EXTENDED SCHAUDER DECOMPOSITIONS OF LOCALLY CONVEX SPACES†

by J. H. WEBB

(Received 26 October, 1973)

Let $E[\tau]$ be a locally convex Hausdorff topological vector space. An *extended decomposition* of $E[\tau]$ is a family $\{E_\alpha\}_{\alpha \in A}$ of closed subspaces of E such that, for each x in E and each α in A, there exists a unique point x_α in E_α, with $\sum_{\alpha \in A} x_\alpha = x$. Here convergence will have the following meaning. Let Φ denote the set of all finite subsets of A. The sum $\sum_{\alpha \in A} x_\alpha$ is said to be convergent to x if for each neighbourhood U of 0 in E, there is an element ϕ_0 of Φ such that $x - \sum_{\alpha \in \phi} x_\alpha \in U$, for all ϕ in Φ containing ϕ_0. It follows that $\sum_{\alpha \in A} x_\alpha$ is Cauchy if and only if, for each neighbourhood U of 0 in E, there is an element ϕ_0 of Φ such that $\sum_{\alpha \in \phi} x_\alpha \in U$, for all ϕ in Φ disjoint from ϕ_0.

Let P_α be defined by $P_\alpha x = x_\alpha$. Then P_α is a projection of E on to E_α. Let $S_\phi = \sum_{\alpha \in \phi} P_\alpha$ for each ϕ in Φ. If each P_α (equivalently, each S_ϕ) is continuous, we say that $\{E_\alpha\}_{\alpha \in A}$ is an *extended Schauder decomposition* of $E[\tau]$.

Motivated by corresponding questions in the theory of (countable) Schauder bases and decompositions (see, for example, [9, §4]), we examine the following problems.

A. When is a weak extended Schauder decomposition strong?

B. For which locally convex spaces are extended decompositions Schauder?

If $\{E_\alpha\}_{\alpha \in A}$ is an extended Schauder decomposition of $E[\sigma(E, E')]$, the partial summation operators S_ϕ are easily seen to be strongly continuous. The problem in (A) is to prove the strong convergence of $\{S_\phi x\}$ to x. In (B), the problem is to establish the continuity of the partial summation operators.

These questions have been examined by Arsove and Edwards [2] and Marti [10] in the case of an extended basis of a locally convex space. This is just the case of an extended decomposition into one-dimensional subspaces. The answers obtained are:

(a) In a barrelled space, any weak extended Schauder basis is strong.

(b) In a Fréchet space, any extended basis is a Schauder basis.

Our results, for decompositions, improve " barrelled " to " σ-barrelled " in (a), and " Fréchet " to " strict (LF) " in (b).

An extended Schauder decomposition of a space $E[\tau]$ is said to be *uniformly bounded* if, for every bounded subset A of E, $\bigcup_{\phi \in \Phi} S_\phi(A)$ is bounded in $E[\tau]$. This is equivalent to: for every $\beta(E', E)$-bounded subset B of E', $\bigcup_{\phi \in \Phi} S'_\phi(B)$ is $\beta(E', E)$-bounded.

A locally convex space $E[\tau]$ is said to be *σ-barrelled* [4] if every $\sigma(E', E)$-bounded sequence

† This research was partially supported by a grant from the South African Council for Scientific and Industrial Research.

https://doi.org/10.1017/S0017089500002366 Published online by Cambridge University Press

in E' is equicontinuous, and to be *σ-quasi-barrelled* if every $\beta(E', E)$-bounded sequence in E is equicontinuous.

We denote by $\gamma(E, E')$ the topology on E of uniform convergence on the $\beta(E', E)$-bounded subsets of E'.

