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ALGEBRAIC HOMOTOPY THEORY

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0. Introduction. Kan and Miller have shown in [**9**] that the homotopy type of a finite simplicial set K can be recovered from its R-algebra of 0-forms A^0K, when R is a unique factorization domain. More precisely, if $\mathscr{S}$ is the category of simplicial sets and $\mathscr{A}$ is the category of R-algebras there is a contravariant functor

$$A^0: \mathscr{S} \rightarrow \mathscr{A},$$

with

$$A^0X = \mathscr{S}(X, \nabla),$$

the simplicial set homomorphisms from X to the simplicial R-algebra ∇, where

$$\nabla_n = R[x_0 \dots x_n] \Big/ \left(\sum_{i=0}^{n} x_i - 1\right), \quad n \geqq 0,$$

and the faces and degeneracies of ∇ are induced by

$$d_i x_j = \begin{cases} x_j & j < i \\ 0 & j = i \\ x_{j-1} & j > i \end{cases}$$

and

$$s_i x_j = \begin{cases} x_j & j < i \\ x_i + x_{i+1} & j = i \\ x_{j+1} & j > i, \end{cases}$$

respectively. Of course, if $R = \mathbf{Q}$, A^0X is just the 0-forms portion of the deRham complex A^*X which is used in rational homotopy theory (see [**4**]), and it certainly seems appropriate to call this more general A^0 the 0-forms functor as well (as is done in [**9**]). There is a contravariant functor

$$F^0: \mathscr{A} \rightarrow \mathscr{S},$$

with n-simplices of F^0B defined by

$$(F^0B)_n = \mathscr{A}(B, \nabla_n), \quad n \geqq 0,$$

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for the R-algebra B, and the obvious faces and degeneracies induced by those of ∇. This functor is a close relative of Sullivan's spatial realization of a differential algebra [**14**] and, as in that situation, there is a natural bijection

$$\mathscr{S}(X, F^0B) \cong \mathscr{A}(B, A^0X),$$

making F^0 and A^0 adjoint on the right. What is actually proved in [**9**] is that the corresponding natural map (i.e., the unit of the adjunction)

$$\eta: K \rightarrow F^0A^0K,$$

is a weak (homotopy) equivalence of $\mathscr{S}$ if the simplicial set K is finite in the sense that it has only finitely many non-degenerate simplices, and if the ring of definition R of A^0 and F^0 is any unique factorization domain.

The discussion is begun in § 1 by showing that this map η is not a weak equivalence in general in the case where R is a finite field. This answers the (obvious) question posed in [**9**] in their "Remark 1.2". The discussion requires a mild excursion into the theory of ultrafilters, for which [**3**] is an excellent general reference.

Starting from the Kan-Miller result, one could ask for a closed model structure (in the sense of [**12**]) on the R-algebra category $\mathscr{A}$ in such a way that A^0 and F^0 would induce an equivalence of the corresponding homotopy categories. § 2 of this paper is concerned with repairing the set-theoretic damage done by § 1 to this idea via a passage to the pro-category pro $\mathscr{A}$. The principal results can be paraphrased as follows:

(1) There are functors $\hat{A}: \mathscr{S} \rightarrow$ pro $\mathscr{A}$ and $\hat{F}$: pro $\mathscr{A} \rightarrow \mathscr{S}$ such that
  (a) $\hat{A}$ and $\hat{F}$ are adjoint on the right, and
  (b) there are natural isomorphisms
    (i) $\hat{A}K \cong A^0K$ for K finite, and
    (ii) $\hat{F}B \cong F^0B$ for R-algebras B.
(2) pro $\mathscr{A}$ is a closed model category in such a way that $\hat{A}$ and $\hat{F}$ induce an equivalence of Ho (pro $\mathscr{A}$) with the "usual" homotopy category Ho($\mathscr{S}$).

$\hat{F}$ and $\hat{A}$ are closely related to F^0 and A^0 respectively. It appears to be most fruitful not to think of F^0 or $\hat{F}$ as a realization of some kind, but rather as a good analogue of the ordinary singular functor (to see this apply the Spec functor to the case of F^0). Thus, one calls $\hat{F}$ the algebraic singular functor, and $\hat{A}$ the algebraic realization functor.

In § 3 a few easy observations are noted in an effort to convince the reader that the connection between this homotopy theory and traditional algebraic geometry is somewhat subtle. This phenomenon is a source of hope for future applications of that which follows.

1. Some set theory. Let k be a finite field and let $\mathscr{A}$ be the corresponding category of k-algebras. Explicitly, for a simplical set X the natural map,

$$\eta: X \rightarrow F^0A^0X$$

is defined by

$$\eta(x) = j \circ A^0(i_x): A^0X \rightarrow \nabla_n$$

for $x \in X_n$ and $n \geqq 0$, where $i_x: \Delta^n \rightarrow X$ classifies x and $j: A^0\Delta^n \rightarrow \nabla_n$ is the obvious isomorphism. Ambiguously, write

$$X = \coprod_{x \in X} \Delta^0 \text{ in } \mathscr{S},$$

where X is an infinite set. There is an obvious natural isomorphism

$$\theta: A^0X \rightarrow \prod_{x \in X} k,$$

where $p_x \circ \theta = \eta(x)$ for all $x \in X$.

