Bull. Aust. Math. Soc. **82** (2010), 446–458
doi:10.1017/S0004972710001802

MULTIPLICITY OF SOLUTIONS FOR A FOURTH-ORDER IMPULSIVE DIFFERENTIAL EQUATION VIA VARIATIONAL METHODS

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(Received 8 June 2009)

Abstract

In this paper, we deal with the multiplicity of solutions for a fourth-order impulsive differential equation with a parameter. Using variational methods and a 'three critical points' theorem, we give some new criteria to guarantee that the impulsive problem has at least three classical solutions. An example is also given in order to illustrate the main results.

2000 *Mathematics subject classification*: primary 34B15; secondary 34B37.

Keywords and phrases: impulsive differential equation, boundary value problem, critical points, variational methods.

1. Introduction

In this paper, we consider the fourth-order boundary value problem with impulses

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)) & \text{a.e. } t \in [0, 1], \\ \Delta(u''(t_j)) = I_j(u'(t_j)) & j = 1, 2, \ldots, l, \\ \Delta(u'''(t_j)) = N_j(u(t_j)) & j = 1, 2, \ldots, l, \\ u(0) = u'(0) = u''(1^-) = 0 \\ u'''(1^-) = g(u(1)) \end{cases} \tag{1.1}$$

where λ is a positive parameter, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, g, I_j, $N_j \in C(\mathbb{R}, \mathbb{R})$ for $1 \le j \le l$, $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = 1$,

$$\Delta(u''(t_j)) = u''(t_j^+) - u''(t_j^-) = \lim_{t \to t_j^+} u''(t) - \lim_{t \to t_j^-} u''(t),$$

$$\Delta(u'''(t_j)) = u'''(t_j^+) - u'''(t_j^-) = \lim_{t \to t_j^+} u'''(t) - \lim_{t \to t_j^-} u'''(t),$$

and where $u''(1^-)$ and $u'''(1^-)$ denote the left limits of $u''(t)$ and $u'''(t)$ at 1.

The first author was supported by the Graduate Degree Thesis Innovation Foundation of Central South University (CX2009B023). The second author was supported by NFSC (10871206).

This kind of problem without impulses arises in the study of deflections of elastic beams on nonlinear elastic foundations. The problem has the following physical description: a thin flexible elastic beam of length 1 is clamped at its left end $t = 0$ and rests on an elastic device at its right end $t = 1$, which is given by g. Then, the problem models the static equilibrium of the beam under a load, along its length, characterized by f. The derivation of the model can be deduced from [26].

In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems (IBVPs), which describe a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics and physical phenomena. For general aspects of impulsive differential equations, we refer the reader to the classical monograph [13]. For some general and recent works on the theory of impulsive differential equations, we refer the reader to [1, 2, 11, 14, 18, 21]. Some classical tools have been used to study such problems in the literature. These classical techniques and tools include the coincidence degree theory of Mawhin [19], the method of upper and lower solutions with the monotone iterative technique [7], and some fixed point theorems in cones [8, 9, 12]. More precisely, in [12], Karaca studied the existence of positive solutions of the fourth-order impulsive differential equation

$$\begin{cases} y^{(4)} - q(t)y'' = f(t, y(t)), & t \in [a, c) \cup (c, b], \\ \alpha_1 y(a) - \beta_1 y'(a) = 0, & \gamma_1 y(b) + \delta_1 y'(b) = 0, \\ y''(c-0) = d_1 y''(c+0), & y'''(c-0) = d_2 y'''(c+0), \\ \alpha_2 y''(a) - \beta_2 y'''(a) = 0, & \gamma_2 y''(b) + \delta_2 y'''(b) = 0, \end{cases} \tag{1.2}$$

and the eigenvalue problem $y^{(4)} - q(t)y'' = \lambda f(t, y(t))$ with the same boundary conditions where $\lambda > 0$. Using the Krasnosel'skiĭ fixed point theorem, the author obtained the existence and multiplicity of positive solutions for the IBVP (1.2) and proved that the IBVP (1.2) with parameter λ has at least one positive solution when λ lies in some positive interval.

On the other hand, in the last few years, many researchers have used variational methods to study the existence and multiplicity of solutions for boundary value problems without impulsive effects [3–6, 10, 15, 25]. For related basic information, we refer the reader to [16, 20, 22].

Recently, some researchers have begun to study the existence of solutions for IBVPs using variational methods. However, to the best of our knowledge with the exception of [17, 23, 24, 28, 29], the study of solutions (in particular, the multiplicity of solutions) for IBVPs using variational methods has received considerably less attention. It may become a new powerful tool to deal with nonlinear problems with some types of discontinuity such as impulses.