THEOREM 1. *Let $E[\tau]$ be a σ-quasi-barrelled space. Then any uniformly bounded extended Schauder decomposition of $E[\sigma(E, E')]$ is an extended Schauder decomposition of $E[\gamma(E, E')]$.*

Proof. Let $\{E_\alpha\}_{\alpha\in A}$ be a uniformly bounded extended Schauder decomposition of $E[\sigma(E, E')]$. Then each S_ϕ is $\sigma(E, E')$-continuous. The adjoints S'_ϕ are then $\beta(E', E)$-continuous, whence each S_ϕ is $\gamma(E, E')$-continuous. The main problem is to prove that, for each x in E, $\{S_\phi x\}$ is $\gamma(E, E')$-convergent to x. We know that $\{S_\phi x\}$ is $\sigma(E, E')$-convergent to x. Since $\gamma(E, E')$ has a base of $\sigma(E, E')$-closed neighbourhoods, it is sufficient to prove that $\{S_\phi x\}$ is $\gamma(E, E')$-Cauchy. Suppose that this is false for some x in E. Then there is a $\beta(E', E)$-bounded set B in E' such that, for each ϕ in Φ, there exists ϕ' in Φ, with $\phi\cap\phi' = \emptyset$ and $S_{\phi'}x \notin B^\circ$. Choose a sequence $\{\phi_k\}$ of mutually disjoint finite subsets of A such that $S_{\phi_k}x \notin B^\circ$ for each k; then choose x'_k in B such that

$$|\langle S_{\phi_k}x, x'_k\rangle| > 1 \quad \text{for each} \quad k. \tag{1}$$

Let $A = \{S'_{\phi_k}x'_k : k = 1, 2, \ldots\}$. Then $A \subset \bigcup_{\phi\in\Phi} S'_\phi(B)$. Since the decomposition is uniformly bounded, A is $\beta(E', E)$-bounded. Since $E[\tau]$ is σ-quasi-barrelled, A is equicontinuous; hence A° is a τ-neighbourhood of 0 in E.

Now $\bigcup_{\phi\in\Phi} S_\phi(E)$ is a $\sigma(E, E')$-dense subspace of E, and hence is also τ-dense. We can therefore find $z \in S_{\phi_0}(E)$, for some ϕ_0 in Φ, such that $x - z \in A^\circ$.

Then

$$\begin{aligned} |\langle S_{\phi_k}x, x'_k\rangle| &= |\langle x, S'_{\phi_k}x'_k\rangle| \\ &\leqq |\langle x-z, S'_{\phi_k}x'_k\rangle| + |\langle z, S'_{\phi_k}x'_k\rangle| \\ &\leqq 1 + |\langle S_{\phi_k}z, x'_k\rangle|. \end{aligned}$$

Now choose k so large that $\phi_k \cap \phi_0 = \emptyset$. (This is possible since ϕ_0 is finite, and the sets ϕ_k are disjoint.) The second term above is then zero, and we have a contradiction to (1).

COROLLARY. *In a σ-barrelled space, any weak extended Schauder decomposition is strong.*

Proof. A σ-barrelled space is σ-quasi-barrelled, and has $\gamma(E, E') = \beta(E, E')$, while any weak extended Schauder decomposition of it is uniformly bounded.

The result stated in the Corollary has been proved by Tweddle **[11]** in the case of extended Schauder bases. The proof of Theorem 1 is modelled on Tweddle's, but is a little simpler.

To examine question B we use a technique formulated by McArthur [9]. Let $E[\tau]$ be a space, such that $E[\sigma(E, E')]$ has an extended decomposition with partial summation operators $\{S_\phi\}$. Let $\mathscr{V}$ be a base at 0 of closed absolutely convex τ-neighbourhoods. If $V \in \mathscr{V}$, let

https://doi.org/10.1017/S0017089500002366 Published online by Cambridge University Press

$V' = \bigcap_{\phi \in \Phi} S_\phi^{-1}(V)$, and let $\mathscr{V}' = \{V' : V \in \mathscr{V}\}$. Then

(1) $\mathscr{V}'$ is a base at 0 for a locally convex topology τ' on E.

(2) $\tau \leqq \tau'$.

(3) Each S_ϕ is τ'-continuous on E.

These results are easily established. It is clear that the problem of proving the τ-continuity of the operators S_ϕ is solved by showing that $\tau = \tau'$. It is also easy to see that

(4) If $E[\tau]$ is metrisable, so is $E[\tau']$.

Finally, we have

(5) If $E[\tau]$ is (sequentially) complete, so is $E[\tau']$.

