neighbourhood of L_0 for $\lambda \in I \backslash \{\lambda_0\}$, and

(ii) if L_0 is isolated, system (1.1) has at least one limit cycle near L_0 as λ varies in a suitable sense but no closed orbits near L_0 as λ varies in the opposite sense.

Based on result (ii), a conjecture follows.

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Conjecture of [14]: *There is at most one limit cycle near L_0 for $\lambda \in I$ satisfying $0 < |\lambda - \lambda_0| \ll 1$ under the conditions of [14, Theorem 2.4], i.e. under the assumptions (H_a) for the vector field of (1.1).*

The following theorem, indicating that two limit cycles can be bifurcated from a cuspidal loop, gives a negative answer to the above conjecture.

THEOREM 3.2. *Assume that system (2.6) with hypotheses (H_1) and (H_2) has a cuspidal loop Γ for $\alpha = \alpha_0$ and the cusp persists if $|\alpha - \alpha_0| \ll 1$. Moreover, the vector field of system (2.6) is analytic in a small neighbourhood of the cusp. Then, we have the following conclusions:*

- **(i)** *If Γ is of odd multiplicity, system (2.6) has a unique limit cycle near Γ for $\alpha \in I$ with $0 < |\alpha - \alpha_0| \ll 1$.*
- **(ii)** *If Γ is of even multiplicity, system (2.6) has exactly two limit cycles near Γ as α varies in a suitable sense and has no closed orbits near Γ as α varies in the opposite sense.*

Proof. Without loss of generality, we can assume that the cusp is located at $(0, 0)$. Since the vector field of system (2.6) at the cusp is analytic, in a small neighbourhood of $(0, 0)$ system (2.6) can be rewritten as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a_k x^k\big(1 + p_1(x)\big) + b_n x^n y\big(1 + p_2(x)\big) + y^2 p_3(x, y) \\ &\quad + (\alpha - \alpha_0) y^l p_4(x, y, \alpha) \end{aligned} \tag{3.1}$$

by [11, Chapter 3] or [30, Chapter 2], where integer $k \geqslant 2$, integers n, l are positive, and functions $p_1(x)$, $p_2(x)$, $p_3(x, y)$ and $p_4(x, y, \alpha)$ are analytic such that $p_1(0) = 0$, $p_2(0) = 0$ and $y^l p_4(x, y, \alpha) \geqslant 0$. By Theorem 3.5 of [11, Chapter 3] or Theorem 7.3 of [30, Chapter 2], the origin $(0, 0)$ of system (3.1) is a cusp when one of the following statements holds:

(a) $k = 2m$ ($m \geqslant 1$), $l \geqslant 2$ and $b_n = 0$.

(b) $k = 2m$ ($m \geqslant 1$), $n \geqslant m$, $l \geqslant 2$ and $b_n \neq 0$.

Therefore, according to the conditions of this theorem, we can define the Poincaré map on both sides of Γ_0 when $|\alpha - \alpha_0| \ll 1$. Moreover, system (3.1) is of the form of system (2.6), the remainder research is similar to the proof of theorem 2.3 and we omit it. □

Remark that in theorem 3.2 we provide a class of singular closed orbits in a family of rotated vector fields for system (1.1) and two limit cycles can be bifurcated from the singular closed orbit.

By theorems 2.3–3.2, we have the following further result.

PROPOSITION 3.3. *When system (2.6) has a singular closed orbit $L_0 \subset \bar{D}$ for some $\lambda_0 \in I$ such that the Poincaré map is well defined on only one side of L_0, conjecture of [14] is correct. Furthermore, when system (2.6) has a singular closed orbit $L_0 \subset$*

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*$\bar{D}$ with even multiplicity for some $\lambda_0 \in I$ such that the Poincaré map is well defined on both sides of L_0, there can exist more than one limit cycle near L_0 for $0 < |\lambda - \lambda_0| \ll 1$, i.e. a negative answer to the conjecture of [**14**].*

For a specific example to proposition 3.3, consider the system

$$\dot{x} = y, \quad \dot{y} = -x^2(x+1) + \delta\left(\alpha + \beta x + x^2\right) y, \tag{3.2}$$

where δ is a perturbation parameter. For $\delta = 0$ system (3.2) is Hamiltonian and has a cuspidal loop Γ_0. As indicated in the main theorem on page 211 of [**10**], limit cycles of system (3.2) have been researched by computing Abelian integrals. Let $\alpha_1 := \delta\alpha$. Then the vector field of (3.2) is rotated with the rotated parameter α_1. Thus, system (3.2) can have two limit cycles near Γ_0 with even multiplicity for $0 < |\alpha_1|$ as the rotated parameter α_1 varies and (α_1, β) is near the point $C_2 : (0, 14/15)$.

