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ISOMORPHIC INDUCED MODULES AND DYNKIN DIAGRAM AUTOMORPHISMS OF SEMISIMPLE LIE ALGEBRAS

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Abstract. Consider a simple Lie algebra $\mathfrak{g}$ and $\overline{\mathfrak{g}} \subset \mathfrak{g}$ a Levi subalgebra. Two irreducible $\overline{\mathfrak{g}}$-modules yield isomorphic inductions to $\mathfrak{g}$ when their highest weights coincide up to conjugation by an element of the Weyl group W of $\mathfrak{g}$ which is also a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$. In this paper, we study the converse problem: given two irreducible $\overline{\mathfrak{g}}$-modules of highest weight μ and ν whose inductions to $\mathfrak{g}$ are isomorphic, can we conclude that μ and ν are conjugate under the action of an element of W which is also a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$? We conjecture this is true in general. We prove this conjecture in type A and, for the other root systems, in various situations providing μ and ν satisfy additional hypotheses. Our result can be interpreted as an analogue for branching coefficients of the main result of Rajan [**6**] on tensor product multiplicities.

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1. Introduction. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\overline{\mathfrak{g}}$ be a Levi subalgebra with the same Cartan subalgebra so that $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ have the same integral weight lattice (all weights considered in this paper are integral). Let μ and ν be two dominant integral weights for $\overline{\mathfrak{g}}$. Denote by $\overline{V}(\mu)$ and $\overline{V}(\nu)$ the associated highest weight $\overline{\mathfrak{g}}$-modules. Let $\overline{V}(\mu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ and $\overline{V}(\nu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ be the $\mathfrak{g}$-modules obtained by induction from $\overline{\mathfrak{g}}$. When μ and ν are conjugate by an element of the Weyl group W of $\mathfrak{g}$ which is also a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$, the modules $\overline{V}(\mu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ and $\overline{V}(\nu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ are isomorphic; see Proposition 4.4. In this paper, we address the following question: assume $\overline{V}(\mu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ and $\overline{V}(\nu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ are isomorphic, can we conclude that μ and ν are conjugate by an element of the Weyl group W of $\mathfrak{g}$ which is also a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$? We conjecture that this is true in general and we prove the conjecture in type A and in various other cases; see Theorem 7.4.

It is interesting to reformulate the problem in terms of the (infinite) matrix $M = (m_\mu^\lambda)$ with columns and rows labelled respectively by the dominant weights λ of $\mathfrak{g}$ and by the dominant weights μ of $\overline{\mathfrak{g}}$. Here m_μ^λ denotes the branching coefficient corresponding to the multiplicity of the irreducible highest weight $\mathfrak{g}$-module $V(\lambda)$ in $\overline{V}(\mu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}}$ (or equivalently the multiplicity of $\overline{V}(\mu)$ in the restriction of $V(\lambda)$ to $\overline{\mathfrak{g}}$). We then ask if two rows of the matrix M can be equal. Note that two distinct columns of M labelled by λ and Λ cannot coincide since this would imply $V(\lambda) \simeq V(\Lambda)$. Indeed, both modules would then have the same weight decomposition and therefore the same character.

We can also address a similar question for tensor product multiplicities. The corresponding matrix, say C, has columns and rows labelled by dominant weights of $\mathfrak{g}$ and k-tuples $(\mu^{(1)}, \dots, \mu^{(k)})$ of such dominant weights. The coefficients $c^{\lambda}_{\mu^{(1)},\dots,\mu^{(k)}}$ is then the multiplicity of $V(\lambda)$ in $V(\mu^{(1)}) \otimes \cdots \otimes V(\mu^{(k)})$. It was proved by Rajan in [**6**] (see also [**8**] for a shorter proof and an extension to the case of Kac–Moody algebras) that two rows of C are equal if and only if the associated k-tuples of dominant weights coincide up to permutation. It is also easy to see that if the columns of C labelled by λ and κ coincide, then $\lambda = \kappa$ (take $(\mu^{(1)}, \dots, \mu^{(k)}) = (\lambda, 0, \dots, 0)$ and $(\mu^{(1)}, \dots, \mu^{(k)}) = (\kappa, 0, \dots, 0)$).

Finally, one can also consider the decomposition matrix D associated to the modular representation theory of the symmetric group in characteristic p. Its columns and rows are indexed by p-restricted partitions and partitions of n, respectively. The study of possible identical rows and columns was considered by Wildon in [**9**]: the columns of D are distinct and its rows can only coincide in characteristic 2 when the underlying partitions are conjugate.

In the present paper, we prove that two rows of the matrix M corresponding to weights conjugate by an element of the Weyl group W of $\mathfrak{g}$ which is also a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$ coincide. We conjecture that the converse is true and prove this conjecture in various cases (see Theorem 7.4). We believe that the study of the matrix M is more complicated than that of the matrix C for two main reasons. First, there could exist infinitely many nonzero coefficients in a row of M (this is not the case for C). Second, the possible transformations relating the labels corresponding to identical rows in M are more complicated than in the case of the matrix C where they simply correspond to permutations of the k-tuples of dominant weights.

The paper is organised as follows. Section 2 is devoted to some classical background on representation theory of Lie algebras. In Section 3, we study the relationships between the roots, the weights and the Weyl chambers of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$. More precisely, we study the set of elements in W which stabilise the positive roots of $\overline{\mathfrak{g}}$. In Section 4, we formulate our conjecture in terms of equality of distinguish functions in the character ring of $\mathfrak{g}$. This permits in Sections 5 and 6 to prove our conjecture when μ and ν satisfy some technical conditions; see Corollary 5.5 and Proposition 6.4 Finally, in Section 7, we prove the conjecture in the case $\mathfrak{g} = \mathfrak{gl}_n$ using the main result of Rajan [**6**]. This also allows us to establish the conjecture when $\mathfrak{g}$ is a classical Lie algebra of type B_n, C_n or D_n and when $\overline{\mathfrak{g}} = \mathfrak{gl}_n$.

2. Background on Lie algebras. This section is a recollection of classical result on representation theory of Lie algebras. We refer to [**1**] and [**3**] for a detailed exposition. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha},$$

so that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and R_+ its set of positive roots. The root system $R = R_+ \sqcup (-R_+)$ of $\mathfrak{g}$ is realised in a real Euclidean space E with inner product $(\cdot, \cdot)$. For any $\alpha \in R$, we write $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ for its coroot. Let $S \subset R_+$ be the subset of simple roots. The set P of integral weights for $\mathfrak{g}$ satisfies $(\beta, \alpha^\vee) \in \mathbb{Z}$ for any $\beta \in P$ and $\alpha \in R$.

We write $P_+ = \{\beta \in P \mid (\beta, \alpha^\vee) \geq 0 \text{ for all } \alpha \in S\}$ for the cone of dominant weights of $\mathfrak{g}$. Let W be the Weyl group of $\mathfrak{g}$ generated by the reflections s_α with $\alpha \in R_+$ (or equivalently by the simple reflections s_α with $\alpha \in S$). Set $C = \{x \in E \mid (x, \alpha^\vee) > 0$ for all $\alpha \in S\}$ and $\mathrm{cl}(C) = \{x \in E \mid (x, \alpha^\vee) \geq 0$ for all $\alpha \in S\}$. For any $w \in W$, we set

$$C_w = w^{-1}(C), \quad \mathrm{cl}(C_w) = w^{-1}(\mathrm{cl}(C)) \quad \text{and} \quad P_+^w = P \cap \mathrm{cl}(C_w).$$

Each set $w^{-1}(S)$ can be chosen as a set of simple roots for R, the corresponding set of positive roots is then $R_+^w = w^{-1}(R_+)$. Given $w \in W$, we define the dominance order $\leq_w$ on P by the following relation: $\gamma \leq_w \beta$ if and only if $\beta - \gamma$ decomposes as a sum of roots in R_+^w. When $w = 1$, we simply write $\leq$ for the order $\leq_1$.