We sketch the proof of this last statement. Let $\{x_n\}$ be a τ'-Cauchy net (or sequence). Then $\{x_n\}$ is τ-Cauchy, hence τ-convergent to $x \in E$. Also $\{S_\phi x_n\}$ is τ-Cauchy, uniformly for $\phi \in \Phi$; so there exists $y_\phi \in E$ (for each $\phi \in \Phi$) such that $\{S_\phi x_n - y_\phi\}$ is τ-convergent to 0, uniformly for $\phi \in \Phi$. Now

$$x - y_\phi = (x - x_n) + (x_n - S_\phi x_n) + (S_\phi x_n - y_\phi).$$

The first and third terms are τ-convergent to zero as $n \to \infty$, uniformly for $\phi \in \Phi$. The middle term is $\sigma(E, E')$-convergent to 0 as $\phi \to \infty$ for fixed n, since the mappings $\{S_\phi\}$ are the partial summation operators for a weak extended decomposition. Hence $y_\phi \to x$ with respect to $\sigma(E, E')$. Since each of the spaces $\{E_\alpha\}_{\alpha \in A}$ of the decomposition is closed, we have $y_\phi \in \sum_{\alpha \in \phi} E_\alpha$. By uniqueness of the decomposition, $S_\phi x = y_\phi$ for each ϕ, and we now see that $x_n \to x$ with respect to τ'.

If $E[\tau]$ is a Fréchet space with a weak extended decomposition, then $E[\tau']$ is also a Fréchet space. Since $\tau \leqq \tau'$, the open mapping theorem for Fréchet spaces gives $\tau = \tau'$, which proves that the decomposition is Schauder. This is what Marti **[10]** proves, for extended bases in Fréchet spaces.

A strict (LF)-space $E[\tau]$ is an inductive limit of a sequence $\{F_n[\tau_n]\}$ of Fréchet spaces, with $F_n \subseteq F_{n+1}$ and $\tau_{n+1} | F_n = \tau_n$. The topology τ is the finest locally convex topology on E such that $\tau | F_n = \tau_n$ for each n. Every bounded subset of E is contained in some E_n. A strict (LF)-space is complete. For proofs of these statements, see **[5]**.

THEOREM 2. *Let $E[\tau] = \lim F_n[\tau_n]$ be a strict (LF)-space. Then any weak extended decomposition of E is an extended Schauder decomposition of* $E[\tau]$.

Proof. Denote the partial summation operators of the decomposition by $\{S_\phi\}_{\phi \in \Phi}$. Let τ' be defined as above. For each n, put $G_n = \bigcap_{\phi \in \Phi} S_\phi^{-1}(F_n)$. Then G_n is a subspace of F_n, and is τ'-closed in E, since F_n is τ'-closed, and each S_ϕ is τ'-continuous. Since $E[\tau']$ is complete, $G_n[\tau' | G_n]$ is complete. If $x \in E$, $\{S_\phi x\}_{\phi \in \Phi}$ is a bounded set and so is contained in some F_n. Hence $E = \bigcup_{n=1}^{\infty} G_n$.

For a fixed m, consider the topology $\tau' | G_m$. It has basic neighbourhoods

$$V' \cap G_m = [\bigcap_{\phi \in \Phi} S_\phi^{-1}(V)] \cap G_m = \bigcap_{\phi \in \Phi} S_\phi^{-1}(V \cap F_m),$$

https://doi.org/10.1017/S0017089500002366 Published online by Cambridge University Press

where V is a basic neighbourhood in $E[\tau]$. Now $\tau|F_m = \tau_m$ is metrisable; hence $\tau'|G_m$ is metrisable. Thus $G_m[\tau'|G_m]$ is a Fréchet space.

So $\{G_n[\tau'|G_n]\}$ is an inductive sequence of Fréchet spaces. Let τ'' denote the associated inductive limit topology on E, making $E[\tau'']$ a strict (LF)-space, with $\tau \leqq \tau' \leqq \tau''$. By the closed graph theorem for strict (LF)-spaces [**5**], $\tau = \tau''$.