4. Application to SD oscillator

The main results in § 2 can be applied to the problem of limit cycles for the SD oscillator. In other words, we will use our theorems to give a positive answer to question **(Q3)**, mentioned in the Introduction for this differential system.

The authors of [**4**] studied the global bifurcation diagram and all phase portraits of the SD oscillator

$$\dot{x} = y - \xi(bx + x^3), \quad \dot{y} = -x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right), \tag{4.1}$$

where $(a, b, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$. It is shown in [**4**] that system (4.1) has three equilibria

$$E_L : (-\sqrt{1-a^2}, (a^2 - 1 - b)\xi\sqrt{1-a^2}), \ E_0 : (0, 0),$$
$$E_R : (\sqrt{1-a^2}, (1 - a^2 + b)\xi\sqrt{1-a^2})$$

if $a < 1$, and only E_0 exists if $a \geqslant 1$. *Large limit cycles* represent the ones surrounding all three equilibria and *small limit cycles* the ones surrounding a single equilibrium. However, the maximum number of small limit cycles of system (4.1) are not proven as parameters belong to the region

$$\mathcal{G} := \Big\{(a, b, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ : 0 < a < \sqrt{3}/3, \ 2\sqrt{3}a - 4 < b < \min\{\varphi(a, \xi), 3a^2 - 3\}\Big\},$$

where $b = \varphi(a, \xi)$ is the homoclinic bifurcation surface and $2\sqrt{3}a - 4 < \varphi(a, \xi) < a^2 - 1$. By the symmetry of system (4.1), we only need to study the maximum number of small limit cycles surrounding E_R. Moreover, the authors of [**4**] conjectured that system (4.1) has at most two small limit cycles surrounding E_R. In other words, the bifurcation surface of double limit cycle can be expressed as a

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function $b = \phi(a, \xi)$, continuous in $a \in (0, \sqrt{3}/3)$, such that $2\sqrt{3}a - 4 < \phi(a, \xi) < \min\{\varphi(a, \xi), 3a^2 - 3\}$, $\phi(1/\sqrt{3}, \xi) = -2$ and $\phi(0, \xi) = \varphi(0, \xi)$.

Recently, Liu and Sun ([**27**]) proved that the conjecture is true in $\mathcal{G}$ for small ξ by computing Abelian integrals. However, the approach of Abelian integrals is not available for general (large) ξ because system (4.1) is no longer near-Hamiltonian. In this paper, we use theorem 2.3 to prove that at most two small limit cycles surround equilibrium E_R of system (4.1) as parameters lie in the region $\mathcal{G}$ for general ξ.

THEOREM 4.1. *System (4.1) has at most two small limit cycles surrounding E_R for $(a, b, \xi) \in \mathcal{G}$ with general ξ. Specially, a path of a small semi-stable limit cycle can be presented by a function $b = \phi(a, \xi)$, where $0 < a < \sqrt{3}/3$, $2\sqrt{3}a - 4 < \phi(a, \xi) < \min\{\varphi(a, \xi), 3a^2 - 3\}$, $\phi(1/\sqrt{3}, \xi) = -2$ and $\phi(0, \xi) = \varphi(0, \xi)$.*

Proof. Let $\tilde{b} = b\xi$ and take $(a, \tilde{b}, \xi)$ as new parameter. With the transformation $(x, y) \to (x, y + \tilde{b}x + \xi x^3)$, system (4.1) can be changed into the following form

$$\dot{x} = y, \quad \dot{y} = -x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) - (\tilde{b} + 3\xi x^2)y. \tag{4.2}$$

