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EIGENVALUE ESTIMATES FOR QUADRATIC POLYNOMIAL OPERATOR OF THE LAPLACIAN

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Abstract. For a bounded domain Ω in a complete Riemannian manifold M, we investigate the Dirichlet weighted eigenvalue problem of quadratic polynomial operator $\Delta^2 - a\Delta + b$ of the Laplacian Δ, where a and b are the nonnegative constants. We obtain an inequality for eigenvalues which contains a constant that only depends on the mean curvature of M. It yields an upper bound of the $(k+1)$th eigenvalue Λ_{k+1}. As their applications, some inequalities and bounds of eigenvalues on a complete minimal submanifold in a Euclidean space and a unit sphere are obtained.

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1. Introduction. Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. The Dirichlet eigenvalue problem of the biharmonic operator is described by

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where ν denotes the outwards unit normal vector field of $\partial\Omega$, and Δ^2 is the biharmonic operator on M. It is also called a clamped plate problem, which describes the characteristic vibrations of a clamped plate. An open question in estimates for eigenvalues of problem (1.1) is to give universal upper bounds of the $(k+1)$-th eigenvalue λ_{k+1} in terms of the first k eigenvalues.

To begin with, people were concerned about the case that Ω is a bounded domain in $\mathbb{R}^n$. In 1956, Payne, Pólya and Weinberger [**15**] established the following universal inequality:

$$\lambda_{k+1} - \lambda_k \le \frac{8(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\lambda_i. \tag{1.2}$$

Some progresses have been made after their work. As a generalization of their result, Hile and Yeh [**10**] obtained

$$\frac{n^2k^{3/2}}{8(n+2)}\left(\sum_{i=1}^{k}\lambda_i\right)^{-\frac{1}{2}} \le \sum_{i=1}^{k}\frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1}-\lambda_i} \tag{1.3}$$

by using an improved method of Hile and Protter [**9**]. In 1990, Hook [**11**] and Chen and Qian [**5**] independently proved

$$\frac{n^2k^2}{8(n+2)} \le \sum_{i=1}^{k}\frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1}-\lambda_i}\sum_{i=1}^{k}\lambda_i^{\frac{1}{2}}. \tag{1.4}$$

In 2006, Cheng and Yang [**7**] obtained the following sharper inequality

$$\lambda_{k+1} \le \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \left[\frac{8(n+2)}{n^2}\right]^{\frac{1}{2}}\frac{1}{k}\sum_{i=1}^{k}\left[\lambda_i(\lambda_{k+1}-\lambda_i)\right]^{\frac{1}{2}}. \tag{1.5}$$

This also gave an affirmative answer for a question introduced by Ashbaugh in [**1**]. And more information about universal eigenvalue inequalities can find in [**2, 3, 16**].

It is natural to consider the estimates for eigenvalues of problem (1.1) on the other Riemannian manifolds. In 2007, Wang and Xia [**17**] proved

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{8(n+2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i \tag{1.6}$$

on an n-dimensional complete minimal submanifold in a Euclidean space, and

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)[n^2+(2n+4)\lambda_i^{\frac{1}{2}}][n^2+4\lambda_i^{\frac{1}{2}}] \tag{1.7}$$

on an n-dimensional unit sphere. In 2009, Cheng and Yang [**8**] proved

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\lambda_i-\frac{(n-1)^2}{4}\right] \tag{1.8}$$

for eigenvalues of problem (1.1) on a bounded domain Ω in a hyperbolic space $H^n(-1)$. Recently, Cheng, Ichikawa and Mametsuka [**6**] proved that, for any complete Riemannian manifold M, there exists a universal bound of the $(k+1)$-th eigenvalue in terms of the first k eigenvalues of (1.1). They obtained the following remarkable inequality of eigenvalues

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)[n^2H_0^2+(2n+4)\lambda_i^{\frac{1}{2}}](n^2H_0^2+4\lambda_i^{\frac{1}{2}}), \tag{1.9}$$

where H_0 is a constant which only depends on the mean curvature of M. It is easy to find that (1.9) contains the inequalities (1.6) and (1.7).