If one restricts these results to the case when the index set A of the decomposition is countable, one obtains known results in the theory of countable Schauder bases and decompositions. Theorem 1 and its corollary are obtained in [**12**] for countable Schauder decompositions, by a completely different method. Theorem 2 is proved by Bennett and Cooper [**3**] for countable Schauder bases. Strictly speaking, when specializing the results above to the countable case, one obtains results in terms of unconditional Schauder bases and decompositions. However, simple modifications of the proofs will adapt them to cover the usual convergence of series.

From now on we consider the special case of an extended Schauder basis. Here the spaces $\{E_\alpha\}_{\alpha \in A}$ are one-dimensional; so we have $E_\alpha = \text{span}\{x_\alpha\}$. The projections P_α are given by elements $x'_\alpha \in E'$ by the rule $P_\alpha x = \langle x, x'_\alpha \rangle x_\alpha$, and the partial summation operators are given by the formula $S_\phi x = \sum_{\alpha \in \phi} \langle x, x'_\alpha \rangle x_\alpha$.

It is clear that each S_ϕ can be extended by this formula to a linear mapping of $(E')^*$ (the algebraic dual of E') into E.

An extended Schauder basis or decomposition is said to be *equi-Schauder* if the family $\{S_\phi\}_{\phi \in \Phi}$ is equicontinuous. This is equivalent to having $\bigcup_{\phi \in \Phi} S'_\phi(K)$ equicontinuous whenever K is equicontinuous.

Let $\tilde{E}[\tilde{\tau}]$ denote the completion of the space $E[\tau]$. Then $\tilde{E}$ is, by Grothendieck's Completion Theorem, the subspace of $(E')^*$ consisting of all elements that are $\sigma(E', E)$-continuous on the equicontinuous subsets of E'.

The first part of the following lemma is well known, but we give a simple proof here for completeness.

LEMMA. *Let $E[\tau]$ have an extended Schauder basis. Then*

(1) *for each x in E, $\{S_\phi x\}_{\phi \in \Phi}$ is bounded,*
(2) *if the basis is equi-Schauder, $\{S_\phi z\}_{\phi \in \Phi}$ is bounded for each $z \in \tilde{E}$.*

Proof. (1) Let $x' \in E'$. If $x \in E$, $\{S_\phi x\}$ is a Cauchy net; hence $\{\langle S_\phi x, x' \rangle\}$ is Cauchy. There is, therefore, an element ϕ_0 of Φ such that $|\langle S_\phi x, x' \rangle| \leqq 1$ whenever $\phi \cap \phi_0 = \emptyset$. If $M = \sum_{\alpha \in \phi_0} |\langle x, x'_\alpha \rangle| \cdot |\langle x_\alpha, x' \rangle|$, then $|\langle S_\phi x, x' \rangle| \leqq 1 + M < \infty$ for all $\phi \in \Phi$.

(2) Let $x' \in E'$. Then $\{S'_\phi x'\}$ is equicontinuous. If K is the closure of $\{S'_\phi x'\}$ for $\sigma(E', E)$, K is $\sigma(E', E)$-compact, and equicontinuous. Now z is $\sigma(E', E)$-continuous and therefore bounded on K; hence

$$\sup_{\phi \in \Phi} |\langle S_\phi z, x' \rangle| = \sup_{\phi \in \Phi} |\langle z, S'_\phi x' \rangle| < \infty. \text{ So } \{S_\phi z\}_{\phi \in \Phi} \text{ is bounded.}$$

It follows from (1) that the partial summation operators of an extended Schauder basis are pointwise bounded. Hence in a barrelled space any extended Schauder basis is equi-Schauder.

THEOREM 3. *Let* $E[\tau]$ *be a quasi-complete space with an extended equi-Schauder basis. Then* $E[\tau]$ *is complete. In particular, a quasi-complete barrelled space with an extended Schauder basis is complete.*

Proof. Let $z \in \tilde{E}$. By part (2) of the Lemma, $\{S_\phi z\}$ is a bounded set in E. By hypothesis, bounded closed sets are complete. So to prove that $z \in E$ it is sufficient to prove that $\{S_\phi z\}$ is a Cauchy net.