We can check that $(y, -x(1 - 1/\sqrt{x^2 + a^2}) - (\tilde{b} + 3\xi x^2)y)$ is a rotated vector field with respect to $\tilde{b}$ and ξ. Moreover, the parameter region $\mathcal{G}$ can be changed into

$$\mathcal{G}_1 := \left\{(a, \tilde{b}, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ : 0 < a < \frac{\sqrt{3}}{3}, (2\sqrt{3}a - 4)\xi \right.$$
$$\left. < \tilde{b} < \min\{\xi\varphi(a, \xi), (3a^2 - 3)\xi\}\right\}.$$

By theorem 2.3, unstable limit cycles of system (4.2) expand and stable limit cycles contract as either $\tilde{b}$ or ξ increases. Assume that system (4.2) has at least three limit cycles Γ_1, Γ_2 and Γ_3 surrounding E_R only. Without loss of generality, we assume that Γ_1, Γ_2 and Γ_3 are innermost limit cycles and Γ_i lies in the interior region surrounded by Γ_{i+1} for $i = 1, 2$.

In the following we prove by reduction to absurdity that at most two small limit cycles surround E_R only as the parameter belongs to region $\mathcal{G}_1$. First, we exhibit the idea of the proof. We assume that there may exist three small limit cycles surrounding E_R only as the parameter belongs to region $\mathcal{G}_1$. Then we enlarge or lessen those rotated parameters $\tilde{b}$ and ξ in order, and the parameters lie in a region at last for such case the number of limit cycles has been obtained which is smaller than three. Thus, it induces a contradiction.

By the instability of E_R, we can consider the case that Γ_1 and Γ_3 are stable and Γ_2 is unstable. Otherwise, we will vary the rotated parameters to obtain such case. Now we go on the following steps:

Step 1: Lessen $\xi = \xi_1 < \xi_0$ till either Γ_1 and Γ_2 coincidence or Γ_3 becomes a homoclinic loop or Γ_3 becomes a semi-stable limit cycle for fixed $a = a_0$ and $\tilde{b} = \tilde{b}_0$. We claim that $(a, \tilde{b}, \xi) = (a_0, \tilde{b}_0, \xi_1) \in \mathcal{G}_1$. Otherwise, system (4.1) has at least three small limit cycles surrounding E_R as parameters lie in the other region except $\mathcal{G}_1$ by the rotated properties of vector fields. By theorem 1 of [**4**], this is a contradiction.

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We also claim that ξ_1 is not small. By the results of [**27**], system (4.1) has at most two small limit cycles surrounding E_R for small ξ_1, also implying a contradiction.

Step 2: Enlarge $\tilde{b} = \tilde{b}_1 > \tilde{b}_0$. We can get three limit cycles, which are still denoted by Γ_1, Γ_2 and Γ_3 by the rotated properties of vector fields. We continue to enlarge $\tilde{b}$ till Γ_2 and Γ_3 coincide. In other words, we get a stable Γ_1 and a semi-stable $\tilde{\Gamma}_{23}$. Moreover, $\tilde{\Gamma}_{23}$ is internally unstable and externally stable. We claim that $(a, \tilde{b}, \xi) = (a_0, \tilde{b}_1, \xi_1) \in \mathcal{G}_1$ and $\tilde{b}_1$ is not small. Otherwise, we can similarly obtain a contradiction as tep 1.

Step 3: Repeat step 1, i.e. lessen $\xi = \xi_2 < \xi_1$ till either Γ_1 and Γ_2 coincide or Γ_3 becomes a homoclinic loop or Γ_3 becomes a semi-stable limit cycle for fixed $a = a_0$ and $\tilde{b} = \tilde{b}_1$. We also claim that $(a, \tilde{b}, \xi) = (a_0, \tilde{b}_1, \xi_2) \in \mathcal{G}_1$ and ξ_2 is not small. Otherwise, we can obtain the similar contradiction as step 1. Then repeat step 2, step 1, step 2,..., up to n times.