Now, consider a subset of simple roots $\overline{S} \subset S$. Write $\overline{R} \subset R$ for the parabolic root system generated by $\overline{S}$ and $\overline{R}_+ = \overline{R} \cap R_+$ the corresponding set of positive roots. Let $\overline{\mathfrak{g}} \subset \mathfrak{g}$ be the Levi subalgebra of $\mathfrak{g}$ with set of positive roots $\overline{R}_+$ and triangular decomposition

$$\overline{\mathfrak{g}} = \bigoplus_{\alpha \in \overline{R}_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \overline{R}_+} \mathfrak{g}_{-\alpha}.$$

In particular, $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ have the same Cartan subalgebra. The algebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ have the same integral weight lattice P. Therefore, the weight decomposition of any $\mathfrak{g}$-module is compatible with the weight decomposition of its restriction as a $\overline{\mathfrak{g}}$-module. The Weyl group $\overline{W}$ of $\overline{\mathfrak{g}}$ is generated by the simple reflections s_α with $\alpha \in \overline{S}$. Denote by $\overline{P}_+ \subset P$ the set of dominant integral weights of $\overline{\mathfrak{g}}$. We shall also need the partial order $\preceq$ on P defined by the following relation: $\gamma \preceq \beta$ if and only if $\beta - \gamma$ decomposes as a sum of roots in $\overline{R}_+$.

EXAMPLE 2.1. Consider $\mathfrak{g} = \mathfrak{sp}_{12}$. We have $P = \bigoplus_{i=1}^{n} \mathbb{Z} e_i$,

$$R_+ = \{e_i - e_j \mid 1 \leq i < j \leq 6\} \cup \{e_i + e_j \mid 1 < i < j \leq 6\} \cup \{2e_i \mid 1 \leq i \leq 6\},$$

and

$$P_+ = \{x = (x_1, \ldots, x_6) \in \mathbb{Z}^6 \mid x_1 \geq \cdots \geq x_6 \geq 0\}.$$

The Levi subalgebra $\overline{\mathfrak{g}} \subset \mathfrak{g}$ such that

$$\overline{R}_+ = \{e_1 - e_2, e_1 - e_3, e_2 - e_3\} \cup \{e_4 \pm e_5, e_4 \pm e_6, e_5 \pm e_6\} \cup \{2e_4, 2e_5, 2e_6\},$$

is then isomorphic to $\mathfrak{gl}_3 \oplus \mathfrak{sp}_6$.

Given $\lambda \in P_+$, we denote by $V(\lambda)$ the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ. Let s_λ be the character of $V(\lambda)$. This is an element of the group algebra $\mathbb{Z}[P]$ with basis $\{e^\beta \mid \beta \in P\}$. More precisely

$$s_\lambda = \sum_{\mu \in P} \dim V(\lambda)_\mu e^\mu,$$

where $V(\lambda)_\mu$ is the weight space in $V(\lambda)$ corresponding to μ. Set $\mathbb{G} = \mathbb{Z}[P]^W$. We then have $s_\lambda \in \mathbb{G}$, that is s_λ is symmetric under the action of W. We also recall the Weyl character formula

$$s_\lambda = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$. Note that, for any $w \in W$ and $\beta \in P$, we have $s_{w(\beta)} = \varepsilon(w)s_{w\circ\beta}$ where $\circ$ is the dot action of the Weyl group defined by $w \circ \beta = w(\beta+\rho) - \rho$.

Using the restriction of $V(\lambda)$ to $\overline{\mathfrak{g}}$ we define the branching coefficients m_μ^λ by

$$s_\lambda = \sum_{\mu\in\overline{P}_+} m_\mu^\lambda \overline{s}_\mu,$$

where $\overline{s}_\mu$ is the character of the irreducible representation $\overline{V}(\mu)$ of $\overline{\mathfrak{g}}$ of highest weight μ. We introduce the partition function $\overline{\mathcal{P}}$ defined by

$$\prod_{\alpha\in R_+\setminus\overline{R}_+} \frac{1}{1-e^\alpha} = \sum_{\beta\in P} \overline{\mathcal{P}}(\beta)e^\beta.$$

Then, the branching coefficient m_μ^λ can be computed in term of $\overline{\mathcal{P}}$ using the Weyl character formula (see Corollary 8.2.1 in [**3**, p. 357]) .

Theorem 2.2. *Let $\lambda \in P_+$ and $\mu \in \overline{P}_+$. Then*

$$m_\mu^\lambda = \sum_{w\in W} \varepsilon(w)\overline{\mathcal{P}}(w(\lambda+\rho) - \mu - \rho),$$

where ε is the sign representation of W.

3. Dominant weights of $\overline{\mathfrak{g}}$ and Weyl chambers. This section is devoted to study the relationship between the various subsets of roots and weights we have defined. To this end, we introduce the following subset which will play an important role in this paper:

$$U = \{u \in W \mid u(\overline{R}_+) \subset R_+\}.$$

Proposition 3.1. *We have*

(1)

$$\overline{P}_+ = \bigcup_{u\in U} u^{-1}(P_+).$$

(2)

$$\overline{R}_+ = \bigcap_{u\in U} u^{-1}(R_+).$$

(3) Each element w in W admits a unique decomposition under the form $w = u\overline{w}$ with $u \in U$ and $\overline{w} \in \overline{W}$.

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Proof. We prove 1. Let $\lambda \in P_+$ and $u \in U$. For all $\alpha \in \overline{R}_+$, we have

$$(u^{-1}(\lambda), \alpha^\vee) = (\lambda, u(\alpha)^\vee) \geq 0,$$

since $\lambda \in P_+$ and $u(\alpha) \in R_+$. It follows that $u^{-1}(\lambda) \in \overline{P}_+$ and $\bigcup_{u \in U} u^{-1}(P_+) \subset \overline{P}_+$.

Next, let $\gamma \in \overline{P}_+$. There exists $u' \in W$ such that $u'(\gamma) \in P_+$. Let $\alpha \in \overline{R}_+$. Then $(\gamma, \alpha) = (u'(\gamma), u'(\alpha)) \geq 0$. If the inequality is strict then we have $u'(\alpha) \in R_+$. We set

$$\begin{aligned} R_{>0} &:= \{\beta \in R \mid (u'(\gamma), \beta) > 0\} \subset R_+, \\ R_0 &:= \{\beta \in R \mid (u'(\gamma), \beta) = 0\}, \\ R_{0,+} &:= \{\beta \in R_+ \mid (u'(\gamma), \beta) = 0\},\ R_{0,-} = -R_{0,+}. \end{aligned}$$

Note that, R_0 is a subroot system of R and that the simple system associated to $R_{0,+}$ consists simply of $R_{0,+} \cap S$. Also, since $u(\gamma) \in P_+$, we have $R_+ = R_{>0} \cup R_{0,+}$. Let $W_0 = \langle s_\beta \mid \beta \in R_0 \rangle$. The group W_0 then acts on R and stabilises both R_0 and $R_{>0}$. Since all the roots in R_0 are orthogonal to $u'(\gamma)$, we have $vu'(\gamma) = u'(\gamma) \in P_+$ for all $v \in W_0$. Now, let u be the element of minimal length in the coset $W_0 u'$. By the previous argument, we have $u'(\gamma) \in P_+$. Let us show that $u \in U$. Let $\alpha \in \overline{R}_+$. First if $u'(\alpha) \in R_{>0}$, then so is $u(\alpha)$ since W_0 stabilises $R_{>0}$ and we are done in this case since $u(\alpha) \in R_{>0} \subset R_+$. Second, if $u'(\alpha) \in R_0$, then so is $u(\alpha)$. Let $\delta \in R_{0,+} \cap S$. Since u is of minimal length, we have $\ell(s_\delta u) > \ell(u)$ (here ℓ is the length function) and this implies that $u^{-1}(\delta) \in R_+$ (see for example [**4**, Section 1.6]). It follows that $u^{-1}(\beta)$ is positive for all $\beta \in R_{0,+}$. Therefore, we cannot have $u(\alpha) = -\beta \in R_{0,-}$ with $\beta \in R_{0,+}$, since this would imply that $u^{-1}(\beta) = -\alpha \in R_-$. We have shown that $u(\alpha) \in R_+$ in both cases, that is $u \in U$ as required.

We prove 2. By definition of U, we have $\overline{R}_+ \subset \bigcap_{u \in U} u^{-1}(R_+)$. Assume $\alpha \in \bigcap_{u \in U} u^{-1}(R_+)$. We then have $u(\alpha) \in R_+$ for any $u \in U$. Consider $\gamma \in \overline{P}_+$. By assertion 1, there exists $u \in U$ such that $\gamma \in u^{-1}(P_+)$. We thus have $(\gamma, \alpha^\vee) = (u(\gamma), u(\alpha)^\vee) \geq 0$ for any $\gamma \in \overline{P}_+$. This implies that α is a positive root of $\overline{R}_+$.