In [17], using variational methods and critical point theory, Nieto and O'Regan studied the existence of solutions of the equation

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)) & \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = I_j(u(t_j)) & j = 1, 2, \ldots, l, \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \ldots, l$, are continuous. They proved that IBVP (1.3) has at least one solution when $\lambda \geq -\pi^2/T^2$.

In [28], Zhang and Ge studied the multiplicity of solutions of the equation

$$\begin{cases} (\Phi_p(u'(t)))' = (a(t)\Phi_p(u) + \lambda f(t, u) + \mu h(u))g(u') & \text{a.e. } t \in [0, 1], \\ \Delta G(u'(t_i)) = I_i(u(t_i)) & i = 1, 2, \ldots, k, \\ \alpha_1 u(0) - \alpha_2 u'(0) = 0 \\ \beta_1 u(1) + \beta_2 u'(1) = 0 \end{cases} \tag{1.4}$$

where $p > 1$, $\Phi_p(u) = |u|^{p-2}u$, λ, μ are positive parameters, and

$$G(u) = \int_0^u \frac{(p-1)|s|^{p-2}}{g(s)}\, ds.$$

They proved that IBVP (1.4) has at least three solutions by using a three critical points theorem.

Motivated by the above facts, in this paper, our aim is to study the multiplicity of solutions for IBVP (1.1). To the best of our knowledge, there has so far no paper concerning fourth-order impulsive differential equations using variational methods. In addition, this paper is a generalization of [15], in which impulsive effects are not involved.

This paper is organized as follows. In Section 2 we present some preliminaries. In Section 3 we discuss the existence of three classical solutions to IBVP (1.1), and present an example to illustrate our main results.

2. Preliminaries

Our main tool is the following three critical points theorem obtained in [5] (see also [6, Theorem 2.1]).

THEOREM 2.1. *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteau differentiable functional whose Gâteau derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteau differentiable functional whose Gâteau derivative is compact. Assume that there exist $r \in \mathbb{R}$ and $x_0, \bar{x} \in X$, with $\Phi(x_0) < r < \Phi(\bar{x})$ and $\Psi(x_0) = 0$ such that:*

(A1)
$$\sup_{\Phi(x) \leq r} \Psi(x) < (r - \Phi(x_0)) \frac{\Psi(\bar{x})}{\Phi(\bar{x}) - \Phi(x_0)};$$

(A2) *for each*
$$\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x}) - \Phi(x_0)}{\Psi(\bar{x})}, \frac{r - \Phi(x_0)}{\sup_{\Phi(x) \leq r} \Psi(x)} \right[,$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Let us recall some basic knowledge. Denote the Hilbert space X by

$$X = \{u \in H^2([0, 1]) : u(0) = u'(0) = 0\}$$

with the inner product

$$(u, v) = \int_0^1 u''(t)v''(t)\, dt, \quad \forall u, v \in X,$$

which induces the norm

$$\|u\|_X = \left[\int_0^1 |u''(t)|^2\, dt\right]^{1/2},$$

where $H^2([0, 1])$ is the Sobolev space of all functions $u : [0, 1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0, 1])$.

We define the norm in $C^1([0, 1])$ and $L^2([0, 1])$ as $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$ and $\|u\|_2 = [\int_0^1 |u|^2\, dt]^{1/2}$, respectively.

For any $u \in X$, u and u' are absolutely continuous and $u'' \in L^2([0, 1])$. In this case, the one-side derivatives $u''(t_j^+)$, $u''(t_j^-)$, $u'''(t_j^+)$ and $u'''(t_j^-)$ may not exist. As a consequence, we need to introduce a concept of solution. Suppose that $u \in C^1([0, 1])$ with $u(0) = u'(0) = 0$. Moreover, assume that for every $j = 0, 1, \ldots, l$, $u_j = u|_{(t_j, t_{j+1})}$ is such that $u_j \in C^4(t_j, t_{j+1})$. We say that u is *a classical solution* of IBVP (1.1) if it satisfies the equation in IBVP (1.1) a.e. on $[0, 1]$, the limits $u''(t_j^+)$, $u''(t_j^-)$, $u'''(t_j^+)$ and $u'''(t_j^-)$, $j = 1, 2, \ldots, l$, exist, $u''(1^-)$, $u'''(1^-)$ exist and $u''(1^-) = u'''(1^-) - g(u(1)) = 0$, and two kinds of impulsive conditions in IBVP (1.1) hold.