On the other hand, by the proof of theorem 2.3, both the variations of $\tilde{b}$ and ξ in the aforementioned steps are not sufficiently small (i.e. there exists a positive d_0 such that the variations are larger than d_0) because none of distances between any two of given Γ_1, Γ_2 and Γ_3 is sufficiently small. Thus, we stop the aforementioned process in finitely many steps when either ξ is sufficiently close to 0 or $\tilde{b} \geqslant \min\{\xi\varphi(a, \xi), (3a^2 - 3)\xi\}$. Then there exists $n_0 \in \mathbb{Z}_+$ such that the number of limit cycles is larger than 2 as $(a, \tilde{b}, \xi) = (a_0, \tilde{b}_n, \xi_n)$ for $n > n_0$, contradicting the result 'a unique limit cycle' for $\tilde{b} = \min\{\xi\varphi(a, \xi), (3a^2 - 3)\xi\}$ and the result 'at most 2 limit cycles' for small ξ. This is a contradiction. Therefore, there are at most two small limit cycles only surrounding E_R as the parameter belongs to region $\mathcal{G}$. By the instability of E_0, the interior small limit cycle is stable and the outer one is unstable respectively if there exist exactly two small limit cycles surrounding E_R. □

Remark that the global dynamics of system (4.1) can be presented completely by [**4**, Theorem 1], results of [**27**] and theorem 4.1.

5. Application to switching system with three zones

The main results in § 2 can also be applied to the problem of limit cycles for piecewise linear differential system with three zones and asymmetry. Consider a piecewise linear differential system

$$\dot{x} = F(x) - y, \qquad \dot{y} = g(x) \tag{5.1}$$

with two parallel lines and asymmetry, which was introduced in [**5**, **13**, **18**, **19**], where

$$F(x) := \begin{cases} t_r(x - v) + t_c v, & \text{if } x > v, \\ t_c x, & \text{if } -u \leqslant x \leqslant v, \\ t_l(x + u) - t_c u, & \text{if } x < -u, \end{cases} \qquad g(x)$$

$$:= \begin{cases} r(x - v) + v, & \text{if } x > v, \\ x, & \text{if } -u \leqslant x \leqslant v, \\ l(x + u) - u, & \text{if } x < -u. \end{cases}$$

The plane is separated by two switching lines $\Gamma_L = \{(x, y) : x = -u\}$ and $\Gamma_R = \{(x, y) : x = v\}$.

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Llibre *et al.* [**18**] discussed system (5.1) in the parameter regions

$$\mathcal{G}_1 := \{(u,v,l,r,t_l,t_r) \in \mathbb{R}^6 : 0 < v < u, \quad l > 0, r = 1,\ 0 < t_r < 2,\ t_l < 0,\ t_r + \frac{t_l}{\sqrt{l}} < 0\},$$

$$\mathcal{G}_2 := \{(u,v,l,r,t_l,t_r) \in \mathbb{R}^6 : 0 < u < v,\ r > 0, l = 1,\ 0 < t_l < 2,\ t_r < 0,\ t_l + \frac{t_r}{\sqrt{r}} < 0\},$$

$$\mathcal{G}_3 := \{(u,v,l,r,t_l,t_r) \in \mathbb{R}^6 : 0 < v < u,\ l > 0, r = 1,\ -2 < t_r < 0,\ t_l > 0,\ t_r + \frac{t_l}{\sqrt{l}} > 0\},$$

$$\mathcal{G}_4 := \{(u,v,l,r,t_l,t_r) \in \mathbb{R}^6 : 0 < u < v,\ r > 0, l = 1,\ -2 < t_l < 0,\ t_r > 0,\ t_l + \frac{t_r}{\sqrt{r}} > 0\}$$

and obtained the following results.

Proposition 5.1 [**18**, Theorems 7-10]. *In the parameter region $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$), the following statements hold.*

(a) *When $0 < t_c \leqslant t_r$ (resp. $0 < t_c \leqslant t_l$, $t_r \leqslant t_c < 0$, $t_l \leqslant t_c < 0$), system* (5.1) *has a unique limit cycle, which is stable (resp. stable, unstable, unstable).*

(b) *When $t_c = 0$, the origin of system* (5.1) *is surrounded by a bounded period annulus. The most external periodic orbit of the period annulus, which is tangent to the line $x = v$, is unstable (resp. unstable, stable, stable). There exists a stable (resp. stable, unstable, unstable) limit cycle surrounding such period annulus.*

(c) *There exists small $\varepsilon > 0$ such that if $-\varepsilon < t_c < 0$ (resp. $-\varepsilon < t_c < 0$, $0 < t_c < \varepsilon$, $0 < t_c < \varepsilon$), then the origin is surrounded by at least two limit cycles, where the smaller is unstable (resp. unstable, stable, stable) and the bigger is stable (resp. stable, unstable, unstable).*