We prove 3. Recall that the stabiliser of ρ under W is $\{1\}$. Consider $w \in W$. There exists $\overline{w} \in \overline{W}$ such that $\overline{w}(w^{-1}\rho) \in \overline{P}_+$. By assertion 1, there exists $u \in U$ such that $u\overline{w}(w^{-1}\rho) \in P_+$. Since ρ is the unique element of the orbit $W\rho$ in P_+, we must have $w = u\overline{w}$. Now, assume that there exist $u_1, u_2 \in U$ and $\overline{w}_1, \overline{w}_2 \in \overline{W}$ such that $u_1\overline{w}_1 = u_2\overline{w}_2$. We have $u_2 = u_1\overline{w}$ with $\overline{w} = \overline{w}_1\overline{w}_2^{-1} \in \overline{W}$. If $\overline{w} \neq 1$, there exists $\alpha \in \overline{R}_+$ such that $\overline{w}(\alpha) = -\beta$ with $\beta \in \overline{R}_+$. Then $(\rho, u_2(\alpha)^\vee) = -(\rho, u_1(\beta)^\vee) < 0$ since $u_1(\beta) \in R_+$. This contradicts the hypothesis $u_2(\alpha) \in R_+$. hence $\overline{w} = 1$, that is $\overline{w}_1 = \overline{w}_2$ and $u_1 = u_2$. □

Denote by $\overline{E}$ the $\mathbb{Q}$-vector space generated by the roots in $\overline{R}_+$. Then, we have $\overline{E} \cap R_+ = \overline{R}_+$; see [**4**, Section 1.10]. We will make frequent use of this fact in the rest of the paper. It is important to notice that this holds because we assumed that $\overline{S} \subset S$.

Lemma 3.2. *Let $u \in U$. Then, $u(\overline{\rho}) = \overline{\rho}$ if and only if $u(\overline{R}_+) = \overline{R}_+$.*

Proof. Assume that there exists $\alpha \in \overline{R}_+$ such that $u(\alpha) \notin \overline{R}_+$. Then, since $u(\alpha) \in R_+$ we have $u(\alpha) \notin \overline{E}$. It follows that there exists a simple root $\alpha_j \notin \overline{R}_+$ such that $u(\alpha) \geq \alpha_j$. On the one hand, since $u(\overline{R}_+) \subset R_+$, we see that $u(\overline{\rho}) \geq \alpha_j$. We also know that $\overline{\rho} \in \overline{E}$. Therefore, the root α_j appears (with a positive coefficient) in the decomposition of $u(\overline{\rho}) - \overline{\rho}$ in the basis S. We get that $u(\overline{\rho}) \neq \overline{\rho}$ as required. The converse is trivial. □

Lemma 3.3. *Let $u \in U$ be such that $u(\overline{\rho}) \neq \overline{\rho}$. Then, $u(\overline{\rho}) \not< \overline{\rho}$.*

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Proof. Since $u(\overline{R}_+) \neq \overline{R}_+$, arguing as in the proof of the previous lemma, we know that there exists a simple root $\alpha_j \notin \overline{E}$ such that α_j appears with a positive coefficient in the decomposition of $u(\overline{\rho}) - \overline{\rho}$ in the basis S. Hence, we cannot have $u(\overline{\rho}) < \overline{\rho}$. □

LEMMA 3.4. *Let $\gamma, \gamma' \in P$ be such that $\gamma \leq_{\overline{R}_+} \gamma'$. Then, we have $u(\gamma) \leq_{R_+} u(\gamma')$ for all $u \in U$.*

Proof. By definition $\gamma \geq_{\overline{R}_+} \gamma'$ implies that $\gamma - \gamma'$ is a sum of roots in $\overline{R}_+$. Since, $u(\overline{R}_+) \subset R_+$ we see that $u(\gamma - \gamma')$ is a sum of roots in R_+. Hence, $u(\gamma - \gamma') = u(\gamma) - u(\gamma') \geq_{R_+} 0$ as required. □

LEMMA 3.5. *Let $\gamma \in P$ be such that $\gamma \notin \overline{P}_+$. Then, we have $u(\gamma) \notin P_+$ for all $u \in U$.*

Proof. Since $\gamma \notin \overline{P}_+$, there exists $\alpha \in \overline{R}_+$ such that $(\gamma, \alpha^\vee) < 0$. It follows that

$$(u(\gamma), u(\alpha)^\vee) = (\gamma, \alpha^\vee) < 0.$$

Since $u(\alpha) \in R_+$, this implies that $u(\gamma) \notin P_+$. □

4. Induced characters.

4.1. The functions H_μ. Given $\mu \in \overline{P}_+$, write $H_\mu := \mathrm{char}(V(\mu) \uparrow_{\overline{\mathfrak{g}}}^{\mathfrak{g}})$ for the induced character of $\overline{V}(\mu)$ from $\overline{\mathfrak{g}}$ to $\mathfrak{g}$. We then have

$$H_\mu := \sum_{\lambda \in P_+} m_\mu^\lambda s_\lambda.$$

Observe there can exist infinitely many weights λ such that $m_\mu^\lambda \neq 0$. When $\overline{\mathfrak{g}} = \mathfrak{h}$ is reduced to the Cartan subalgebra, we have $\overline{R}_+ = \emptyset$ and we set $m_\mu^\lambda = K_{\lambda,\mu} = \dim V(\lambda)_\mu$ so that

$$h_\mu := \sum_{\lambda \in P_+} K_{\lambda,\mu} s_\lambda. \tag{1}$$

Also when $\overline{\mathfrak{g}} = \mathfrak{g}$, we have $H_\mu = s_\mu$. So the function H_μ interpolates between the functions h_μ and s_μ. Since $K_{\lambda,\mu} = K_{\lambda,w(\mu)}$ for any $w \in W$, we have $h_\mu = h_{w(\mu)}$ (for the usual action of W on P). Moreover, $K_{\mu,\mu} = 1$ and $K_{\lambda,\mu} \neq 0$ if and only if $\lambda \geq \mu$ (i.e. $\lambda - \mu$ decomposes as a sum of simple roots). The sets $\{s_\lambda \mid \lambda \in P_+\}$ and $\{h_\lambda \mid \lambda \in P_+\}$ are bases of $\mathbb{G}$ and the corresponding transition matrix is unitriangular for the order $\leq$.

We now define two $\mathbb{Z}$-linear maps H and S by

$$H : \begin{cases} \mathbb{Z}[P] \to \mathbb{G} \\ e^\beta \mapsto h_\beta \end{cases} \text{and } S : \begin{cases} \mathbb{Z}[P] \to \mathbb{G} \\ e^\beta \mapsto s_\beta \end{cases}.$$

Set

$$\Delta = \prod_{\alpha \in R_+} (1 - e^\alpha)$$

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PROPOSITION 4.1. *The maps H and S satisfy the relations*

$$S(e^{\beta}) = H(\Delta e^{\beta}) \text{ and } H(e^{\beta}) = S(\Delta^{-1}e^{\beta}),$$

for any $\beta \in P$. Therefore, $S = H \circ \Delta$ and $H = S \circ \Delta^{-1}$ (by writing for short Δ and Δ^{-1} for the multiplication by Δ and Δ^{-1} in $\mathbb{Z}[[P]]$).