Let $U = K^\circ$ be a τ-neighbourhood of 0 in E, where K is an equicontinuous set. For any $x \in E$, $\sup_{x' \in K} |\langle x, S'_\phi x' \rangle| = \sup_{x' \in K} |\langle S_\phi x, x' \rangle|$; so $\{S'_\phi x'\}$ is $\sigma(E', E)$-Cauchy, uniformly for $x' \in K$. Now $\bigcup_{\phi \in \Phi} S'_\phi(K) = L$ is equicontinuous. Since z is $\sigma(E', E)$-continuous on L, we can find a $\sigma(E', E)$-neighbourhood W of 0 in E', such that $|\langle z, x' \rangle| \leqq 1$ whenever $x' \in W \cap L$. Since $\{S'_\phi x'\}$ is uniformly $\sigma(E', E)$-Cauchy on K, we can find ϕ_0 in Φ such that $S'_\phi x' \in W$ for all $x' \in K$, if $\phi \cap \phi_0 = \emptyset$. Thus

$$\begin{aligned} \phi \cap \phi_0 = \emptyset \Rightarrow S'_\phi x' \in W \cap L \qquad & \text{for all} \quad x' \in K \\ \Rightarrow |\langle z, S'_\phi x' \rangle| \leqq 1 \qquad & \text{for all} \quad x' \in K \\ \Rightarrow |\langle S_\phi z, x' \rangle| \leqq 1 \qquad & \text{for all} \quad x' \in K \\ \Rightarrow S_\phi z \in U, \qquad & \end{aligned}$$

and we have proved that $\{S_\phi z\}$ is a Cauchy net.

This result has been proved by Kalton [7] for countable Schauder bases. Kalton's result is a little better in this case, for he shows that sequential completeness implies completeness for a space with an equi-Schauder basis. However, Theorem 3 cannot be improved in this way. In an uncountable product of lines, let E be the subspace consisting of elements with at most countably many nonzero coordinates, with the product topology. Then E is sequentially complete and barrelled, and has an extended equi-Schauder basis, but is not complete.

Let $E[\tau]$ be a separable non-complete Montel space. (An incorrect construction of such a space was given by Amemiya and Kōmura [1], and corrected by Knowles and Cook [8].) Such a space, as Garling and Kalton [6] pointed out, has no Schauder basis. In fact, since a Montel space is barrelled and quasi-complete, Theorem 3 shows that $E[\tau]$ has no extended Schauder basis.

REFERENCES

1. I. Amemiya and Y. Kōmura, Über nicht-vollständige Montelräume, *Math. Ann.* **177** (1968), 273–277.

2. M. G. Arsove and R. E. Edwards, Generalized bases in topological linear spaces, *Studia Math.* **19** (1960), 95–113.

3. G. Bennett and J. B. Cooper, Weak bases in (*F*)- and (*LF*)-spaces, *J. London Math. Soc.* **44** (1969), 505–508.

4. M. de Wilde and C. Houet, On increasing sequences of absolutely convex sets in locally convex spaces, *Math. Ann.* **192** (1971), 257–261.

5. J. Dieudonné and L. Schwartz, La dualité dans les espaces (*F*) et (*LF*), *Ann. Inst. Fourier* **1** (1950), 61–101.

6. D. J. H. Garling and N. J. Kalton, Colloquium on nuclear spaces and ideals in operator algebras, *Studia Math.* **38** (1970), 474.

7. N. J. Kalton, Schauder decompositions and completeness, *Bull. London Math. Soc.* **2** (1970), 34–36.

8. R. J. Knowles and T. A. Cook, Non-complete reflexive spaces without Schauder bases, *Proc. Cambridge Philos. Soc.* **74** (1973), 83–86.

9. C. W. McArthur, Developments in Schauder basis theory, *Bull. Amer. Math. Soc.* **78** (1972), 877–908.

10. J. T. Marti, A weak basis theorem for non-separable Fréchet spaces, *J. London Math. Soc.* (2) **5** (1972), 8–10.

11. I. Tweddle, Unconditional convergence and bases, *Proc. Edinburgh Math. Soc.*; to appear.

12. J. H. Webb, Schauder bases and decompositions in locally convex spaces, *Proc. Cambridge Philos Soc.* **76** (1974), 145–152.

UNIVERSITY OF CAPE TOWN
RONDEBOSCH
C.P., SOUTH AFRICA