In order to study the hyperbolicity and exact number of limit cycles obtained in theorem 5.1, we further recall the following results, where

$$h_1 := \min\left\{\frac{t_r(v-u)}{(u+v)}, \frac{t_r v + t_l u/l}{u+v}\right\}, \quad h_2 := \min\left\{\frac{t_l(u-v)}{(u+v)}, \frac{t_l u + t_r v/r}{u+v}\right\},$$
$$h_3 := \max\left\{\frac{t_r(v-u)}{(u+v)}, \frac{t_r v + t_l u/l}{u+v}\right\}, \quad h_4 := \max\left\{\frac{t_l(u-v)}{(u+v)}, \frac{t_l u + t_r v/r}{u+v}\right\}.$$

Proposition 5.2 [**5**, Theorem 1.2]. *When $0 \leqslant t_c \leqslant t_r$ (resp. $0 \leqslant t_c \leqslant t_l$, $t_r \leqslant t_c \leqslant 0$, $t_l \leqslant t_c \leqslant 0$) and parameters lie in the region $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$), system* (5.1) *has a unique limit cycle, which is hyperbolic. Moreover, the limit cycle is stable in $\mathcal{G}_1$, $\mathcal{G}_2$ and unstable in $\mathcal{G}_3$, $\mathcal{G}_4$.*

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Proposition 5.3 [**5**, Theorem 1.3]. *In the parameter region $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$), the following statements hold.*

(1) *When* $t_c \leqslant h_1$ (resp. $t_c \leqslant h_2,\ t_c \geqslant h_3,\ t_c \geqslant h_4$), *system* (5.1) *exhibits no limit cycles.*

(2) *When* $-\varepsilon < t_c < 0$ *(resp.* $-\varepsilon < t_c < 0$, $0 < t_c < \varepsilon$, $0 < t_c < \varepsilon$*) for small* $\varepsilon > 0$, *system* (5.1) *exhibits exactly two limit cycles, where the inner limit cycle which only intersects* Γ_R *is hyperbolic and unstable (resp. unstable, stable, stable) and the outer one which intersects* Γ_L *and* Γ_R *is hyperbolic and stable(resp. stable, unstable, unstable).*

By propositions 5.2 and 5.3, we naturally have the following question:
(Q4) *When* $h_1 < t_c < -\varepsilon$ *(resp.* $h_2 < t_c < -\varepsilon,\ \varepsilon < t_c < h_3,\ \varepsilon < t_c < h_4$*) for small* $\varepsilon > 0$, *how many limit cycles does system* (5.1) *exhibit?*
The following theorem can answer question **(Q4)**.

Theorem 5.4. *In the parameter region $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$), there are four functions* $t_c = \varphi_1(t_r)$ *(resp.* $\varphi_2(t_r),\ \varphi_3(t_l),\ \varphi_4(t_l)$*) for any fixed* (t_l, l, r, u, v) *(resp.* (t_l, l, r, u, v), (t_r, l, r, u, v), (t_r, l, r, u, v)*) such that*

(1) *system* (5.1) *has exactly two limit cycles when* $\varphi_1(t_r) < t_c < 0$ *(resp.* $\varphi_2(t_r) < t_c < 0$, $0 < t_c < \varphi_3(t_l)$, $0 < t_c < \varphi_4(t_l)$*) for small* $\varepsilon > 0$, *where the inner limit cycle which only intersects* Γ_R *is unstable (resp. unstable, stable, stable) and the outer one which intersects* Γ_L *and* Γ_R *is stable (resp. stable, unstable, unstable);*

(2) *system* (5.1) *has exactly one semi-stable limit cycle when* $t_c = \varphi_1(t_r)$ *(resp.* $\varphi_2(t_r)$, $\varphi_3(t_l)$, $\varphi_4(t_l)$*), where the limit cycle which intersects* Γ_L *and* Γ_R *is externally stable (resp. stable, unstable, unstable);*

(3) *system* (5.1) *has no limit cycles when* $t_c < \varphi_1(t_r)$ *(resp.* $t_c < \varphi_2(t_r)$, $t_c > \varphi_3(t_l)$, $t_c > \varphi_4(t_l)$*).*

Proof. Without loss of generality, we only discuss $\mathcal{G}_1$ since the remainder cases $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ can be discussed similarly. With the transformation $(x, y) \to (x, y + F(x))$, system (5.1) is changed into the following discontinuous system

$$\dot{x} = -y, \qquad \dot{y} = g(x) + f(x)y, \tag{5.2}$$

where

$$f(x) = \begin{cases} t_r, & \text{if } x > v, \\ t_c, & \text{if } -u \leqslant x \leqslant v, \\ t_l, & \text{if } x < -u. \end{cases}$$

We can check that the vector field $(-y, g(x) + f(x)y)$ of system (5.2) is rotated about t_c, t_l and t_r.