Proof. The partition function $\mathcal{P}$ is defined by

$$\Delta^{-1} = \prod_{\alpha\in R_+} \frac{1}{1-e^{\alpha}} = \sum_{\gamma\in P} \mathcal{P}(\gamma)e^{\gamma},$$

and we have by definition $h_\beta = \sum_\lambda K_{\lambda,\beta}s_\lambda$ where $K_{\lambda,\beta} = \sum_w \varepsilon(w)\mathcal{P}(w\circ\lambda - \beta)$. This gives

$$S(\Delta^{-1}e^{\beta}) = \sum_{\gamma\in P} \mathcal{P}(\gamma)s_{\beta+\gamma}.$$

Let $\gamma \in P$. Then either $s_{\beta+\gamma} = 0$ or there exists $\lambda \in P_+$ and $w \in W$ such that $w^{-1} \circ (\beta+\gamma) = \lambda$, that is $\gamma = w\circ\lambda - \beta$. This yields $s_{\beta+\gamma} = \varepsilon(w)s_\lambda$ and in turn we obtain

$$S(\Delta^{-1}e^{\beta}) = \sum_{\lambda\in P_+}\sum_{w\in W} \varepsilon(w)\mathcal{P}(w\circ\lambda-\beta)s_\lambda = \sum_{\lambda\in P_+} K_{\lambda,\beta}s_\lambda = h_\beta,$$

as desired. Note that, we have for any $U \in \mathbb{Z}[P]$, $H(U) = S(\Delta^{-1}U)$. Then if we set $U = \Delta e^{\beta}$, we get the relation $H(\Delta e^{\beta}) = S(e^{\beta})$, as required. □

Define the $\mathbb{Z}$-linear map

$$\overline{H} : \begin{cases} \mathbb{Z}[P] \to \mathbb{G} \\ e^{\mu} \mapsto H_\mu \end{cases}.$$

Set

$$\overline{\Delta} = \prod_{\alpha\in R_+\setminus\overline{R}_+} (1-e^{\alpha}) \text{ and } \overline{\nabla} = \prod_{\alpha\in\overline{R}_+}(1-e^{\alpha}),$$

PROPOSITION 4.2.

(1) The maps $\overline{H}$ and S satisfy the relation

$$\overline{H}(e^{\mu}) = S(\overline{\Delta}^{-1}e^{\mu}),$$

for any $\mu \in P$. We write for short $\overline{H} = S \circ \overline{\Delta}^{-1}$.

(2) We have $\overline{H}(e^{\mu}) = H(\overline{\nabla}e^{\mu})$.

Proof. The first assertion is proved as in the previous proof by replacing the partition function $\mathcal{P}$ by $\overline{\mathcal{P}}$. For the second one, we combine the first part with the previous proposition. □

We have, using the Weyl character formula for $\overline{\mathfrak{g}}$:

$$\overline{\nabla} = \prod_{\alpha\in\overline{R}_+}(1-e^{\alpha}) = \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w})e^{\overline{\rho}-\overline{w}(\overline{\rho})},$$

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where $\overline{\rho}$ is the half sum of positive roots of $\overline{\mathfrak{g}}$. By the second assertion of the previous proposition, we get for all $\mu \in P$

$$H_\mu = \overline{H}(e^\mu) = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) h_{\mu+\overline{\rho}-\overline{w}(\overline{\rho})}.$$

4.2. Irreducible components of $\overline{R}$. Now, assume the reductive Lie algebra $\overline{\mathfrak{g}}$ decomposes in the form

$$\overline{\mathfrak{g}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

where each $\mathfrak{g}_k$, $k = 1, \ldots, r$ is a Lie subalgebra of $\mathfrak{g}$ with irreducible root system $R_k \subset \overline{R}$ and $\overline{R} = \bigsqcup_{k=1}^r R^{(k)}$. We also assume that we have $P = P^{(1)} \oplus \cdots \oplus P^{(r)}$ where $P^{(k)}$ is the weight lattice of $\mathfrak{g}_k$. In particular, each weight $\mu \in P_+$ decomposes in the form $\mu = \mu^{(1)} + \cdots + \mu^{(r)}$ with $\mu^{(k)} \in P_+^{(k)}$. We then have additional properties for the functions H_μ we shall need in Section 7.1. For instance

$$\overline{\nabla} = \prod_{\alpha \in \overline{R}_+} (1 - e^\alpha) = \prod_{k=1}^{r} \prod_{\alpha \in R_+^{(k)}} (1 - e^\alpha),$$

and

$$H_\mu = \prod_{k=1}^{r} \prod_{\alpha \in R_+^{(k)}} (1 - e^\alpha) h_{\mu^{(1)}+\cdots+\mu^{(r)}}.$$

Combining (1) and Proposition 4.1 (for each root system R_k), we get for any $k = 1, \ldots, r$

$$\prod_{\alpha \in R_+^{(k)}} (1 - e^\alpha) h_{\mu^{(1)}+\cdots+\mu^{(r)}} = \sum_{\lambda^{(k)} \in P_+^{(k)}} K^{-1}_{\lambda^{(k)},\mu^{(k)}} h_{\mu^{(1)}+\cdots\lambda^{(k)}+\cdots+\mu^{(r)}},$$

where the coefficients $K^{-1}_{\lambda^{(k)},\mu^{(k)}}$ are those of the inverse matrix of $(K_{\lambda^{(k)},\mu^{(k)}})_{\lambda^{(k)},\mu^{(k)} \in P_+^{(k)}}$. By an easy induction, we obtain

$$H_\mu = \sum_{\lambda^{(1)} \in P_+^{(k)}} \cdots \sum_{\lambda^{(r)} \in P_+^{(r)}} K^{-1}_{\lambda^{(1)},\mu^{(1)}} \cdots K^{-1}_{\lambda^{(r)},\mu^{(r)}} h_{\lambda^{(1)}+\cdots+\lambda^{(r)}}. \tag{2}$$

4.3. The conjecture. We start with an easy observation.

LEMMA 4.3. *Consider $u \in W$. Then the two following statements are equivalent :*

(1) $u(\overline{R}_+) = \overline{R}_+$
(2) *u is a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$*

Proof. When u is a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$, we clearly have $u(\overline{R}_+) = \overline{R}_+$. Now, assume $u(\overline{R}_+) = \overline{R}_+$. Then, we have $u(\overline{R}) = \overline{R}$ and u is an automorphism of the root system $\overline{R}$. It is known (see [7]) that $\mathrm{Aut}(\overline{R}) = \overline{W} \ltimes \mathrm{Aut}(\overline{\Gamma})$ where $\overline{\Gamma}$ is the Dynkin diagram of $\overline{R}$ i.e. $\mathrm{Aut}(\overline{R})$ is the semidirect product of $\overline{W}$ (which is normal

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in $\mathrm{Aut}(\overline{R})$) with $\mathrm{Aut}(\overline{\Gamma})$. Since, $u(\overline{R}_+) = \overline{R}_+$ the element u belongs in fact in $\mathrm{Aut}(\overline{\Gamma})$ (otherwise u would send at least a positive root of $\overline{R}_+$ on a negative root). □

PROPOSITION 4.4. *Let $\mu, \nu \in \overline{P}_+$. Assume that there exists $u \in W$ such that $u(\overline{R}_+) = \overline{R}_+$ and $\nu = u(\mu)$ (or equivalently, μ and ν are conjugate by a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$ lying in the Weyl group of $\mathfrak{g}$). Then, $H_\mu = H_\nu$.*

Proof. With the previous notation, we have

$$H_\mu = H\Big(\prod_{\alpha\in\overline{R}_+}(1-e^\alpha)e^\mu\Big) \quad \text{and } H_\nu = H\Big(\prod_{\alpha\in\overline{R}_+}(1-e^\alpha)e^\nu\Big).$$

Since $u(\overline{R}_+) = \overline{R}_+$, we see that $u(\overline{\rho}) = \overline{\rho}$ and that $u\overline{W}u^{-1} = \overline{W}$ (indeed, $us_\alpha u^{-1} = s_{u\alpha}$ for all $\alpha \in \overline{R}$). Therefore

$$\prod_{\alpha\in\overline{R}_+}(1-e^\alpha)e^\nu = \sum_{w\in\overline{W}}\varepsilon(w)e^{\nu+\overline{\rho}-w(\overline{\rho})} = \sum_{w\in\overline{W}}\varepsilon(w)e^{u(\mu)+u(\overline{\rho})-uw(u^{-1}(\overline{\rho}))}$$
$$= \sum_{w\in\overline{W}}\varepsilon(w)e^{u(\mu+\overline{\rho}-w(\overline{\rho}))}.$$

It follows that

$$H_\nu = H\Big(\sum_{w\in\overline{W}}\varepsilon(w)e^{u(\mu+\overline{\rho}-w(\overline{\rho}))}\Big) = \sum_{w\in\overline{W}}\varepsilon(w)h_{u(\mu+\overline{\rho}-w(\overline{\rho}))} = \sum_{w\in\overline{W}}\varepsilon(w)h_{\mu+\overline{\rho}-w(\overline{\rho})} = H_\mu,$$

since $h_{w(\beta)} = h_\beta$ for any $w \in W$. □

We conjecture that the converse is true:

CONJECTURE 4.5. Consider $\mu, \nu \in \overline{P}_+$. Then, we have $H_\mu = H_\nu$ if and only if there exists u in W such that $u(\overline{R}_+) = \overline{R}_+$ and $\nu = u(\mu)$ or equivalently, μ and ν are conjugate by a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$ lying in the Weyl group of $\mathfrak{g}$.