For each $u \in X$, put

$$\Phi(u) := \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{l} \int_0^{u'(t_j)} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{u(t_j)} N_j(s)\, ds + \int_0^{u(1)} g(s)\, ds, \tag{2.1}$$

$$\Psi(u) := \int_0^1 F(t, u)\, dt, \tag{2.2}$$

where $F(t, u) = \int_0^u f(t, s)\, ds$.

Clearly, Φ is a Gâteau differentiable functional whose Gâteau derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_0^1 u''v''\, dt + \sum_{j=1}^{l} I_j(u'(t_j))v'(t_j) - \sum_{j=1}^{l} N_j(u(t_j))v(t_j) + g(u(1))v(1) \tag{2.3}$$

for any $v \in X$, and $\Phi' : X \to X^*$ is continuous.

https://doi.org/10.1017/S0004972710001802 Published online by Cambridge University Press

It is also easy to see that $\Psi : X \to \mathbb{R}$ is a Gâteau differentiable functional whose Gâteau derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$ given by

$$\Psi'(u)(v) = \int_0^1 f(t, u(t))v(t)\,dt \tag{2.4}$$

for any $v \in X$.

LEMMA 2.2. *If $u \in X$ is a critical point of $\Phi - \lambda\Psi$, then u is a classical solution of IBVP (1.1).*

PROOF. Obviously, $u(0) = u'(0) = 0$ for $u \in X$. Since $u \in X$ is a critical point of $\Phi - \lambda\Psi$,

$$\begin{aligned}\int_0^1 u''v''\,dt + \sum_{j=1}^{l} I_j(u'(t_j))v'(t_j) - \sum_{j=1}^{l} N_j(u(t_j))v(t_j)\\ + g(u(1))v(1) - \lambda\int_0^1 f(t, u(t))v(t)\,dt = 0\end{aligned} \tag{2.5}$$

for any $v \in X$. For $j \in \{0, 1, 2, \ldots, l\}$, choose any $v \in X$ such that $v(t) \equiv 0$ for $t \in [0, t_j] \cup [t_{j+1}, 1]$. Then $v'(t_j) = 0$. Equation (2.5) implies that

$$\int_{t_j}^{t_{j+1}} u''v''\,dt = \lambda\int_{t_j}^{t_{j+1}} f(t, u(t))v(t)\,dt.$$

This means, for any $w \in X_j := H^2(t_j, t_{j+1}) \cap H_0^1(t_j, t_{j+1})$, that

$$\int_{t_j}^{t_{j+1}} u_j''w''\,dt = \lambda\int_{t_j}^{t_{j+1}} f(t, u_j(t))w(t)\,dt,$$

where $u_j = u|_{(t_j,t_{j+1})}$. Thus u_j is a weak solution of the equation

$$u^{(iv)} = \lambda f(t, u), \quad t \in (t_j, t_{j+1}), \tag{2.6}$$

and $u_j \in X_j \subset C^1([t_j, t_{j+1}])$. Let $v_1 := u''$ and $q(t) := \lambda f(t, u)$. Then (2.6) has the form

$$v_1''(t) = q(t) \quad \text{on } (t_j, t_{j+1}). \tag{2.7}$$

Then the solution of (2.7) can be written as

$$v_1(t) = C_1 + C_2 t + \int_{t_j}^{t}\int_{t_j}^{s} q(r)\,dr\,ds, \quad t \in (t_j, t_{j+1}),$$

where C_1 and C_2 are two constants. Then $u_j'' \in C(t_j, t_{j+1})$ and $u_j^{(iv)} \in C(t_j, t_{j+1})$. Therefore, $u_j \in C^4(t_j, t_{j+1})$ and u satisfies the equation in IBVP (1.1) a.e. on [0, 1].

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By the previous equation, we find that the limits $u''(t_j^+)$, $u''(t_j^-)$, $u'''(t_j^+)$, $u'''(t_j^-)$, $j = 1, 2, \ldots, l$, $u''(1^-)$ and $u'''(1^-)$ exist. Integrating (2.5) leads to

$$\begin{aligned}
&\int_0^1 u''v''\,dt + \sum_{j=1}^{l} I_j(u'(t_j))v'(t_j) - \sum_{j=1}^{l} N_j(u(t_j))v(t_j) \\
&\quad + g(u(1))v(1) - \lambda \int_0^1 f(t,u)v\,dt \\
&\quad\quad = \int_0^1 (u^{(iv)} - \lambda f(t,u))v\,dt + \sum_{j=1}^{l}(I_j(u'(t_j)) - \Delta(u''(t_j)))v'(t_j) \\
&\quad\quad\quad - \sum_{j=1}^{l}(N_j(u(t_j)) - \Delta(u'''(t_j)))v(t_j) + u''(1^-)v'(1) \\
&\quad\quad\quad + (g(u(1)) - u'''(1^-))v(1) \\
&\quad\quad = 0,
\end{aligned} \tag{2.8}$$

and combining with (2.6) gives

$$\begin{aligned}
&\sum_{j=1}^{l}(I_j(u'(t_j)) - \Delta(u''(t_j)))v'(t_j) - \sum_{j=1}^{l}(N_j(u(t_j)) - \Delta(u'''(t_j)))v(t_j) \\
&\quad + u''(1^-)v'(1) + (g(u(1)) - u'''(1^-))v(1) = 0.
\end{aligned} \tag{2.9}$$