Assume that system (5.2) has at least three limit cycles as parameters lie in the region $\mathcal{G}_1$ and $(t_c, t_r) = (t_c^0, t_r^0)$ for $h_1 < t_c^0 < -\varepsilon$ and $0 < t_r^0 < 2$, and Γ_1, Γ_2 and Γ_3 are the three innermost limit cycles, where Γ_2 lies in the annular region surrounded

https://doi.org/10.1017/prm.2024.21 Published online by Cambridge University Press

by inner boundary Γ_1 and outer boundary Γ_3. Without loss of generality, we can let Γ_1, Γ_3 be unstable and Γ_2 be stable. By theorem 2.3, Γ_1, Γ_3 contract and Γ_2 expands when one of t_c, t_l, t_r increases. Now we make a similar process to the proof of theorem 4.1 as follows.

Step 1: There is $t_c^1 \in (t_c^0, 0)$ such that Γ_2, Γ_3 coincide when the other parameters are fixed. We claim that $|t_c^1|$ is not small. Otherwise, system (5.2) has more than one limit cycle as $t_c = 0$ by the rotated properties of vector fields. This contradicts proposition 5.2 and what we claimed is true.

Step 2: There is $t_r^1 \in (0, t_r^0)$ such that Γ_1, Γ_2 coincide or Γ_3 becomes a semi-stable limit cycle when the other parameters are fixed. In this step, we claim that t_r^1 is not small. Otherwise, system (5.2) has more than one limit cycle as $t_r = 0$. By [**17**], system (5.2) has at most one limit cycle for $t_r = 0$. This is a contradiction. We claim that t_c^1 keeps in $(h_1, -\varepsilon)$. Otherwise, assume that $t_c^1 \leqslant h_1$. From proposition 5.3 system (5.2) has no limit cycles if $t_c^1 \leqslant h_1$. This is also a contradiction. Then, we turn to step 1 again.

Finally, for arbitrary small $\varepsilon > 0$ there exists $n_0 \in \mathbb{Z}_+$ such that $|t_c^n| < \varepsilon$ or $t_r^n < \varepsilon$ as $n > n_0$ step by step. We can find that system (5.2) still has at least three limit cycles as $|t_c^n|$ or t_r^n is small, which contradicts proposition 5.3. Therefore, in the parameter region $\mathcal{G}_1$ for $h_1 < t_c < -\varepsilon$ system (5.1) has at most two limit cycles. Applying the rotated properties from theorem 2.3, the continuity of vector fields and propositions 5.1–5.3, there exists a function $t_c = \varphi_1(t_r)$ such that system (5.1) has exactly two limit cycles when $\varphi_1(t_r) < t_c < 0$, where the inner limit cycle which only intersects Γ_R is unstable and the outer one which intersects Γ_L and Γ_R is stable; system (5.1) has exactly one semi-stable limit cycle when $t_c = \varphi_1(t_r)$, where the limit cycle which intersects Γ_L and Γ_R is externally stable; and system (5.1) has no limit cycles when $t_c < \varphi_1(t_r)$. The proof is completed. □

Therefore, the exact number of limit cycles of system (5.1) can be obtained completely when parameters lie in regions $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ or $\mathcal{G}_4$ by propositions 5.1–5.3 and theorem 5.4.

Acknowledgements

This work is financially supported by the National Key R&D Program of China (No. 2022YFA1005900). The first author is supported by the National Natural Science Foundation of China (Nos. 12322109, 12171485) and Science and Technology Innovation Program of Hunan Province (No. 2023RC3040). The second author is supported by the National Natural Science Foundations of China (Nos. 11931016, 12271355, 12161131001). The third author is supported by the National Natural Science Foundation of China (Nos. 12171336, 11831012).

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