5. Triangular decomposition of H_μ.

5.1. Decomposition on the h-basis. Let $\mu \in \overline{P}_+$ and let $w \in U$ be such that $\mu \in \mathrm{cl}(C_w)$. Recall that $R_+^w = w^{-1}(R_+)$. Since $w \in U$, we have $w(\overline{R}_+) \subset R_+$ which in turn implies that $\overline{R}_+ \subset R_+^w$. It follows that $\preceq$ is finer than $\leq_w$, that is $\alpha \preceq \beta \implies \alpha \leq_w \beta$ for all $\alpha, \beta \in P$.

PROPOSITION 5.1. *Let $w \in U$. We have for all $\mu \in \overline{P}_+$*

$$H_\mu = h_\mu + \sum_{\substack{\lambda\in P_+^w \\ \mu<_w\lambda}} a_{\lambda,\mu}h_\lambda,$$

where for any $\lambda \in P_+^w$

$$a_{\lambda,\mu} = \sum_{\substack{\overline{w}\in\overline{W} \\ \mu+\overline{\rho}-\overline{w}(\overline{\rho})\in W\lambda}} \varepsilon(\overline{w}).$$

Proof. Since $\preceq$ is finer than $\leq_w$, we have

$$H_\mu = h_\mu + \sum_{\overline{w} \in \overline{W} \setminus \{1\}} \varepsilon(\overline{w}) h_{\mu+\overline{\rho}-\overline{w}(\overline{\rho})} \text{ with } \mu <_w \mu + \overline{\rho} - \overline{w}(\overline{\rho}) \text{ for } \overline{w} \neq 1.$$

Now for each $w \neq 1$, the orbit of each $\gamma = \mu + \overline{\rho} - \overline{w}(\overline{\rho})$ intersects P_+^w at one point (say λ) and we can use the relations $h_{w(\gamma)} = h_\gamma$ for any $w \in W$. Moreover, we then have $\gamma \leqslant_w \lambda$. We thus obtain $\mu <_w \mu + \overline{\rho} - \overline{w}(\overline{\rho}) \leqslant_w \lambda$ which gives the unitriangularity of the decomposition. The coefficients $a_{\lambda,\mu}$ are then obtained by gathering the contributions in h_λ for each $\lambda \in P_+^w$. □

REMARK 5.2.

(1) For $\mathfrak{g} = \overline{\mathfrak{g}}$, the coefficients $a_{\lambda,\mu}$ are the entries of the inverse matrix K^{-1} where $K = (K_{\lambda,\mu})_{\lambda,\mu \in P_+}$. In type A, K is the Kostka matrix. Obtaining a combinatorial formula for the coefficients of K^{-1} is already a nontrivial problem (see **[2]** and the references therein). As far as we are aware, no such description for the coefficients of K^{-1} exists for other root systems (and thus also for the coefficients $a_{\lambda,\mu}$ associated to a general Levi subalgebra).
(2) We can also deduce from Propositions 3.1 and 5.1 that for any $u \in U$, the set $\{H_\lambda \mid \lambda \in P_+^u\}$ is a basis of $\mathbb{G}$.

5.2. Consequences.

PROPOSITION 5.3. *Let μ and ν be dominant weights in $\overline{P}_+$ such that $H_\mu = H_\nu$. Then, there exists $\tau \in W$ such that $\tau(\nu) = \mu$. In particular, if μ and ν belong to the same closed Weyl chamber for $\mathfrak{g}$, we have $\tau = 1$ and $\mu = \nu$.*

Proof. Assume that μ belongs to $\overline{P}_+^w$ and ν belongs to $\overline{P}_+^{w'}$ with w, w' in U. Let $\tau \in W$ be such that $w' = w\tau$. We then have $R_+^{w'} = \tau^{-1}(R_+^w)$ and $P_+^{w'} = \tau^{-1}(P_+^w)$. Moreover, $\mu <_w \gamma$ if and only if $\tau^{-1}(\mu) <_{w'} \tau^{-1}(\gamma)$. On the one hand, using Proposition 5.1, we get

$$H_\nu = h_\nu + \sum_{\substack{\lambda \in P_+^{w'} \\ \nu <_{w'} \lambda}} a_{\lambda,\nu} h_\lambda = h_\nu + \sum_{\substack{\lambda \in P_+^{w} \\ \tau(\nu) <_{w} \lambda}} a_{\tau^{-1}(\lambda),\nu} h_{\tau^{-1}(\lambda)}.$$

Since $h_{w(\beta)} = h_\beta$ for all $w \in W$ and $\beta \in P$, this can be rewritten under the form

$$H_\nu = h_\nu + \sum_{\substack{\lambda \in P_+^{w} \\ \tau(\nu) <_{w} \lambda}} a_{\tau^{-1}(\lambda),\nu} h_\lambda.$$

On the other hand, we have

$$H_\mu = h_\mu + \sum_{\substack{\lambda \in P_+^{w} \\ \mu <_{w} \lambda}} a_{\lambda,\mu} h_\lambda.$$

So, $H_\nu = H_\mu$ implies that $h_{\tau(\nu)} = h_\mu$ by comparing the indices of the basis vectors of $\{h_\lambda \mid \lambda \in P_+^w\}$ which are minimal for the order $\leq_w$. Hence, $\mu = \tau(\nu)$ as desired. □

REMARK 5.4. If $H_\mu = H_0$ (i.e. we have $\nu = 0$), then $\mu = 0$ since μ and 0 always belong to the same closed Weyl chamber.

For any weight $\mu \in \overline{P}_+$, define the set

$$E_\mu = \{\mu + \overline{\rho} - \overline{w}(\overline{\rho}) \mid \overline{w} \in \overline{W}\}.$$

Since the stabiliser of $\overline{\rho}$ under the action of $\overline{W}$ is $\{1\}$, the cardinality of E_μ is equal to that of $\overline{W}$. The following corollary shows that the conjecture holds when each of the sets E_μ and E_ν is contained in a closed Weyl chamber. This happens in particular when μ and ν are sufficiently far from the walls of the Weyl chambers in which they appear.

COROLLARY 5.5. *Let μ and ν be two dominant weights in $\overline{P}_+$. Assume that there exist $w \in W$ such that $E_\mu \subset P_+^w$ and $w' \in W$ such that $E_\nu \subset P_+^{w'}$. Then, $H_\mu = H_\nu$ implies that $\nu = \tau(\mu)$ and $\tau(\overline{R}_+) = \overline{R}_+$ where $\tau = w^{-1}w'$.*

Proof. All the elements of E_μ belong to P_+^w. They thus belong to distinct W-orbits. Hence, the decomposition of H_μ in the basis $\{h_\lambda \mid \lambda \in P_+^w\}$ is

$$H_\mu = h_\mu + \sum_{\overline{w} \in \overline{W} \setminus \{1\}} \varepsilon(\overline{w}) h_{\mu + \overline{\rho} - \overline{w}(\overline{\rho})}.$$

Similarly, the elements of E_ν belong to distinct W-orbits. Hence, the decomposition of H_ν in the basis $\{h_\lambda \mid \lambda \in P_+^{w'}\}$ is

$$H_\nu = h_\nu + \sum_{\overline{w}' \in \overline{W} \setminus \{1\}} \varepsilon(\overline{w}') h_{\nu + \overline{\rho} - \overline{w}'(\overline{\rho})}.$$

Since $H_\nu = H_\mu$, we see that there exists $\tau \in W$ such that $\tau(\nu) = \mu$ by the previous proposition. Further, we know that τ is such that $P_+^{w'} = \tau^{-1}(P_+^w)$ thus we have $\tau(E_\nu) = E_\mu$. Let $\alpha \in \overline{R}_+$ and $\overline{w} = s_\alpha$. Then, $\overline{w}(\overline{\rho}) - \overline{\rho} = \alpha$ and we see that there exists an element $\overline{w}' \in \overline{W}$ such that $\tau(\nu + \alpha) = \mu + \overline{\rho} - \overline{w}'(\overline{\rho})$. In turn, this implies $\tau(\alpha) = \overline{\rho} - \overline{w}'(\overline{\rho})$ as $\tau(\nu) = \mu$ and $\tau(\alpha)$ is a sum of positive roots in $\overline{R}_+$. But $\tau(\alpha)$ also lies in R, hence $\tau(\alpha) \in \overline{R}_+$; see Section 3. We have shown that τ maps $\overline{R}_+$ onto itself as expected. □

6. The functions M_μ. We now give an equivalent formulation of our problem in terms of parabolic analogues of monomial functions.