In this paper, for any complete Riemannian manifold M, we consider the following Dirichlet eigenvalue problem of quadratic polynomial operator of the Laplacian

$$\begin{cases} \Delta^2 u - a\Delta u + bu = \Lambda \rho u, & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.10}$$

where ρ is a positive function continuous on $\overline{\Omega}$ and the constants $a, b \geq 0$. In general, (1.10) is a more ideal model which is abstracted from the problems of physics and mechanics. In fact, the weight function ρ denotes the density. And weighted estimates are intelligent in filtering and identification problems (see **[12, 13]**). Moreover, problem (1.1) is only a special case of problem (1.10).

The main goal of this paper is to give some estimates for eigenvalues of problem (1.10). In Section 2, we prove a general inequality for eigenvalues of problem (1.10) on a complete Riemannian manifold. Then, by using this general inequality, we derive the following result.

THEOREM 1.1. *For a domain Ω in an n-dimensional complete Riemannian manifold M, denote by Λ_i the i-th eigenvalue of the eigenvalue problem (1.10). Set $\sigma = (\inf_{\Omega} \rho)^{-1}$ and $\tau = (\sup_{\Omega} \rho)^{-1}$. Then there exists a constant H_0 which only depends on the mean curvature of M such that the following inequality*

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{1}{n^2\tau^2}\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\left[n^2\sigma H_0^2 + (2n+4)E_i + na\sigma\right]\left(n^2\sigma^2 H_0^2 + 4\sigma E_i\right) \tag{1.11}$$

holds, where

$$E_i = \frac{1}{2}\left[-a\sigma + \sqrt{a^2\sigma^2 + 4\sigma(\Lambda_i - b\tau)}\right].$$

The inequality (1.10) is a quadratic inequality of Λ_{k+1}. It yields a more explicit inequality which gives an upper bound of Λ_{k+1}.

THEOREM 1.2. *Under the same assumptions as Theorem 1.1, we have*

$$\Lambda_{k+1} \leq A_k + \sqrt{A_k^2 - B_k}, \tag{1.12}$$

where

$$A_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Lambda_i + \frac{1}{2n^2\tau^2}\sum_{i=1}^{k}\left[n^2\sigma H_0^2 + (2n+4)E_i + na\sigma\right]\left(n^2\sigma^2 H_0^2 + 4\sigma E_i\right)\right\},$$

$$B_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Lambda_i^2 + \frac{1}{n^2\tau^2}\sum_{i=1}^{k}\Lambda_i\left[n^2\sigma H_0^2 + (2n+4)E_i + na\sigma\right]\left(n^2\sigma^2 H_0^2 + 4\sigma E_i\right)\right\}.$$

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Putting $a = b = 0$ and $\sigma = \tau = 1$ in (1.11), we can get (1.9) in **[6]**. Namely, it is a corollary of Theorem 1.1. As applications of Theorems 1.1 and 1.2, we also obtain some results for an n-dimensional complete minimal submanifold M in an Euclidean space, and an n-dimensional unit sphere $M = S^n(1)$ (see Corollary 3.1–3.4). Moreover, these results also contain the inequalities (1.6) and (1.7) .

2. Proof of Theorem 1.1. The main goal of this section is to give the proof of Theorem 1.1. Firstly, we establish a lemma which will be used to estimate some terms in the proof of Theorem 1.1.

LEMMA 2.1. *Under the same assumptions as Theorem 1.1, let u_i be i-th weighted orthonormal eigenfunctions of problem (1.10) corresponding to eigenvalues Λ_i, $i = 1, 2, \dots, k$. Namely, u_i satisfies*

$$\begin{cases} \Delta^2 u_i - a\Delta u_i + bu_i = \Lambda_i \rho u_i, & \text{in } \Omega, \\ u_i|_{\partial\Omega} = \left.\dfrac{\partial u_i}{\partial \nu}\right|_{\partial\Omega} = 0, \\ \displaystyle\int_\Omega \rho u_i u_j = \delta_{ij}. \end{cases} \tag{2.1}$$

Then we have

$$\int_\Omega |\nabla u_i|^2 \leq E_i, \tag{2.2}$$

where

$$E_i = \frac{1}{2}\Big[- a\sigma + \sqrt{a^2\sigma^2 + 4\sigma(\Lambda_i - b\tau)}\Big].$$

Proof. According to the assumptions, it is easy to find

$$0 < \tau = \tau \int_\Omega \rho u_i^2 \leq \int_\Omega u_i^2 \leq \sigma \int_\Omega \rho u_i^2 = \sigma. \tag{2.3}$$

