A HAHN-BANACH THEOREM FOR SEMIFIELDS

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Iseki and Kasahara (see [3]) have given a Hahn-Banach type theorem for semifield-valued linear functionals on real linear spaces. We shall generalize their result by considering linear spaces over semifields.

The theory of topological semifields was developed by Antonovskii, Boltyanskii, and Sarymsakov (see [1] and [2]) who showed that every semifield is a subsemifield of a Tichonov product of reals under coordinatewise addition and multiplication. There is no restriction in our work to consider only this type of semifield which is called a 'Tichonov' semifield by those authors.

In a semifield R^I, we say that a is weakly less than b, written $a \ll b$, iff $\pi_i(a) \leqq \pi_i(b)$ for all $i \in I$. This partial ordering is used in the characterization of semifields. By e_i we denote the point of R^I such that $e_i(j) = 0$ if $j \neq i$ and $e_i(i) = 1$. If R^I and R^J are semifields, we will always identify the points of I and J according to their cardinality so that one is a subset of the other. Then one of the semifields is identified with a subsemifield of the other.

Suppose that X is a linear space over the semifield R^I. A linear functional on X is a function $f : X \to R^J$ such that

$$f(ax+by) = af(x)+bf(y)$$

for every $x, y \in X$ and $a, b \in R^I$. We note that the right-hand side of the equation yields a well-defined element of R^J with our above identification. By a sublinear functional on X is meant a function $p : X \to R^J$ satisfying the conditions

(i) $p(x+y) \ll p(x)+p(y)$,

(ii) $p(ax) = ap(x)$ for $a \gg 0$.

THEOREM. *Let p be a sublinear functional on X, f a linear functional defined on a linear subspace S of X. If $f(s) \ll p(s)$ for every $s \in S$, then f has a linear extension F on X such that $F(x) \ll p(x)$ for all $x \in X$.*

PROOF. Let $\mathscr{F}$ be the set of all linear functionals on subspaces of X which are weakly less than p on their domain. Now $\mathscr{F}$ is nonempty and can be partially ordered by setting $g < h$ if h is an extension of g. By Hausdorff's maximal principle, there exists a maximal totally ordered subset $\mathscr{H}$ of $\mathscr{F}$ which possesses f as an element. Let T be the linear subspace of X which is the union of the domains of the members of $\mathscr{H}$. We then define a function $F : T \to R^J$ by setting $F(s) = g(s)$ whenever s lies in the domain of $g \in \mathscr{H}$. It is clear that F is a well-defined linear functional which belongs to $\mathscr{F}$ and is an extension of every element of $\mathscr{H}$. We shall show that $T = X$ so that F is the desired extension. To do this we shall show that every $g \in \mathscr{F}$ whose domain is a proper subset of X has a proper extension so that, by the obvious maximality of F in $\mathscr{F}$, T is not a proper subset of X. Let the domain of $g \in \mathscr{F}$ be $S \neq X$ so there exists a point $y \in X-S$ and we shall show that g can be extended to the subspace U spanned by $S \cup \{y\}$. Let $s, t \in S$, then

$$g(s+t) = g(s)+g(t) \ll p(s+t) = p(s+y+t-y) \ll p(s+y)+p(t-y)$$

so that $-p(t-y)+g(t) \ll p(s+y)-g(s)$. Hence

$$\sup\{-p(s-y)+g(s)\} \ll \inf\{p(s+y)-g(s)\}.$$

Let $\alpha \in R^J$ be between this supremum and this infimum. We can then define a functional h on U by setting

$$h(s+ky) = g(s)+k\alpha$$

for every $k \in R^I$. h is clearly linear and an extension of g. We shall show that $h \in \mathscr{F}$. We must show, therefore, that $g(s)+k\alpha \ll p(s+ky)$ for every $s \in S$ and $k \in R^I$. To this end let $i \in I$ and suppose that $\pi_i(k) > 0$. Then let $a \in R^I$ be such that $\pi_j(a) = 0$ whenever $j \neq i$ and $\pi_i(a) = 1/\pi_i(k)$. Then $ka = e_i$, so that

$$\begin{aligned} e_i[g(s)+k\alpha] &= e_i g(s)+e_i k\alpha = e_i kag(s)+e_i k\alpha = e_i k[ag(s)+\alpha] \\ &= e_i k[g(as)+\alpha] \ll e_i k[g(as)+p(as+y)-g(as)] \\ &= e_i kp(as+y) = p(e_i kas+e_i ky) = p(e_i s+e_i ky) = e_i p(s+ky). \end{aligned}$$

Next suppose that $\pi_i(k) = 0$, so that $e_i k = 0$. Then

$$\begin{aligned} e_i[g(s)+k\alpha] &= e_i g(s)+e_i k\alpha = e_i g(s) \ll e_i p(s) = p(e_i s) \\ &= p(e_i s+e_i ky) = e_i p(s+ky). \end{aligned}$$

Finally, suppose that $\pi_i(k) < 0$. Then let $m = -k$ so that $\pi_i(m) > 0$. Let $a \in R^I$ be such that $\pi_j(a) = 0$ whenever $j \neq i$ and $\pi_i(a) = 1/\pi_i(m)$. Then $ma = e_i$, so that

$$\begin{aligned} e_i[g(s)+k\alpha] &= e_i[g(s)-m\alpha] = e_i g(s)-e_i m\alpha = e_i mag(s)-e_i m\alpha \\ &= e_i m[ag(s)-\alpha] \ll e_i m[g(as)+p(as-y)-g(as)] \\ &= e_i mp(as-y) = p(e_i mas-e_i my) = p(e_i s+e_i ky) = e_i p(s+ky) \end{aligned}$$

as desired. q.e.d.

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