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ON A THEOREM OF D. W. BARNES

BY
ERNEST L. STITZINGER

Let L be a solvable Lie algebra and A be an abelian ideal of L. For $a \in A$, let d_a be the (right) inner derivation of L generated by a and let $\exp d_a = 1 + d_a$. Since A is abelian, $\exp d_a$ is an automorphism and $\exp d_a \exp d_b = \exp d_{a+b}$ for all $a, b \in A$. Let $I(L, A)$ be the subgroup of the automorphism group of L generated by $\exp d_a$ for all $a \in A$. Clearly each element of $I(L, A)$ is of the form $\exp d_a$ for some $a \in A$. Barnes has shown in [1] that if A is a minimal ideal of L and A is its own centralizer in L, then A is complemented in L and all complements are conjugate under $I(L, A)$. It seems natural to ask if A is an arbitrary minimal ideal in L, then how many conjugate classes of complements to A exist in L and if furthermore A is complemented in L, then what are necessary and sufficient conditions such that two complements are conjugate under $I(L, A)$. All Lie algebras considered here are solvable, finite dimensional and have arbitrary ground field. We denote the centralizer of A in L by $C_L(A)$.

THEOREM 1. *Let A be a minimal ideal of the solvable Lie algebra L. There exists a one to one correspondence between the distinct conjugate classes of complements to A under $I(L, A)$ and the L-invariant complement to A in $C_L(A)$.*

Proof. Suppose that M and N are complements to A in L which are conjugate under $I(L, A)$. The $M' = C_L(A) \cap M$ and $N' = C_L(A) \cap N$ are complements to A in $C_L(A)$ and, as is easily verified, M' and N' are each L-invariant. Hence if $a \in A$, then $\exp d_a$ leave M' invariant and it follows that $M' = N'$.

On the other hand, let M' be an L-invariant complement of A in $C_L(A)$. Then $C_{L/M'}(A+M'/M') = A+M'/M'$ and, by the result of Barnes mentioned above, $A+M'/M'$ is complemented in L/M' and all such complements are conjugate under $I(L/M', A+M'/M')$. Let M and N be subalgebras of L containing M' such that M/M' and N/M' are complements to $A+M'/M'$ in L/M'. Clearly M and N are complements to A in L and since each element of $I(L/M', A+M'/M')$ is induced by an element of $I(L, A)$, M and N are conjugate under $I(L, A)$.

COROLLARY. *Let L be a solvable Lie algebra and A be a minimal ideal of L with complements M and N in L. Then M and N are conjugate under $I(L, A)$ if and only if $M \cap C_L(A) = N \cap C_L(A)$.*

THEOREM 2. *Let L be a solvable Lie algebra and A be a minimal ideal of L with complements M and N in L. Then M and N are not conjugate under $I(L, A)$ if and only if $L/(N \cap C_L(A)) \not\cong N/(N \cap M \cap C_L(A))$.*

Proof. Let $N' = N \cap C_L(A)$ and $M' = M \cap C_L(A)$. If M and N are conjugate, then by the Corollary $M' = N'$, hence $L/M' \not\cong N/M' \cap N'$. On the other hand, if M and N are not conjugate, then $N' \neq M'$. In the natural way we may consider A, $C_L(A)/N'$, $M'/(M' \cap N')$, and $C_L(A)/(A+(M' \cap N'))$ as (right) $L/C_L(A)$-modules and these are all $L/C_L(A)$-isomorphic. Now $C_L(A)/N'$ is a minimal ideal which is its own centralizer in L/N'. Consequently L/N' is the split extension of $C_L(A)/N'$ with $L/C_L(A)$. Also, $C_L(A)/(A+(M' \cap N'))$ is a minimal ideal which is its own centralizer in $L/(A+(N' \cap M'))$. Hence $L/(A+(M' \cap N'))$ is the split extension of $C_L(A)/(A+(M' \cap N'))$ by $L/C_L(A)$. From the $L/C_L(A)$-isomorphism of $C_L(A)/N'$ and $C_L(A)/(A+(N' \cap M'))$ follows the isomorphism of L/N' with $L/(A+(N' \cap M'))$ and, consequently, the isomorphism of L/N' with $N/(N' \cap M')$.

Reference

1. D. W. Barnes, *On the cohomology of solvable Lie algebras*, Math. Z. **101** (1967), 343–349.

North Carolina State University,
Raleigh, North Carolina