Noticing the constants $a, b \geq 0$ and the weight function $\rho > 0$, and utilizing

$$\int_\Omega |\nabla u_i|^2 = \int_\Omega u_i(-\Delta u_i) \leq \left[\int_\Omega u_i^2 \int_\Omega (\Delta u_i)^2\right]^{\frac{1}{2}} \leq \left[\sigma \int_\Omega (\Delta u_i)^2\right]^{\frac{1}{2}}, \tag{2.4}$$

we know

$$a^2\sigma^2 + 4\sigma(\Lambda_i - b\tau) \geq 0.$$

Substituting (2.4) into

$$\Lambda_i = \int_\Omega u_i(\Delta^2 u_i - a\Delta u_i + bu_i) = \int_\Omega (\Delta u_i)^2 + a\int_\Omega |\nabla u_i|^2 + b\int_\Omega u_i^2, \tag{2.5}$$

we have

$$\left(\int_\Omega |\nabla u_i|^2\right)^2 + a\sigma \int_\Omega |\nabla u_i|^2 - \sigma(\Lambda_i - b\tau) \le 0.$$

This is a quadratic inequality of $\int_\Omega |\nabla u_i|^2$ which yields (2.2). □

Now we prove a general inequality for eigenvalues of problem (1.10) which plays an important role in the proof of Theorem 1.1.

THEOREM 2.2. *Under the same assumptions as Lemma 2.1, for any function $h \in C^4(M) \bigcap C^3(M)$, we have*

$$\begin{aligned} &-2\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \int_\Omega h u_i(\langle \nabla h, \nabla u_i\rangle + \frac{1}{2} u_i \Delta h) \\ &\le \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 w_i + \sum_{i=1}^k \frac{1}{\delta_i}(\Lambda_{k+1} - \Lambda_i) \int_\Omega \frac{1}{\rho}\left(\langle \nabla h, \nabla u_i\rangle + \frac{1}{2} u_i \Delta h\right)^2, \end{aligned} \tag{2.6}$$

where the positive constants δ_i $(i = 1, \ldots, k, \ldots \to \infty)$ construct a monotonic decreasing sequence and

$$w_i = \int_\Omega \left[(u_i \Delta h + 2\langle \nabla h, \nabla u_i\rangle)^2 - 2u_i \Delta u_i |\nabla h|^2 - 2ahu_i\langle \nabla h, \nabla u_i\rangle - au_i^2 h\Delta h\right].$$

Proof. Define the trial functions φ_i by

$$\varphi_i = hu_i - \sum_{j=1}^k r_{ij} u_j, \quad i = 1, \ldots, k, \tag{2.7}$$

where

$$r_{ij} = \int_\Omega \rho h u_i u_j = r_{ji}. \tag{2.8}$$

Then, it is easy to check

$$\int_\Omega \rho \varphi_i u_j = 0 \tag{2.9}$$

and

$$\int_\Omega \rho \varphi_i h u_j = \int_\Omega \rho \varphi_i^2. \tag{2.10}$$

It yields from (2.1) that

$$\Delta^2 \varphi_i - a\Delta\varphi_i + b\varphi_i = q_i + h\Lambda_i \rho u_i - \sum_{j=1}^k r_{ij}\Lambda_j \rho u_j, \tag{2.11}$$

where

$$q_i = u_i\Delta^2 h + 2\langle \nabla\Delta h, \nabla u_i\rangle + 2\langle \nabla h, \nabla\Delta u_i\rangle + 2\Delta(\langle \nabla h, \nabla u_i\rangle) \\ + 2\Delta h\Delta u_i - 2a\langle \nabla h, \nabla u_i\rangle - au_i\Delta h.$$

From (2.9), (2.10) and (2.11), we have

$$\begin{aligned}\int_\Omega \varphi_i(\Delta^2\varphi_i - a\Delta\varphi_i + b\varphi_i) &= \int_\Omega \varphi_i q_i + \Lambda_i\int_\Omega \rho\varphi_i hu_i - \sum_{j=1}^k r_{ij}\Lambda_j\int_\Omega \rho u_j\varphi_i \\ &= \int_\Omega hu_iq_i - \sum_{j=1}^k r_{ij}s_{ij} + \Lambda_i\int_\Omega \rho\varphi_i^2, \end{aligned} \tag{2.12}$$

where

$$s_{ij} = \int_\Omega q_iu_j.$$

Substituting (2.12) into the Rayleigh–Ritz inequality

$$\Lambda_{k+1} \le \frac{\int_\Omega \varphi_i(\Delta^2\varphi_i - a\Delta\varphi_i + b\varphi_i)}{\int_\Omega \rho\varphi_i^2}, \tag{2.13}$$

it follows that

$$(\Lambda_{k+1} - \Lambda_i)\int_\Omega \rho\varphi_i^2 \le \int_\Omega hu_iq_i - \sum_{j=1}^k r_{ij}s_{ij}. \tag{2.14}$$