Now recall that if I is an ideal of $\prod_{x \in X} k$, then the assignment to I of the set

$$F(I) = \{J \in \mathscr{P}(X) | J \text{ is the set of zeros of some } V \in I\},$$

induces an order-preserving bijective correspondence between ideals of $\prod_{x \in X} k$ and filters on X. If an ideal $Q \subset \prod_{x \in X} k$ is prime it is easily shown that $F(Q)$ has the property that for all $J \subset X$, either $J \in F(Q)$ or $X - J \in F(Q)$. It follows [**3**, p. 20] that $F(Q)$ is an ultrafilter, whence Q is a maximal ideal. This essentially proves

LEMMA 1.1. *Every n-simplex of $F^0(\prod_{x \in X} k)$ is degenerate for positive n.*

It follows from Lemma 1.1 that, if $\eta: X \rightarrow F^0A^0X$ is a weak equivalence, then the function

$$\eta_*: X \rightarrow \mathscr{A}(\textstyle\prod_{x \in X} k, k)$$

is a bijection, where $\eta_*(x) = p_x$, the projection to the xth factor. This is because η_* is just $\pi_0(\eta)$ after a few identifications have been made. But η_* cannot be surjective, in view of

PROPOSITION 1.2. *Let k be a finite field and let X be an infinite set with cardinality α. Then* card $(\mathscr{A}(\prod_{x \in X} k, k)) = 2^{2^\alpha}$.

Proof. The rational points $\mathscr{A}(A, k)$ of any k-algebra A can be identified with the collection of ideals $I \subseteq A$ satisfying the property that for all $x \in A$ there is a unique $a \in k$ such that $x - a \in I$. If M is any maximal ideal of $\prod_{x \in X} k$, if $z \in \prod_{x \in X} k$, and if $a_1 \ldots a_q$ is a list of the elements of k, then

$$(z - a_1) \ldots (z - a_q) = 0,$$

and so there is an i such that $z - a_i \in M$. Moreover, this i is unique since $\phi \notin F(M)$. Thus, the rational points of $\prod_{x\in X} k$ are precisely its maximal ideals. According to a standard theorem [**3**, p. 108] there are 2^{2^α} non-principal ultrafilters in $\mathscr{P}(X)$. Thus, there are 2^{2^α} rational points of $\prod_{x\in X} k$ outside of the image of η_*, which has cardinality α. The lemma follows.

One can see that η_* is not onto more easily by observing that $\bigoplus_{x\in X} k$ is an ideal of $\prod_{x\in X} k$, and that any maximal ideal containing it cannot be the kernel of a projection.

It is possible to sketch what happens when k is a countable field. An ultrafilter $F \subset \mathscr{P}(X)$ is *ω-complete* if for every countable partition $X = \coprod_{n\in\omega} X_n$ of X there is a unique n such that $X_n \in F$.

Using Theorem 3.13 of [**3**, Chapter 6] one can show that the rational points of $\prod_{x\in X} k$ are precisely the ω-complete ultrafilters of $\mathscr{P}(X)$. It follows that η_* is not a bijection if and only if there is a non-principal ω-complete ultrafilter in $\mathscr{P}(X)$. The question of the existence or non-existence of such "measurable cardinals" X (i.e., sets X having a non-principal ω-complete ultrafilter in $\mathscr{P}(X)$) is a venerable unsolved problem of analysis. Much more can be said; see [**3**] and [**6**].

2. The pro-category. This section contains the main results of this paper. Some terminology is established at the outset, but no attempt is made for the exposition to be self-contained. The reader who finds a paucity of detail here should consult the Appendix of [**1**].

Recall that a *pro-object* in $\mathscr{A}$ (i.e., an object of the category pro $\mathscr{A}$) is a contravariant functor $T: I \to \mathscr{A}$, where I is a small filtered category (see [**1**, p. 154]). Given another pro-object $S: J \to \mathscr{A}$, a pro-map $\phi: T \to S$ is an element of the set

$$\lim_{\leftarrow j} \lim_{\to i} \mathscr{A}(T_i, S_j).$$

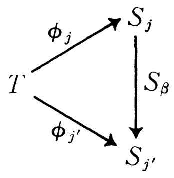
$\mathscr{A}$ is a full subcategory of pro $\mathscr{A}$ in such a way that for $A \in \mathscr{A}$,

$$\text{pro } \mathscr{A}(T, A) = \lim\nolimits_{\to i} \mathscr{A}(T_i, A).$$

This is a set of equivalence classes of maps $\theta: T_i \to A$; the class that θ represents is denoted by $[\theta]$. Thus, a pro-map $\phi: T \to S$ can be thought of as a collection of the simpler kinds of pro-maps

$$\phi_j: T \to S_j, \quad j \in J,$$

such that for each $\beta: j' \to j$ in J, the diagram



commutes in the sense that

$$\phi_{j'} = [S_\beta \circ \tau],$$

if τ represents ϕ_j. This information is summarized by saying that a sequence of maps

$$\phi_j\colon T_{i(j)} \to S_j, \quad j \in J,$$

represents the pro-map ϕ if (ambiguously)

$$[\phi_j] = \phi_j \text{ for every } j \in J.$$

With this terminology it is convenient to say, given another pro-object $U\colon K \to \mathscr{A}$ and a sequence of maps

$$\psi_k\colon S_{j(k)} \to U_k, \quad k \in K,$$

representing the pro-map $\psi\colon S \to U$, that the sequence of compositions

$$T_{i(j(k))} \xrightarrow{\phi_{j(k)}} S_{j(k)} \xrightarrow{\psi_k} U_k, \quad k \in K,$$

represents the composite $\psi \circ \phi\colon T \to U$ in pro $\mathscr{A}$.