6.1. Decomposition on the monomial functions. For any weight $\gamma \in P$, set $\mathrm{m}_\gamma = \sum_{w \in W} e^{w(\gamma)}$ so that m_γ is the image of e^γ by the symmetrisation operator

$$\mathcal{M} : \begin{cases} \mathbb{Z}[P] \to \mathbb{Z}[P]^W \\ e^\gamma \mapsto \mathrm{m}_\gamma \end{cases}.$$

Note that, our function m_γ slightly differs from the usual monomial function $m_\gamma = \frac{1}{|W_\gamma|} \sum_{w \in W} e^{w(\gamma)}$ where W_γ is the stabiliser of γ under the action of W. We clearly have

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$\mathrm{m}_{w(\gamma)} = \mathrm{m}_\gamma$ for any $w \in W$. Also, $\{\mathrm{m}_\lambda \mid \lambda \in P_+^w\}$ is a basis of $\mathbb{G}$. Given $\mu \in P$, set

$$M_\mu := \mathcal{M}\Big(\prod_{\alpha \in \overline{R}_+}(1-e^\alpha)e^\mu\Big) = \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w})\mathrm{m}_{\mu+\overline{\rho}-\overline{w}(\overline{\rho})}.$$

LEMMA 6.1.

(1) We have

$$M_\mu = \sum_{\lambda\in P_+} a_{\lambda,\mu}\mathrm{m}_\lambda \text{ with } a_{\lambda,\mu} = \sum_{\substack{\overline{w}\in\overline{W}\\ \mu+\overline{\rho}-\overline{w}(\overline{\rho})\in W\lambda}} \varepsilon(\overline{w}).$$

(2) Consider $\mu,\nu \in \overline{P}_+$. Then, $H_\mu = H_\nu$ if and only if $M_\mu = M_\nu$.

Proof. Assertion 1 follows from the identity $\mathrm{m}_{w(\gamma)} = \mathrm{m}_\gamma$ for any $\gamma \in P$ and any $w \in W$. By Proposition 5.1, the coefficients of the expansion of M_μ on the basis $\{\mathrm{m}_\lambda \mid \lambda \in P_+\}$ are the same as those appearing in the expansion of H_μ on the basis $\{h_\lambda \mid \lambda \in P_+^w\}$. Assertion 2 follows. □

6.2. A simple expression for the functions M_λ. For any $\gamma \in P$, set

$$\overline{a}_\gamma = \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w})e^{\overline{w}(\gamma)}.$$

We thus have $\overline{a}_{\overline{w}(\gamma)} = \varepsilon(\overline{w})\overline{a}_\gamma$ and $\overline{w}(\overline{a}_\gamma) = \varepsilon(\overline{w})\overline{a}_\gamma$ for any $\overline{w} \in \overline{W}$ and $\overline{a}_{\overline{w}_0(\overline{\rho})} = \varepsilon(\overline{w}_0)\overline{a}_{\overline{\rho}}$ where $\overline{w}_0$ is the element of maximal length in $\overline{W}$.

PROPOSITION 6.2. *Let $\mu \in \overline{P}_+$.*

(1) We have

$$M_\mu = \varepsilon(\overline{w}_0)\sum_{u\in U} u(\overline{a}_{\mu+\overline{\rho}}\overline{a}_{\overline{\rho}}).$$

(2) Let Λ be the unique element lying in $\{u(\mu+2\overline{\rho}) \mid u \in U\} \cap P_+$. Then, we have

$$M_\mu = \varepsilon(\overline{w}_0)e^\Lambda + \sum_{\substack{\gamma\in P\\ \gamma<\Lambda}} b_{\lambda,\mu}e^\gamma.$$

Proof. We prove (1). We have

$$M_\mu = \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w})\mathrm{m}_{\mu+\overline{\rho}-\overline{w}(\overline{\rho})} = \sum_{w\in W} w\left(e^{\mu+\overline{\rho}}\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})e^{-\overline{w}(\overline{\rho})}\right).$$

This gives

$$M_\mu = \sum_{w\in W} w\left(e^{\mu+\overline{\rho}}\overline{a}_{-\overline{\rho}}\right) = \varepsilon(\overline{w}_0)\sum_{w\in W} w\left(e^{\mu+\overline{\rho}}\overline{a}_{\overline{\rho}}\right) = \varepsilon(\overline{w}_0)\sum_{u\in U} u\left(\sum_{\overline{w}\in\overline{W}}\overline{w}\left(e^{\mu+\overline{\rho}}\overline{a}_{\overline{\rho}}\right)\right),$$

by using Assertion 3 of Proposition 3.1. Hence

$$\begin{aligned} M_\mu &= \varepsilon(\overline{w}_0)\sum_{u\in U} u\left(\sum_{\overline{w}\in\overline{W}} e^{\overline{w}(\mu+\overline{\rho})}\overline{w}(\overline{a}_{\overline{\rho}})\right) \\ &= \varepsilon(\overline{w}_0)\sum_{u\in U} u\left(\overline{a}_{\overline{\rho}}\sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w})e^{\overline{w}(\mu+\overline{\rho})}\right) \\ &= \varepsilon(\overline{w}_0)\sum_{u\in U} u(\overline{a}_{\mu+\overline{\rho}}\overline{a}_{\overline{\rho}}), \end{aligned}$$

since $\overline{a}_{\overline{w}(\overline{\rho})} = \varepsilon(\overline{w})\overline{a}_{\overline{\rho}}$.

We prove (2). The monomials $e^{\mu+\overline{\rho}}$ and $e^{\overline{\rho}}$ are the monomials of highest weight (with respect to $\leq_{\overline{R}_+}$) appearing in the expression of $\overline{a}_{\mu+\overline{\rho}}$ and $\overline{a}_{\overline{\rho}}$, respectively. It follows that the monomial $e^{\mu+2\overline{\rho}}$ is of highest weight among those appearing in $\overline{a}_{\mu+\overline{\rho}}\overline{a}_{\overline{\rho}}$. Thus, using (1) we get an expression of the form

$$M_\mu = \varepsilon(\overline{w}_0)\sum_{u\in U} u\left(e^{\mu+2\overline{\rho}} + \sum_{\nu<_{\overline{R}_+}\mu+2\overline{\rho}} \mathbb{Z}e^{\nu}\right).$$

By Lemma 3.4, $\nu <_{\overline{R}_+} \mu + 2\overline{\rho}$ implies that $u(\nu) < u(\mu + 2\overline{\rho})$. Finally, the maximal weight with respect to $\leq$ in the set $\{u(\mu + 2\overline{\rho}) \mid u \in U\}$ is the unique element Λ lying in $\{u(\mu + 2\overline{\rho}) \mid u \in U\} \cap P_+$. Therefore, we have

$$M_\mu = \varepsilon(\overline{w}_0)e^{\Lambda} + \sum_{\substack{\gamma\in P\\ \gamma<\Lambda}} b_{\lambda,\mu}e^{\gamma},$$

as required. □

6.3. Proof of the conjecture for $\mu + 2\overline{\rho}$ dominant.

LEMMA 6.3. *Let $\mu \in \overline{P}_+$ be such that $\mu + 2\overline{\rho}$ belongs to P_+. Then $\mu \in P_+$.*