Moreover, using Stokes' theorem, we obtain

$$\int_\Omega hu_iq_i = w_i. \tag{2.15}$$

At the same time, from the definitions of s_{ij} and q_i, we can deduce

$$\begin{aligned} s_{ij} &= \int_\Omega u_j\left[\Delta^2(hu_i) - a\Delta(hu_i) + bhu_i - h\Lambda_i\rho u_i\right] \\ &= \int_\Omega hu_i(\Delta^2u_j - a\Delta u_j + bu_j) + \Lambda_i\int_\Omega \rho hu_iu_j \\ &= (\Lambda_j - \Lambda_i)r_{ij}. \end{aligned} \tag{2.16}$$

Therefore, substituting (2.15) and (2.16) into (2.14), we get

$$(\Lambda_{k+1} - \Lambda_i)\int_\Omega \rho\varphi_i^2 \le w_i + \sum_{j=1}^k(\Lambda_i - \Lambda_j)r_{ij}^2. \tag{2.17}$$

Then it follows from (2.7) that

$$-2\int_\Omega \varphi_i\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)$$
$$=-2\int_\Omega hu_i\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)+2\sum_{j=1}^k r_{ij}t_{ij}, \tag{2.18}$$

where

$$t_{ij}=\int_\Omega u_j\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)=-t_{ji}.$$

Multiplying $(\Lambda_{k+1}-\Lambda_i)^2$ in the both sides of (2.18), taking sum on i from 1 to k, and using the following inequality

$$2\sum_{i,j=1}^k(\Lambda_{k+1}-\Lambda_i)^2r_{ij}t_{ij}=-2\sum_{i,j=1}^k(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)r_{ij}t_{ij}$$
$$\geq-\sum_{i,j=1}^k\delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2r_{ij}^2-\sum_{i,j=1}^k\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)t_{ij}^2,$$

we have

$$-2\sum_{i=1}^k(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega\varphi_i\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)$$
$$\geq-2\sum_{i=1}^k(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega hu_i\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)$$
$$-\sum_{i,j=1}^k\delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2r_{ij}^2-\sum_{i,j=1}^k\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)t_{ij}^2. \tag{2.19}$$

On the other hand, utilizing (2.17), we can get

$$(\Lambda_{k+1}-\Lambda_i)^2\left[-2\int_\Omega\varphi_i\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)\right]$$
$$=-2(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega\sqrt{\rho}\varphi_i\left[\frac{1}{\sqrt{\rho}}\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)-\sqrt{\rho}\sum_{j=1}^k t_{ij}u_j\right]$$
$$\leq\frac{\Lambda_{k+1}-\Lambda_i}{\delta_i}\int_\Omega\left[\frac{1}{\sqrt{\rho}}\left(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_i\Delta h\right)-\sqrt{\rho}\sum_{j=1}^k t_{ij}u_j\right]^2$$
$$+\delta_i(\Lambda_{k+1}-\Lambda_i)^3\int_\Omega\rho\varphi_i^2$$

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$$\leq \frac{\Lambda_{k+1}-\Lambda_i}{\delta_i}\left[\int_\Omega \frac{1}{\rho}\left(\langle \nabla h, \nabla u_i\rangle + \frac{1}{2}u_i\Delta h\right)^2 - \sum_{j=1}^k t_{ij}^2\right]$$
$$+\,\delta_i(\Lambda_{k+1}-\Lambda_i)^2\left[w_i + \sum_{j=1}^k(\Lambda_i-\Lambda_j)r_{ij}^2\right]. \tag{2.20}$$

Since $\{\delta_i\}$ is monotonic decreasing, it follows

$$\sum_{i,j=1}^k \delta_i(\Lambda_{k+1}-\Lambda_i)^2(\Lambda_i-\Lambda_j)r_{ij}^2 \leq -\sum_{i,j=1}^k \delta_i(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)^2 r_{ij}^2. \tag{2.21}$$