Now we describe $\hat{A}$. For $X \in \mathscr{S}$, let $\mathscr{F}(X)$ be the small filtered category which has all finite subcomplexes K of X as objects and all inclusions between them as morphisms. The contravariant functor

$$\hat{A}X\colon \mathscr{F}(X) \to \mathscr{A}$$

is defined on morphisms $i\colon K \to L$ by

$$\hat{A}X(i) = A^0(i)\colon A^0L \to A^0K.$$

If $f\colon X \to Y$ is a simplicial map and K is a finite subcomplex of X then $f(K)$ is a finite subcomplex of Y. Let $f|_K\colon K \to f(K)$ be the restriction of f to K. Then it is easy to see that the sequence

$$A^0(f|_K)\colon A^0(f(K)) \to A^0K, \quad K \in \mathscr{F}(X),$$

represents a pro-map $\hat{A}f\colon \hat{A}Y \to \hat{A}X$, and that the assignment $f \mapsto \hat{A}f$ determines a contravariant functor $\hat{A}\colon \mathscr{S} \to \text{pro } \mathscr{A}$. It is worth noting that $\hat{A}$ is the right Kan extension of the restriction of the composition

$$\mathscr{S} \xrightarrow{A^0} \mathscr{A} \subset \text{pro } \mathscr{A}$$

to the full subcategory $\mathscr{F}$ of finite simplicial sets, along the inclusion of $\mathscr{F}$ in $\mathscr{S}$.

$\hat{F}$: pro $\mathscr{A} \to \mathscr{S}$ is easier to define. For $T \in$ pro $\mathscr{A}$, $\hat{F}T$ is the simplicial set with n-simplices defined by

$$(\hat{F}T)_n = \text{pro } \mathscr{A}(T, \nabla_n), \quad n \geqq 0,$$

and faces and degeneracies induced by those in ∇. Obviously $\hat{F}A = F^0A$ for $A \in \mathscr{A}$, and it is straightforward to show

LEMMA 2.1. *There is a natural map f: $A^0X \to \hat{A}X$ for $X \in \mathscr{S}$, with $f_K = A^0(i)$: $A^0X \to A^0K$ for $K \in \mathscr{F}(X)$, where i is the inclusion of K in X. f is an isomorphism of* pro $\mathscr{A}$ *if X is finite; in this case f^{-1} is represented by 1_{A^0X}.*

There is a natural map

$$\psi\colon \text{pro}\,\mathscr{A}(T, \hat{A}X) \to \mathscr{S}(X, \hat{F}T)$$

such that, for g: $T \to \hat{A}X$, and $x \in X_n$, $n \geqq 0$, $\psi g(x)$ is the composition

$$T \xrightarrow{g} \hat{A}X \xrightarrow{\hat{A}i_x} \hat{A}\Delta^n \underset{\cong}{\xrightarrow{f^{-1}}} A^0\Delta^n \underset{\cong}{\xrightarrow{j}} \nabla_n.$$

In fact, we have

PROPOSITION 2.2. *ψ is a natural bijection, so $\hat{A}$ and $\hat{F}$ are adjoint on the right.*

Proof. By definition, if $X \in \mathscr{S}$ and T: $I \to \mathscr{A}$ is a pro-object, then

$$\text{pro}\,\mathscr{A}(T, \hat{A}X) = \lim_{\leftarrow K} \lim_{\to i} \mathscr{A}(T_i, A^0K),$$

the limits being taken over $K \in \mathscr{F}(X)$ and $i \in I$. But there is a natural isomorphism

$$\lim_{\leftarrow K} \lim_{\to i} \mathscr{A}(T_i, A^0K) \overset{\phi}{\cong} \lim_{\leftarrow K} \lim_{\to i} \mathscr{S}(K, F^0T_i),$$

in view of the adjointness of A^0 and F^0, and a natural map

$$\lim_{\leftarrow K} \lim_{\to i} \mathscr{S}(K, F^0T_i) \xrightarrow{l} \mathscr{S}(X, \hat{F}T)$$

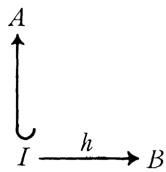
which is gotten by taking colimits. This map l is an isomorphism in view of the fact that

$$\lim_{\to i} \mathscr{S}(K, F^0T_i) \cong \mathscr{S}(K, \hat{F}T)$$

for each finite K. It is an exercise to show that ψ is the composition of the isomorphisms l and ϕ.

Statement (1) of the introduction has now been established. The essence of (2) appears to be

LEMMA 2.3. *Consider the situation*



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where A and B are R-algebras, B is a unique factorization domain, I is a non-zero ideal of A, and h is a non-zero multiplicative R-module homomorphism. Then there is a unique R-algebra homomorphism $h_*: A \to B$ extending h.