Proof. For any simple root $\alpha_i \in S$, we have $(\mu + 2\overline{\rho}, \alpha_i^\vee) \geq 0$ since $\mu + 2\overline{\rho} \in P_+$. Also for any simple root $\alpha_i \in \overline{S}$, we have $(\mu, \alpha_i^\vee) \geq 0$ since $\mu \in \overline{P}_+$. Now consider $\alpha_j \in S \setminus \overline{S}$. Since $2\overline{\rho}$ decomposes as a sum of simple roots in $\overline{S}$, we must have $(2\overline{\rho}, \alpha_j^\vee) \leq 0$. Indeed for any $\alpha_i \in \overline{S}$, $(\alpha_i, \alpha_j^\vee) = 0$ or is negative since distinct simple roots are always at an angle greater than $\pi/2$. Therefore, $(\mu, \alpha_j^\vee) \geq (\mu + 2\overline{\rho}, \alpha_j^\vee) \geq 0$. □

PROPOSITION 6.4. *Let $\mu, \nu \in \overline{P}_+$ be such that $H_\mu = H_\nu$ and assume that $\mu + 2\overline{\rho} \in P_+$. Then, there exists $v \in U$ such that $\nu = v(\mu)$ and $v(\overline{R}_+) = \overline{R}_+$.*

Proof. By the previous lemma, we see that $\mu \in P_+$. Let $v \in U$ be such that $\nu \in P_+^v$. Then by (the proof of) Proposition 5.3, we know that $v(\nu) = \mu$. Next Lemma 6.1 implies that $M_\mu = M_\nu$ and, in particular, M_μ and M_ν have the same maximal monomial with respect to $<$. Hence

$$\{u(\mu + 2\overline{\rho}) \mid u \in U\} \cap P_+ = \{u(v^{-1}(\mu) + 2\overline{\rho}) \mid u \in U\} \cap P_+.$$

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But $\mu + 2\overline{\rho} \in P_+$ so we have $\{u(v^{-1}(\mu) + 2\overline{\rho}) \mid u \in U\} \cap P_+ = \{\mu + 2\overline{\rho}\}$. Hence, there exists $u \in U$ such that $u(v^{-1}(\mu) + 2\overline{\rho}) = \mu + 2\overline{\rho}$. We have

$$\begin{gathered} \mu + 2\overline{\rho} = u(v^{-1}(\mu) + 2\overline{\rho}) \\ \Updownarrow \\ u^{-1}(\mu + 2\overline{\rho}) = v^{-1}(\mu) + 2\overline{\rho} \\ \Updownarrow \\ vu^{-1}(\mu + 2\overline{\rho}) = \mu + 2v(\overline{\rho}) \\ \Updownarrow \\ vu^{-1}(\mu + 2\overline{\rho}) - (\mu + 2\overline{\rho}) = 2(v(\overline{\rho}) - \overline{\rho}). \end{gathered}$$

Since $\mu + 2\overline{\rho} \in P_+$, we have $vu^{-1}(\mu + 2\overline{\rho}) - (\mu + 2\overline{\rho}) \leq 0$. Hence $v(\overline{\rho}) \leq \overline{\rho}$. By Lemma 3.3, this implies that $v(\overline{\rho}) = \overline{\rho}$. Finally by Lemma 3.2, we have $v(\overline{R}_+) = \overline{R}_+$. □

REMARK 6.5. We will see in the next section (Remark 7.2) that we can have μ and ν in the same W-orbit, $\mu + 2\overline{\rho}$ and $\nu + 2\overline{\rho}$ in the same W-orbit but $H_\mu \neq H_\nu$. So, the hypothesis $\mu + 2\overline{\rho} \in P_+$ is crucial in the above proposition.

7. The classical Lie algebras.

7.1. Proof of the conjecture for $\mathfrak{gl}_n$. We now prove our conjecture in type A. We shall work in fact with $\mathfrak{gl}_n$ rather than $\mathfrak{sl}_n$. The main tool is a duality result between the branching coefficients m_μ^λ and some generalised Littlewood–Richardson coefficients together with the main result of **[6]**. Each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$ with $d \leq n$ can be regarded as a dominant weight of $\mathfrak{gl}_n$ by adding $n - d$ coordinates equal to 0. We will use this convention in this section. For any partition $\mu = (\mu_1 \geq \cdots \geq \mu_d)$, we have in fact

$$s_\mu = \sum_{\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)} K^{-1}_{\lambda,\mu} h_\lambda, \tag{3}$$

that is, the coefficients appearing in the expansion of s_μ on the h-basis are inverse Kostka numbers indexed by pairs (λ, μ) of partitions with at most d nonzero parts. When $\mathfrak{g} = \mathfrak{gl}_n$, the h-functions have also an additional property (which does not hold for the other root systems). Consider $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$, then $h_\beta = h_{\beta_1} \times \cdots \times h_{\beta_n}$.

Recall that the dominant weights of $\mathfrak{gl}_n$ can be regarded as non-increasing sequences of integers (possibly negative) with length n. We will realise $\overline{\mathfrak{g}} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ as the subalgebra of $\mathfrak{gl}_m$ of block matrices with block sizes $m_1, \ldots, m_r$. Now, consider $\mu \in \overline{P}$ such that $\mu = \mu^{(1)} + \cdots + \mu^{(r)}$ where $\mu^{(k)} \in P_+^{(k)}$ as in Section 4.2. Then, each $\mu^{(k)}$ is a non-increasing sequence of integers of length m_k. We will assume temporary that the coordinates of μ are nonnegative so that each $\mu^{(k)}$ is a partition with m_k parts. We then have according to (2)

$$H_\mu = \sum_{\lambda^{(1)} \in P_+^{(1)}} \cdots \sum_{\lambda^{(r)} \in P_+^{(r)}} K^{-1}_{\lambda^{(1)},\mu^{(1)}} \cdots K^{-1}_{\lambda^{(r)},\mu^{(r)}} h_{\lambda^{(1)} + \cdots + \lambda^{(r)}},$$

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where each $\lambda^{(k)}$ is a partition. In particular, we have $h_{\lambda^{(1)}+\cdots+\lambda^{(r)}} = h_{\lambda^{(1)}} \times \cdots \times h_{\lambda^{(r)}}$ which yields

$$H_\mu = \prod_{i=1}^{k} \left(\sum_{\lambda^{(k)} \in P_+^{(k)}} K^{-1}_{\lambda^{(k)},\mu^{(k)}} h_{\lambda^{(k)}} \right).$$

Finally by using (3), we obtain

$$H_\mu = \prod_{i=1}^{k} s_{\mu^{(k)}}.$$

We can now prove our conjecture for induced representations of $\mathfrak{gl}_n$

Proposition 7.1. *Consider μ and ν any dominant weights of $\overline{\mathfrak{g}}$. Assume $H_\mu = H_\nu$. Then, there exists a permutation σ of $\{1, \ldots, n\}$ such that $\sigma(\overline{R}_+) = \overline{R}_+$.*

Proof. By Theorem 2.2, we have $m_\mu^\lambda = \sum_{\sigma \in S_n} \varepsilon(\sigma)\overline{\mathcal{P}}(\sigma(\lambda+\rho) - \mu - \rho)$. Set $\delta = (1, \ldots, 1) \in \mathbb{Z}^n$. Since δ is fixed by S_n, we have for any nonnegative integer a, $m_{\lambda+a\delta}^{\mu+a\delta} = m_\lambda^\mu$. Observe also that P_+ is invariant by translation by δ. Therefore

$$H_{\mu+\delta a} = \sum_{\nu\in P_+} m_\nu^{\mu+a\delta} s_\nu = \sum_{\lambda\in P_+} m_{\lambda+a\delta}^{\mu+a\delta} s_{\lambda+a\delta} = \sum_{\lambda\in P_+} m_\lambda^\mu s_{\lambda+a\delta},$$

by setting $\nu = \lambda + a\delta$ in the leftmost sum. So, $H_\mu = H_\nu$ if and only if $H_{\mu+a\delta} = H_{\nu+a\delta}$. We can now choose a sufficiently large so that $\mu \in \mathbb{Z}^n_{>0}$ and $\nu \in \mathbb{Z}^n_{>0}$. Decompose $\mu = \mu^{(1)} + \cdots + \mu^{(r)}$ and $\nu = \nu^{(1)} + \cdots + \nu^{(r)}$ as in Section 4.2. For any $k = 1, \ldots, r$, set $\delta_k = (1, \ldots, 1) \in \mathbb{Z}^{m_k}$. The similar decompositions of $\mu + a\delta$ and $\nu + a\delta$ verify $(\mu + a\delta)^{(k)} = \mu^{(k)} + a\delta^{(k)}$ and $(\nu + a\delta)^{(k)} = \nu^{(k)} + a\delta^{(k)}$ for any $k = 1, \ldots, r$. We thus obtain

$$\prod_{i=1}^{k} s_{\mu^{(k)}+a\delta^{(k)}} = \prod_{i=1}^{k} s_{\nu^{(k)}+a\delta^{(k)}}.$$