Taking sum on i from 1 to k in (2.20), and using (2.19) and (2.21), we can obtain (2.6). $\square$

By using Lemma 2.1 and Theorem 2.2, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Nash's theorem [**14**] says that an n-dimensional complete Riemannian manifold M can be isometrically immersed in $\mathbb{R}^N$. For an arbitrary point $p\in M$, let $(x^1,\ldots,x^n)$ be an arbitrary coordinate system in a neighborhood U of $p\in M$. Let y be the position vector of $p\in M$ which is defined by

$$y = (y^1(x^1,\ldots,x^n),\ldots,y^N(x^1,\ldots,x^n)).$$

Putting $h = y^\alpha$ in (2.6), we have

$$-2\sum_{i=1}^k(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega y^\alpha u_i\left(\langle\nabla y^\alpha,\nabla u_i\rangle + \frac{1}{2}u_i\Delta y^\alpha\right)$$
$$\leq \sum_{i=1}^k \frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\frac{1}{\rho}\left(\langle\nabla y^\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta y^\alpha\right)^2 + \sum_{i=1}^k\delta_i(\Lambda_{k+1}-\Lambda_i)^2\omega_i^\alpha, \tag{2.22}$$

where

$$w_i^\alpha = \int_\Omega\big[(u_i\Delta y^\alpha + 2\langle\nabla y^\alpha,\nabla u_i\rangle)^2 - 2ay^\alpha u_i\langle\nabla y^\alpha,\nabla u_i\rangle$$
$$-\,2u_i\Delta u_i|\nabla y^\alpha|^2 - au_i^2 y^\alpha\Delta y^\alpha\big].$$

By calculating, one can get the following equalities (see [**4**]):

$$\sum_{\alpha=1}^N(\langle\nabla y^\alpha,\nabla u\rangle)^2 = |\nabla u|^2, \quad \sum_{\alpha=1}^N|\nabla y^\alpha|^2 = n,$$
$$\sum_{\alpha=1}^N(\Delta y^\alpha)^2 = n^2|H|^2, \quad \sum_{\alpha=1}^N\Delta y^\alpha\nabla y^\alpha = 0, \tag{2.23}$$

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where $|H|$ is the mean curvature of M. Utilizing Lemma 2.1 and (2.23), we have

$$\begin{aligned}\int_\Omega \frac{1}{\rho}\sum_{\alpha=1}^{N}\left(\langle\nabla y^\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta y^\alpha\right)^2 &= \int_\Omega \frac{1}{\rho}|\nabla u_i|^2+\frac{1}{4}n^2\int_\Omega \frac{1}{\rho}u_i^2|H|^2\\ &\le \sigma E_i+\frac{1}{4}n^2\sigma^2\sup_\Omega|H|^2, \end{aligned}\tag{2.24}$$

$$\begin{aligned}\sum_{\alpha=1}^{N} w_i^\alpha &= \int_\Omega\left[\sum_{\alpha=1}^{N}(u_i\Delta y^\alpha+2\langle\nabla y^\alpha,\nabla u_i\rangle)^2-2u_i\Delta u_i\sum_{\alpha=1}^{N}|\nabla y^\alpha|^2\right]\\ &\quad -2a\sum_{\alpha=1}^{N}\int_\Omega y^\alpha u_i\left(\langle\nabla y^\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta y^\alpha\right)\\ &= n^2\int_\Omega |H|^2u_i^2+4\int_\Omega|\nabla u_i|^2-2n\int_\Omega u_i\Delta u_i+na\int_\Omega u_i^2\\ &\le n^2\sigma\sup_\Omega|H|^2+(2n+4)E_i+na\sigma, \end{aligned}\tag{2.25}$$

and

$$-2\sum_{\alpha=1}^{N}\int_\Omega y^\alpha u_i\left(\langle\nabla y^\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta y^\alpha\right)=\int_\Omega u_i^2\sum_{\alpha=1}^{N}|\nabla y^\alpha|^2=n\int_\Omega u_i^2\ge n\tau. \tag{2.26}$$