Remark. This is a generalization of a Lemma of [**9**]. In fact, the main result of that paper will be obtained here as a corollary of Lemma 2.4.

Proof of Lemma 2.3. Choose $u \in I$ such that $h(u) \neq 0$. Then, for all $x \in A$, there is a $\beta_x \in B$ such that

$$h(xu) = \beta_x h(u).$$

In effect, if $h(xu) = 0$ then obviously $\beta_x = 0$, and if $h(xu) \neq 0$ then use the equations

$$h(xu)^k = h(x^k u)h(u)^{k-1}, \quad k \geqq 2,$$

as in [**9**]. Moreover, since B is an integral domain β_x is unique. Observe that if there is another $v \in I$ such that $h(v) \neq 0$, with corresponding identity

$$h(xv) = \gamma_x h(v),$$

then

$$\beta_x h(u)h(v) = h(xuv) = \gamma_x h(v)h(u),$$

and so $\beta_x = \gamma_x$. Now define a function $h_*: A \to B$ by $h_*(x) = \beta_x$. Clearly h_* is R-linear and extends h. $h_*(xy) = h_*(x)h_*(y)$ if $h_*(x) \neq 0$ and $h_*(y) \neq 0$ since

$$\beta_{xy} h(u) = h(xyu) = \beta_x h(yu) = \beta_x \beta_y h(u).$$

If, for example, $h(yu) = 0$ then $\beta_y = 0$ and so $\beta_x \beta_y = 0$, while

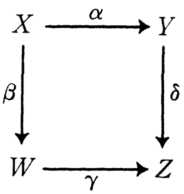
$$h(xyu)h(u) = h(xu)h(yu) = 0,$$

implies that $h(xyu) = 0$, and so $\beta_{xy} = 0$. Finally, h_* is unique from the equations

$$h(xu) = h_*(xu) = h_*(x)h_*(u) = h_*(x)h(u).$$

Some technical lemmas for pro $\mathscr{A}$ will now be listed.

LEMMA 2.4. *Let I be a small filtered category and consider the pullback diagram*



in the category $\mathscr{A}^{I^0}$ *of contravariant functors* $I \to \mathscr{A}$, *where* δ_i: $Y_i \to Z_i$ *is surjective for every* $i \in I$. *Let* $B \in \mathscr{A}$ *be a unique factorization domain. Then the diagram*

$$\begin{array}{ccc} \mathrm{pro}\,\mathscr{A}(Z, B) & \xrightarrow{\gamma^*} & \mathrm{pro}\,\mathscr{A}(W, B) \\ \delta^* \downarrow & & \downarrow \beta^* \\ \mathrm{pro}\,\mathscr{A}(Y, B) & \xrightarrow{\alpha^*} & \mathrm{pro}\,\mathscr{A}(X, B) \end{array}$$

is a pushout of sets.

Proof. Suppose that there is a commutative diagram of sets,

$$\begin{array}{ccc} \mathrm{pro}\,\mathscr{A}(Z, B) & \xrightarrow{\gamma^*} & \mathrm{pro}\,\mathscr{A}(W, B) \\ \delta^* \downarrow & & \downarrow g \\ \mathrm{pro}\,\mathscr{A}(Y, B) & \xrightarrow[f]{} & E \end{array}$$

Take a pro-map ϕ: $X \to B$ and let

$$K_i = \ker \{\delta_i\colon Y_i \twoheadrightarrow Z_i\} = \ker \{\beta_i\colon X_i \twoheadrightarrow W_i\}.$$

Then:

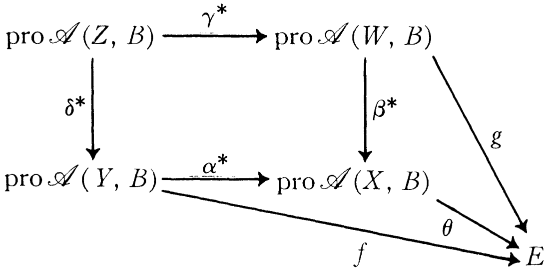
(i) If there is a representative ϕ_i: $X_i \to B$ of ϕ such that $\phi_i(K_i) = 0$, then there is a unique pro-map ψ: $W \to B$ such that $\psi\beta = \phi$.

(ii) If there is no representative ϕ_i of ϕ which kills K_i, then there is a unique pro-map η: $Y \to B$ such that $\eta\alpha = \phi$.

To see (i), observe that β_i: $X_i \to W_i$ is surjective, since it is the pull-back of a surjective map with kernel K_i. Thus, there is a unique ψ_i: $W_i \to B$ such that $\psi_i\beta_i = \phi_i$. Set $\psi = [\psi_i]$. The uniqueness of ψ follows from the fact that β is an epi of pro $\mathscr{A}$ (see Lemma 2.5). For (ii) consider the diagram

$$\begin{array}{ccccc} & & K_i & & \\ & \swarrow & & \searrow & \\ X_i & & \xrightarrow[\alpha_i]{} & & Y_i \\ & \searrow^{\phi_i} & & & \\ & & B & & \end{array}$$

By Lemma 2.3 there is a unique η_i: $Y_i \to B$ such that $\eta_i\alpha_i = \phi_i$. Lemma 2.3 also guarantees that $\eta = [\eta_i]$ is independent of the choice of representative ϕ_i of ϕ, and that η is the unique pro-map such that $\eta\alpha = \phi$. Now, in order to get a commutative diagram



we are obliged to define

$$\theta(\phi) = \begin{cases} g(\psi) & \text{if } \phi \text{ satisfies } (i) \\ f(\eta) & \text{if } \phi \text{ satisfies } (ii). \end{cases}$$

Thus it is clear that all that has to be done in order to finish the proof of the Lemma is to show that $\theta\alpha^* = f$ and $\theta\beta^* = g$, but this a straightforward case check.