Now by the main result of [**6**], since the partitions $\mu^{(k)} + a\delta^{(k)}$ and $\nu^{(k)} + a\delta^{(k)}$ appearing above have positive parts, we know that the set of partitions

$$\{\mu^{(k)} + a\delta^{(k)}, k = 1, \ldots r\} \quad \text{and} \quad \{\nu^{(k)} + a\delta^{(k)}, k = 1, \ldots r\},$$

should coincide. There, thus exists a permutation $\tau \in S_r$ such that $\mu^{(k)} + a\delta^{(k)} = \nu^{(\tau(k))} + a\delta^{(\tau(k))}$. The permutation τ preserves the lengths of the partitions (recall the partitions considered here have positive parts) so $m_k = m_{\tau(k)}$ and $\delta^{(k)} = \delta^{(\tau(k))}$ for any $k = 1, \ldots, r$. We obtain $\mu^{(k)} = \nu^{(\tau(k))}$. For any $k = 1, \ldots, r$, set $I_k = \{m_{k-1} + 1, \ldots, m_k\}$ (with $m_0 = 0$). Then, I_k and $I_{\tau(k)}$ have the same cardinality because $m_k = m_{\tau(k)}$. Let $\sigma \in S_n$ be such that $\sigma(m_{k-1} + j) = m_{\tau(k)-1} + j$ for any $j \in \{1, \ldots, k\}$ and any $k \in \{1, \ldots, r\}$. Then, σ is a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$. We have $\sigma(\mu) = \nu$ and $\sigma(\overline{R}_+) = \overline{R}_+$ as desired. □

Remark 7.2. Observe that we can have μ and ν in the same W-orbit, $\mu + 2\overline{\rho}$ and $\nu + 2\overline{\rho}$ in the same W-orbit but $H_\mu \neq H_\nu$. Consider for example $\overline{\mathfrak{g}} = \mathfrak{gl}_4 \oplus \mathfrak{gl}_2$ in $\mathfrak{gl}_6$

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and $\mu = (5, 2, 2, 1 \mid 4, 3)$ and $\nu = (5, 4, 3, 1 \mid 2, 2)$. We have $2\overline{\rho} = (3, 1, -1, -3 \mid 1, -1)$ so $\mu + 2\overline{\rho} = (8, 3, 1, -2 \mid 5, 2)$ and $\nu + 2\overline{\rho} = (8, 5, 2, -2 \mid 3, 1)$ belong to the same W-orbit. By the previous proposition, we have $H_\mu \neq H_\nu$. We cannot apply Proposition 6.4 since neither $\mu + 2\overline{\rho}$ or $\nu + 2\overline{\rho}$ belong to P_+.

7.2. Polarisation. Assume $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$ and $\overline{\mathfrak{g}} = \mathfrak{gl}_n$. Each dominant weight $\mu \in \overline{P}_+$ defines a pair of partitions (μ_+, μ_-) of length $\leq n$ obtained by ordering decreasingly the positive and negative coordinates of μ, respectively. Recall also that to each partition λ of length $\leq n$ corresponds a dominant weight of P_+. The branching coefficients m_μ^λ were obtained by Littlewood (see [**5**]). They can be expressed in terms of the Littlewood–Richardson coefficients as follows :

$$m_\mu^\lambda = \begin{cases} \sum_{\gamma,\delta} c^\gamma_{\mu_+,\mu_-} c^\lambda_{\gamma,\delta} & \text{for } \mathfrak{g} = \mathfrak{so}_{2n+1}, \\ \sum_{\gamma,\delta} c^\gamma_{\mu_+,\mu_-} c^\lambda_{\gamma,2\delta} & \text{for } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ \sum_{\gamma,\delta} c^\gamma_{\mu_+,\mu_-} c^\lambda_{\gamma,(2\delta)^*} & \text{for } \mathfrak{g} = \mathfrak{so}_{2n}, \end{cases}$$

where γ and δ runs over the set of partitions with length $\leq n$ and $(2\delta)^*$ is the conjugate partition of 2δ.

PROPOSITION 7.3. *Conjecture 4.5 is true for* $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ *or* $\mathfrak{so}_{2n}$ *and* $\overline{\mathfrak{g}} = \mathfrak{gl}_n$.

Proof. Consider μ and ν in $\overline{P}_+$ such that $H_\mu = H_\nu$. We have $m_\mu^\lambda = m_\nu^\lambda$ for any $\lambda \in P_+$. For any partition λ, write $|\lambda|$ the size of λ, that is the sum of its parts. Observe first that $m_\mu^\lambda = 0$ when $|\lambda| < |\mu_+| + |\mu_-|$. Also, when $|\lambda| = |\mu_+| + |\mu_-|$ in the above branching coefficients, we get $\delta = \emptyset, \gamma = \lambda$ and $m_\mu^\lambda = c^\lambda_{\mu_+,\mu_-}$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$.

Assume $|\mu_+| + |\mu_-| < |\nu_+| + |\nu_-|$. Then for $\lambda = \mu_+ + \mu_-$, we have $m_\mu^\lambda = c^\lambda_{\mu_+,\mu_-} = 1$ whereas $m_\nu^\lambda = 0$ since $|\lambda| = |\mu_+| + |\mu_-| < |\nu_+| + |\nu_-|$. So we obtain a contradiction. Similarly, we cannot have $|\mu_+| + |\mu_-| > |\nu_+| + |\nu_-|$. Therefore $|\mu_+| + |\mu_-| = |\nu_+| + |\nu_-|$. Then for any λ such that $|\lambda| = |\mu_+| + |\mu_-| = |\nu_+| + |\nu_-|$, we have $c^\lambda_{\mu_+,\mu_-} = c^\lambda_{\nu_+,\nu_-}$. By the main result of [**6**], we obtain the equality of sets $\{\mu_+, \mu_-\} = \{\nu_+, \nu_-\}$. When $\mu_+ = \nu_+$ and $\mu_- = \nu_-$, we have $\mu = \nu$ and the conjecture holds. When $\mu_+ = \nu_-$ and $\mu_- = \nu_+$, we have $\mu = -\overline{w}_0\nu$ where $\overline{w}_0$ is the longest element of $\overline{W}$ that is, the permutation of $\{1, \dots, n\}$ such that $w_0(k) = n - k + 1$. Since $-\overline{w}_0 \in W$ and $-w_0(\overline{R}_+) = \overline{R}_+$ we are done. $\square$

We now summarise our results.

THEOREM 7.4. *Consider* $\mu, \nu \in \overline{P}_+$.

(1) When μ and ν are conjugate under the action of a Dynkin diagram automorphism of $\overline{\mathfrak{g}}$ lying in W, we have $H_\mu = H_\nu$.

(2) Conversely, if we assume $H_\mu = H_\nu$, then μ and ν are conjugate under the action of a Dynkin diagram automorphism lying in W when one of the following hypotheses is satisfied.

- *μ and ν belong to the same Weyl chamber of $\mathfrak{g}$ (in which case $\mu = \nu$),*
- *the sets $E_\mu = \{\mu + \overline{\rho} - \overline{w}(\overline{\rho}) \mid \overline{w} \in \overline{W}\}$ and $E_\nu = \{\nu + \overline{\rho} - \overline{w}(\overline{\rho}) \mid \overline{w} \in \overline{W}\}$ are entirely contained in a Weyl chamber,*

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- $\mu + 2\overline{\rho}$ *or* $\nu + 2\overline{\rho}$ *belongs to* P_+,
- $\mathfrak{g} = \mathfrak{gl}_n$,
- $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ *or* $\mathfrak{so}_{2n}$ *and* $\overline{\mathfrak{g}} = \mathfrak{gl}_n$.

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