Taking sum on α from 1 to N in (2.22) and using (2.24)–(2.26), we have

$$\begin{aligned} n\tau\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 &\le \sum_{i=1}^{k}\delta_i(\Lambda_{k+1}-\Lambda_i)^2\left[n^2\sigma\sup_\Omega|H|^2+(2n+4)E_i+na\sigma\right]\\ &\quad+\sum_{i=1}^{k}\frac{1}{\delta_i}(\Lambda_{k+1}-\Lambda_i)\left[\frac{1}{4}n^2\sigma^2\sup_\Omega|H|^2+\sigma E_i\right]. \end{aligned}\tag{2.27}$$

Putting

$$\delta_i=\frac{\delta}{n^2\sigma\sup_\Omega|H|^2+(2n+4)E_i+na\sigma}$$

in (2.27), where δ is a positive constant, it yields

$$\begin{aligned} &n\tau\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\le\delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\\ &+\frac{1}{\delta}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[n^2\sigma\sup_\Omega|H|^2+(2n+4)E_i+na\sigma\right]\left[\frac{1}{4}n^2\sigma^2\sup_\Omega|H|^2+\sigma E_i\right]. \end{aligned}\tag{2.28}$$

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Then putting

$$\delta = \left\{ \frac{\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\left[n^2\sigma \sup_{\Omega}|H|^2 + (2n+4)E_i + na\sigma\right]\left[n^2\sigma^2 \sup_{\Omega}|H|^2 + 4\sigma E_i\right]}{4\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2} \right\}^{\frac{1}{2}}$$

in (2.28), we have

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \le \frac{1}{n^2\tau^2}\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\left[n^2\sigma \sup_{\Omega}|H|^2 + (2n+4)E_i + na\sigma\right]\left[n^2\sigma^2 \sup_{\Omega}|H|^2 + 4\sigma E_i\right]. \tag{2.29}$$

Now we define

$$\mathfrak{F} = \{\phi|\phi \textit{ is an isometric immersion from } M \textit{ into a Euclidean space}\}.$$

Putting

$$H_0 = \inf_{\phi\in\mathfrak{F}} \sup_{\Omega}|H|^2$$

in (2.29), we infer (1.11).

□

3. Some Applications. When M is an n-dimensional complete minimal submanifold in a Euclidean space, and an n-dimensional unit sphere $S^n(1)$, it yields $H_0 = 0$ and $H_0 = 1$, respectively. Therefore, as the applications of Theorem 1.1, we easily give the following corollaries.

COROLLARY 3.1. *Under the same assumptions as Theorem 1.1, assume that M is an n-dimensional complete minimal submanifold in a Euclidean space. Then we have*

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \le \frac{4\sigma}{n^2\tau^2}\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)[(2n+4)E_i + na\sigma]E_i. \tag{3.1}$$

COROLLARY 3.2. *Under the same assumptions as Corollary 3.1, we have*

$$\Lambda_{k+1} \le C_k + \sqrt{C_k^2 - D_k}, \tag{3.2}$$

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where

$$C_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Lambda_i + \frac{2\sigma}{n^2\tau^2}\sum_{i=1}^{k}\left[(2n+4)E_i + na\sigma\right]E_i\right\},$$

$$D_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Lambda_i^2 + \frac{4\sigma}{n^2\tau^2}\sum_{i=1}^{k}\left[(2n+4)E_i + na\sigma\right]E_i\Lambda_i\right\}.$$

COROLLARY 3.3. *Under the same assumptions as Theorem 1.1, assume that M is an n-dimensional unit sphere $S^n(1)$. Then we have*

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{1}{n^2\tau^2}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[n^2\sigma + (2n+4)E_i + na\sigma\right](n^2\sigma^2 + 4\sigma E_i). \tag{3.3}$$

COROLLARY 3.4. *Under the same assumptions as Corollary 3.3, we have*

$$\Lambda_{k+1} \leq F_k + \sqrt{F_k^2 - G_k}, \tag{3.4}$$

where

$$F_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Lambda_i + \frac{\sigma}{2n^2\tau^2}\sum_{i=1}^{k}\left[(2n+4)E_i + n^2\sigma + na\sigma\right](4E_i + n^2\sigma)\right\},$$

$$G_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Lambda_i^2 + \frac{\sigma}{n^2\tau^2}\sum_{i=1}^{k}\left[(2n+4)E_i + n^2\sigma + na\sigma\right](4E_i + n^2\sigma)\Lambda_i\right\}.$$

When $a = b = 0$ and $\rho = 1$, it follows $E_i = \Lambda_i^{\frac{1}{2}}$. As the special cases of Corollary 3.1 and 3.3, we can easily get (1.6) and (1.7).

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