COROLLARY 2.4.1. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ W & \xrightarrow[\gamma]{} & Z \end{array}$$

be a pullback diagram of R-algebras in which δ is surjective and let the R-algebra B be a unique factorization domain. Then the diagram

$$\begin{array}{ccc} \mathscr{A}(Z, B) & \xrightarrow{\gamma^*} & \mathscr{A}(W, B) \\ {\scriptstyle\delta^*}\downarrow & & \downarrow{\scriptstyle\beta^*} \\ \mathscr{A}(Y, B) & \xrightarrow[\alpha^*]{} & \mathscr{A}(X, B) \end{array}$$

is a pushout of sets.

Thus, since every $\nabla_n \cong R[x_1 \ldots x_n]$ is a unique factorization domain, we have

COROLLARY 2.4.2. (Kan, Miller). *The natural map*

$$\eta: K \to F^0A^0K$$

is a weak equivalence if K is a finite complex.

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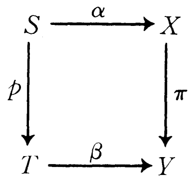
The missing link in Lemma 2.5 was

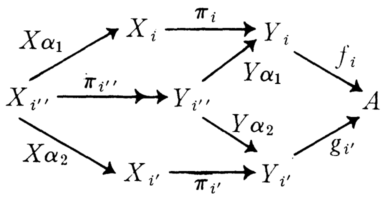
LEMMA 2.5. (i) *Take* $X, Y: I \to \mathscr{A}$ *in* pro $\mathscr{A}$ *and let* $\pi: X \to Y$ *be a natural transformation such that* $\pi_i: X_i \to Y_i$ *is surjective for every* $i \in I$. *Then* π *represents an epimorphism of* pro $\mathscr{A}$.

(ii) *In every pullback diagram*



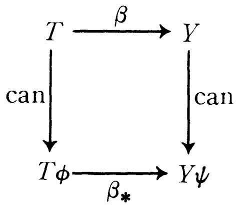
in pro $\mathscr{A}$ *with* π *as above,* p *is epi.*

Proof. (i) Take $f, g: Y \to A$, $A \in \mathscr{A}$, with $f\pi = g\pi$, and let $f_i: Y_i \to A$, $g_{i'}: Y_{i'} \to A$ represent f and g respectively. There is a commutative diagram

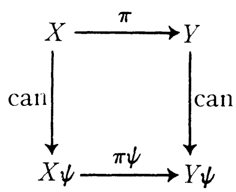


so that $f_i Y\alpha_1 = g_{i'} Y\alpha_2$ and $f = g$. This argument suffices for the general case.

(ii) Suppose that T is the pro-object $T: J \to \mathscr{A}$. By Section 3 of the Appendix of [**1**], there is a small filtered category M_β, cofinal functors $\phi: M_\beta \to J$, $\psi: M_\beta \to I$, a natural transformation $\beta_*: T\phi \to Y\psi$, and a commutative diagram



in pro $\mathscr{A}$, where the vertical arrows are canonical isomorphisms represented by identity maps. Moreover, the diagram



commutes in pro $\mathscr{A}$, where $\pi\psi$ is represented by the obvious natural transformation. Thus, by [**1**, A.4.1] our pullback is isomorphic to the pullback

$$\begin{array}{ccc} S' & \xrightarrow{\alpha'} & X\psi \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi\psi} \\ T\phi & \xrightarrow[\beta_*]{} & Y\psi \end{array}$$

of $\mathscr{A}^{M_0\beta}$. $\pi\psi_m$: $X\psi_m \to Y\psi_m$ is surjective for every $m \in M_\beta$, so p' is an epi of pro $\mathscr{A}$ by (i).

It is necessary at this point to briefly recall the construction of filtered inverse limits in pro $\mathscr{A}$ from [**1**, A.4.4]. Let J be a small filtered category and consider a contravariant functor T: $J \to$ pro $\mathscr{A}$. In particular, we have $T(j)$: $I(j) \to \mathscr{A}$ for every $j \in J$. Let K be the category having as objects all pairs (j, i) with $j \in J$ and $i \in I(j)$, and such that a morphism (α, ϕ): $(j, i) \to (j', i')$ consists of an arrow α: $j \to j'$ of J, together with an $\mathscr{A}$-morphism ϕ: $T(j')_{i'} \to T(j)_i$ representing $T(\alpha)_i$. K is small and filtered. Let L: $K \to \mathscr{A}$ be the pro-object which is defined on morphisms by $L(\alpha, \phi) = \phi$. The pro-maps π_j: $L \to T(j)$ with $(\pi_j)_i$ represented by $1_{T(j)_i}$ form a limiting cone in pro $\mathscr{A}$.