Next we show that u satisfies the second kind of impulsive conditions in IBVP (1.1). If not, without loss of generality, we assume that there exists $i \in \{1, 2, \ldots, l\}$ such that

$$N_i(u(t_i)) - \Delta(u'''(t_i)) \neq 0. \tag{2.10}$$

Let

$$v(t) = (t^2 - 2t_i t) \prod_{j=0, j\neq i}^{l+1} \left(\frac{1}{3}t^3 - \frac{1}{2}(t_i + t_j)t^2 + t_i t_j t + \frac{1}{6}t_j^3 - \frac{1}{2}t_i t_j^2\right).$$

Then

$$\begin{aligned}
v'(t) = 2(t - t_i) &\prod_{j=0, j\neq i}^{l+1} \left(\frac{1}{3}t^3 - \frac{1}{2}(t_i + t_j)t^2 + t_i t_j t + \frac{1}{6}t_j^3 - \frac{1}{2}t_i t_j^2\right) \\
&+ (t^2 - 2t_i t) \sum_{k=0, k\neq i}^{l+1} \Bigg\{(t^2 - (t_k + t_i)t + t_k t_i) \\
&\quad \times \prod_{j=0, j\neq i,k}^{l+1} \left(\frac{1}{3}t^3 - \frac{1}{2}(t_i + t_j)t^2 + t_i t_j t + \frac{1}{6}t_j^3 - \frac{1}{2}t_i t_j^2\right)\Bigg\}.
\end{aligned}$$

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Obviously, $v \in X$. By simple calculations, we obtain $v(t_j)=0$, $j=1,2,\ldots,i-1,i+1,\ldots,l,l+1$, $v(t_i)\neq 0$ and $v'(t_j)=0$, $j=1,2,\ldots,l,l+1$. Thus, substituting these into (2.9) leads to

$$\begin{aligned}&\sum_{j=1}^{l}(I_j(u'(t_j))-\Delta(u''(t_j)))v'(t_j)-\sum_{j=1}^{l}(N_j(u(t_j))-\Delta(u'''(t_j)))v(t_j)\\&\quad+u''(1^-)v'(1)+(g(u(1))-u'''(1^-))v(1)\\&\quad=(N_i(u(t_i))-\Delta(u'''(t_i)))v(t_i)\\&\quad=\frac{t_i^2}{6}(N_i(u(t_i))-\Delta(u'''(t_i)))\prod_{j=0,j\neq i}^{l+1}(t_i-t_j)^3=0,\end{aligned}$$

which contradicts (2.10). So u satisfies the second kind of impulsive conditions in IBVP (1.1). Similarly, we can prove that u satisfies the first kind of impulsive conditions in IBVP (1.1). Then by (2.9), we find that

$$u''(1^-)v'(1)+(g(u(1))-u'''(1^-))v(1)=0$$

holds for all $v \in X$. Since $v(1), v'(1)$ are arbitrary, it follows from the last equality that $u''(1^-)=u'''(1^-)-g(u(1))=0$. Therefore u is a classical solution of IBVP (1.1). □

LEMMA 2.3. *If $u \in X$, then $\|u\| \le \|u\|_X$.*

PROOF. For any $u \in X$,

$$|u'(t)|=\left|\int_0^t u''(s)\,ds\right|\le\int_0^1|u''(s)|\,ds\le\left(\int_0^1|u''(s)|^2\,ds\right)^{1/2}=\|u\|_X. \quad (2.11)$$

Similarly, by (2.11),

$$|u(t)|=\left|\int_0^t u'(s)\,ds\right|\le\int_0^1|u'(s)|\,ds\le\int_0^1\|u\|_X\,ds=\|u\|_X.$$

Therefore, we obtain $\|u\| \le \|u\|_X$ for any $u \in X$. □

LEMMA 2.4. *Assume that the following condition holds:*

(H1) *$g(u)$ and $I_j(u)$ are nondecreasing, $N_j(u)$ are nonincreasing and $g(u)u \ge 0$, $I_j(u)u \ge 0$, and $N_j(u)u \le 0$ for any $u \in \mathbb{R}$.*

Then Φ defined by (2.1) is sequentially weakly lower semicontinuous, coercive and its derivative admits a continuous inverse on X^*.