As an example, take $A_i \in \mathscr{A}$, where the i ranges over some index set X. The product of the A_i's in pro $\mathscr{A}$, denoted by $\hat{\prod}_{i \in X} A_i$, is the functor

$$P: \mathscr{F}(X) \to \mathscr{A},$$

where $\mathscr{F}(X)$ denotes the finite subsets of X considered as a small filtered category, and with

$$P(K) = \prod_{i \in K} A_i \quad \text{for } K \in \mathscr{F}(X),$$

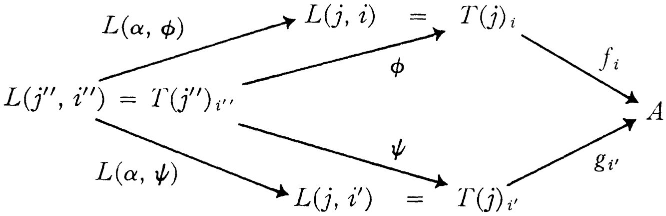
this finite product being taken in $\mathscr{A}$.

LEMMA 2.6. *For* T: $J \to$ pro $\mathscr{A}$ *as above, if* $T(\alpha)$: $T(j') \to T(j)$ *is an epi of* pro $\mathscr{A}$ *for every* α: $j \to j'$ *of* J, *then the maps*

$$\pi_j: L \to T(j), \quad j \in J,$$

are also epi in pro $\mathscr{A}$.

Proof. Take f, g: $T(j) \to A$, with $A \in \mathscr{A}$, such that $f\pi_j = g\pi_j$, and let f_i: $T(j)_i \to A$ and $g_{i'}$: $T(j)_{i'} \to A$ represent f and g respectively. By using the filtered condition on J if necessary we can assume that there is a commutative diagram in $\mathscr{A}$ of the form



But ϕ and ψ both represent $T\alpha$, so $fT\alpha = gT\alpha$ and $f = g$. This argument suffices again for the general case.

COROLLARY 2.6.1. *For T with L as in* 2.6 *and $A \in \mathscr{A}$,*

$$\text{pro } \mathscr{A}(L, A) = \text{pro } \mathscr{A}(\lim_{\leftarrow j} T_j, A) \cong \lim_{\rightarrow j} \text{pro } \mathscr{A}(T_j, A).$$

Proof.

$$\text{pro } \mathscr{A}(\lim_{\leftarrow j} T_j, A) = \bigcup_{j \in J} \text{pro } \mathscr{A}(T_j, A),$$

since every map $L \to A$ is represented by a map $T(j)_i \to A$ of $\mathscr{A}$.

COROLLARY 2.6.2. *Let $B \in \mathscr{A}$ be an integral domain. Then*

$$\text{pro } \mathscr{A}(\textstyle\prod_{i \in I} A_i, B) \cong \coprod_{i \in I} \mathscr{A}(A_i, B)$$

naturally. This isomorphism is natural in B as well.

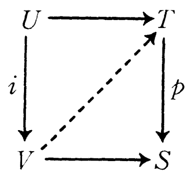
The reader will recall [**12, 13**] that specifying a closed model structure for a category C requires the definition of three classes of maps, called fibrations, cofibrations, and weak equivalences respectively, such that the following axioms are satisfied:

CM1. C is closed under finite direct and inverse limits.

CM2. Given $U \xrightarrow{f} T \xrightarrow{g} S$ in C, if any two of g, f and $g \circ f$ are weak equivalences, then so is the third.

CM3. If f is a retract of g (in the category of arrows of C) and g is a cofibration, fibration, or weak equivalence, then so is f.

CM4. Given a solid arrow diagram



where i is a cofibration and p is a fibration, then a dotted arrow exists making the diagram commutative if either i or p is a weak equivalence.

CM5. Any map f may be factored as

(i) $f = pi$ where i is a cofibration and a weak equivalence and p is a fibration, and

(ii) $f = qj$ where j is a cofibration and q is a fibration and a weak equivalence.

It should be pointed out that the notions of trivial fibration and cofibration, and right and left lifting property have their customary meanings here (see [**12**]). Now say that, in pro $\mathscr{A}$:

(1) $f \colon T \to S$ is a *cofibration* if f has the left lifting property with respect to all maps of the form

$$A^0(i) \colon A^0(\Delta^n) \to A^0(\Lambda_k^n), \quad n \geqq 1, \; 0 \leqq k \leqq n,$$

(2) $f \colon T \to S$ is a *weak equivalence* provided that $\hat{F}f$ is a weak equivalence of $\mathscr{S}$, and

(3) $f \colon T \to S$ is a *fibration* if f has the right lifting property with respect to all cofibrations which are also weak equivalences.

Then we have

THEOREM 2.7. *With these definitions*, pro $\mathscr{A}$ *is a closed model category.*

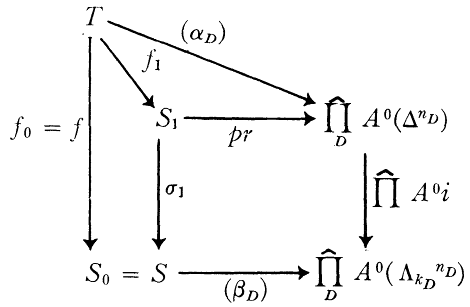
Proof. CM1 comes from [**1**, A.4.2]. CM2 and CM3 are trivial. One shows first of all that the factorization axiom CM5 (ii) holds. Take $f \colon T \to S$ and consider the set of all diagrams D of the form

$$\begin{array}{ccc} T & \xrightarrow{\alpha_D} & A^0(\Delta^{n_D}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle A^0 i} \\ S & \xrightarrow{\beta_D} & A^0(\Lambda_{k_D}^{n_D}) \end{array}$$

Form the pullback

$$\begin{array}{ccc} S_1 & \xrightarrow{pr} & \hat{\prod}_D A^0(\Delta^{n_D}) \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \hat{\prod} A^0 i} \\ S_0 = S & \longrightarrow & \hat{\prod}_D A^0(\Lambda_{k_D}^{n_D}) \end{array}$$

where $\hat{\prod} A^0 i$ is the obvious natural transformation. All of the maps occurring in $\hat{\prod} A^0 i$ are surjective and so σ_1 is epi by 2.5. 2.6, together with the Kan-Miller result, ensures that $\hat{F}(\hat{\prod} A^0 i)$ is a trivial cofibration of $\mathscr{S}$. Thus, by 2.4, so is $\hat{F}\sigma_1$. Let f_1: $T \rightarrow S_1$ be the unique map such that



commutes, and now iterate the construction to produce a tower of maps

$$\ldots \rightarrow S_3 \xrightarrow{\sigma_3} S_2 \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = S,$$

together with a cone

$$f_i \colon T \rightarrow S_i, \quad i \geqq 0.$$

Let $S_\infty = \lim_{\leftarrow i} S_i$, with limiting cone

$$\pi_i \colon S_\infty \rightarrow S_i, \quad i \geqq 0.$$

Then f has a factorization $f = \pi_0 f_\infty$, where f_∞ is the unique map induced by the f_i, $i \geqq 0$. f_∞ is a cofibration, since any β: $S_\infty \rightarrow A^0(\Lambda_k^n)$ factors through some S_m, according to the construction of filtered inverse limits in pro $\mathscr{A}$. It is easy enough to see that π_0 has the right lifting property with respect to all cofibrations, so in particular π_0 is a fibration. Finally, one uses 2.6 to show that $\hat{F}\pi_0$ is a trivial cofibration of $\mathscr{S}$, and so π_0 is a weak equivalence as well as a fibration. The factorization CM5 (i) is obtained similarly, by using the fact that f: $T \rightarrow S$ is a trivial cofibration of pro $\mathscr{A}$ if and only if f has the left lifting property with respect to all maps of the form

$$A^0 i \colon A^0(\Delta^n) \rightarrow A^0(\partial\Delta^n), \quad n \geqq 0,$$

where, by convention, $\partial\Delta^0 = \phi$. The nontrivial part of CM4 is a standard consequence of the construction used for the proof of CM5 (ii).

This section is concluded with

THEOREM 2.8. *$\hat{F}$ and $\hat{A}$ induce an equivalence of* Ho($\mathscr{S}$) *with* Ho(pro $\mathscr{A}$).

Proof. We begin by showing that the unit of the adjunction,

$$\hat{\eta}_X \colon X \rightarrow \hat{F}\hat{A}X,$$

is a weak equivalence for arbitrary $X \in \mathscr{S}$. First of all, one uses the

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naturality of the map f of 2.1 together with the fact that $\hat{\eta}_X(x)$ is the composition

$$\hat{A}X \xrightarrow{\hat{A}i_x} \hat{A}\Delta^n \xrightarrow{f^{-1}} A^0\Delta^n \xrightarrow{j} \nabla_n,$$

to show that there is, for every $X \in \mathscr{S}$, a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{\eta}_X} & \hat{F}\hat{A}X \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \hat{F}f} \\ F^0A^0X & = & \hat{F}A^0X \end{array}$$

Thus, by 2.1 and 2.4.2, $\hat{\eta}_X$ is a weak equivalence for finite $X \in \mathscr{S}$. For the general case, X can be regarded as a filtered colimit

$$X = \varinjlim_{K \in \mathscr{F}(X)} K$$

in $\mathscr{S}$, and by 2.6.1

$$\hat{F}\hat{A}X = \varinjlim_{K \in \mathscr{F}(X)} \hat{F}\hat{A}K.$$

Thus, $\hat{\eta}_X \colon X \to \hat{F}\hat{A}X$ is a weak equivalence, since it is a "filtered colimit" of weak equivalences $\hat{\eta}_K \colon K \to \hat{F}\hat{A}K$, $K \in \mathscr{F}(X)$. Now, the counit of the adjunction

$$\hat{\epsilon}_T \colon T \to \hat{A}\hat{F}T$$

is a weak equivalence of pro $\mathscr{A}$ by the triangle identity

$$\hat{F}\hat{\epsilon}_T \circ \hat{\eta}_{\hat{F}T} = 1_{\hat{F}T}.$$

$\hat{A}$ preserves weak equivalences since $\hat{\eta}$ is a natural weak equivalence, and $\hat{F}$ preserves weak equivalences by definition. The theorem follows easily.

3. Some observations and questions. As pointed out in the introduction, the relationship between this homotopy theory and affine algebraic geometry over an algebraically closed field k is still mysterious, even though they live together, so to speak, within a pro-category. The usual homotopical analysis of a closed model category involves finding a convenient class of fibrant cofibrant objects, like CW complexes in the topological setting, which invades every weak equivalence class. This has not yet been done here. Neither is there a reasonable working model for a homotopy between two maps in pro $\mathscr{A}$, even though such a thing is formally defined. The problem of finding such a model seems to defy ordinary geometric intuition.

A possible reason for this problem is that very natural constructions coming from the simplicial category quickly take one outside the realm of algebras which are finitely generated over k. One might like to believe that for every finite $K \in \mathcal{S}$, A^0K is a finitely generated k-algebra. This is certainly the case if K is an oriented simplicial complex. However, let S^n be the simplicial n-sphere for $n \geqq 1$. It is defined, of course, by the requirement that the following diagram should be a push-out of $\mathcal{S}$:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ \Delta^n & \longrightarrow & S^n \end{array}$$

Then we have

PROPOSITION 3.1. *A^0S^1 is finitely generated as a k-algebra, but A^0S^n is not even Noetherian for $n \geqq 2$.*

Proof. It is easy to check, using the fact that A^0 takes pushouts to pullbacks, that

$$A^0S^n = k + (T),$$

where $k + (T)$ is the smallest subalgebra of $k[x_1 \ldots x_n]$ containing the principal ideal (T), and where

$$T = x_1 \ldots x_n\left(1 - \sum_{i=1}^{n} x_i\right).$$

If $n = 1$, then we have the situation

$$k \subsetneqq A^0S^1 \subset k[x_1],$$

where $k[x_1]$ is an integral extension of A^0S^1 and is finitely generated over k. Then, by [**2**, p. 81] A^0S^1 is finitely generated over k. If $n \geqq 2$, let I_m be the ideal of A^0S^n generated by the set $\{T, x_1T \ldots, x_1^mT\}$, for $m \geqq 0$. If $x_1^{m+1}T \in I_m$ then $x_1^{m+1}T$ has the form

$$x_1^{m+1}T = \sum_{i=0}^{m} (a_i + f_iT)x_1^iT$$

in A^0S^n, where $a_i \in k$ and $f_i \in k[x_1 \ldots x_n]$, so

$$x_1^{m+1} = \sum_{i=0}^{m} (a_i + f_iT)x_1^i$$

in $k[x_1 \ldots x_n]$. Thus, either

(i) $\sum_{i=0}^{m} f_ix_1^i \neq 0,$

and a polynomial in x_1 is divisible by T in $k[x_1, \ldots, x_n]$, or

$$\text{(ii)} \quad \sum_{i=0}^{m} f_i x_1^{\,i} = 0 \quad \text{and} \quad x_1^{m+1} = \sum_{i=0}^{m} a_i x_1^{\,i}.$$

Each is obvious nonsense, so $I_{m+1} \supsetneqq I_m$ for every $m \geqq 0$, and A^0S^n is not Noetherian.

Incidentally, this proposition provides a counterexample to a lemma of [**8**, p. 194], which is incorrect as stated.

It also appears that it is quite difficult to determine what is measured by the homotopy groups and even by the path components of the simplicial set $\hat{F}A$, for an algenra A of finite type over k. For example, think of a curve A as being an integral domain over k subject to the conditions

(i) A is finitely generated as a k-algebra, and
(ii) the transcendance degree of A over k is 1.

For any such object A the k-scheme Spec (A) is connected in the sense that its underlying topological space is irreducible. However, the Laurent polynomial ring $k[x, x^{-1}]$ is a curve and it is easily seen that

$$\hat{F}(k[x, x^{-1}]) = k^*,$$

where k^* denotes the units of k considered as a discrete simplicial group. Clearly then, a distinction must be made between curves A over k which are connected in the geometric sense and those which are path-connected in the sense that $\pi_0(\hat{F}A) = *$. The example $k[x, x^{-1}]$ also makes it clear that path-connectedness is not a local property. Perhaps this is the reason for the apparent unnaturalness of the following classification result.

PROPOSITION 3.2. *The path-connected curves over k are precisely the non-trivial subalgebras of the polynomial ring $k[x_1]$, up to isomorphism.*

Proof. The Noether Normalization Lemma [**2**, p. 69] guarantees that there are no curves which have only one rational point. Thus, if A is connected then there are distinct rational points $\epsilon, \eta: A \rightarrow K$, together with a 1-simplex $\pi: A \rightarrow k[x_1]$ such that $d_0\pi = \epsilon$ and $d_1\pi = \eta$. The image im π is not k, so im π is a curve over k. But then the kernel of π is 0 by [**15**, p. 101], and so $A \cong$ im π. On the other hand, if we have a subalgebra

$$k \subsetneqq A \subset k[x_1],$$

then $k[x_1]$ is an integral extension of A, and so every rational point of A extends over $k[x_1]$ since k is algebraically closed. $k[x_1]$ is obviously path-connected, so A is path-connected as well.

It is the case, however, that interesting computations of some homotopy groups of $\hat{F}A$ for an algebra A of finite type over k can be carried out if one assumes that A has some additional structure, such as that of a Hopf algebra over k. This will be the subject of a future paper.

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