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PROOF. Let $\{u_n\} \subset X$, $u_n \rightharpoonup u$ in X. Then $\{u_n\}$ converges to u on $C^1([0, 1])$ and $\liminf_{n\to\infty} \|u_n\|_X \geq \|u\|_X$. Thus

$$\begin{aligned}\liminf_{n\to\infty} \Phi(u_n) &= \liminf_{n\to\infty}\bigg(\frac{1}{2}\|u_n\|_X^2 + \sum_{j=1}^{l}\int_0^{u_n'(t_j)} I_j(s)\,ds \\ &\qquad - \sum_{j=1}^{l}\int_0^{u_n(t_j)} N_j(s)\,ds + \int_0^{u_n(1)} g(s)\,ds\bigg) \\ &\geq \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{l}\int_0^{u'(t_j)} I_j(s)\,ds - \sum_{j=1}^{l}\int_0^{u(t_j)} N_j(s)\,ds \\ &\quad + \int_0^{u(1)} g(s)\,ds = \Phi(u).\end{aligned}$$

Therefore, Φ is sequentially weakly lower semicontinuous. Moreover, by (2.1) and (H1),

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{l}\int_0^{u'(t_j)} I_j(s)\,ds - \sum_{j=1}^{l}\int_0^{u(t_j)} N_j(s)\,ds + \int_0^{u(1)} g(s)\,ds \\ &\geq \frac{1}{2}\|u\|_X^2.\end{aligned}$$

Hence Φ is coercive.

Next, we show that Φ' admits a continuous inverse on X^*. For any $u \in X \setminus \{0\}$, it follows from (2.3) that

$$\begin{aligned}\langle \Phi'(u), u\rangle &= \int_0^1 |u''|^2\,dt + \sum_{j=1}^{l} I_j(u'(t_j))u'(t_j) - \sum_{j=1}^{l} N_j(u(t_j))u(t_j) + g(u(1))u(1) \\ &\geq \|u\|_X^2.\end{aligned}$$

So $\lim_{\|u\|_X\to+\infty}\langle \Phi'(u), u\rangle/\|u\|_X = +\infty$, that is, Φ' is coercive.

For any $u, v \in X$,

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u - v\rangle &= \int_0^1 (u''(t) - v''(t))(u''(t) - v''(t))\,dt \\ &\quad + \sum_{j=1}^{l}(I_j(u'(t_j)) - I_j(v'(t_j)))(u'(t_j) - v'(t_j)) \\ &\quad - \sum_{j=1}^{l}(N_j(u(t_j)) - N_j(v(t_j)))(u(t_j) - v(t_j)) \\ &\quad + (g(u(1)) - g(v(1)))(u(1) - v(1)) \\ &\geq \|u - v\|_X^2,\end{aligned}$$

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so Φ' is uniformly monotone. By [27, Theorem 26.A(d)], $(\Phi')^{-1}$ exists and is continuous on X^*. □

REMARK 2.5. We can obtain the same conclusion if condition (H1) is replaced by the following condition:

(H1)′ $g(u)$, $I_j(u)$ are odd and nondecreasing, and $N_j(u)$ are odd and nonincreasing.

LEMMA 2.6. *Ψ defined by (2.2) is sequentially weakly upper semicontinuous and its derivative is compact.*

PROOF. It is easily verified that Ψ is weakly continuous. Therefore, Ψ is sequentially weakly upper semicontinuous.

Next we will show that Ψ' is strongly continuous on X. For this, let $u_n \rightharpoonup u$ as $n \to \infty$ on X. Then $u_n \to u$ in $C^1([0, 1])$. Since $f(t, u)$ is continuous on u, then $f(t, u_n) \to f(t, u)$ as $n \to \infty$. So $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. That is, Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [27, Proposition 26.2]. Moreover, Ψ' is continuous since it is strongly continuous. □

3. Main results

In this section we state and prove our main results.

THEOREM 3.1. *Suppose that (H1) or (H1)′ holds and there exist four positive constants a, b, c and p with $p < 1$ such that:*

(H2)
$$\frac{c^2}{2} < 2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds;$$

(H3)
$$\frac{2\int_0^1 \max_{|\xi|\le c} F(t, \xi)\, dt}{c^2} < \frac{\int_0^1 F(t, t^2)\, dt}{2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds};$$

(H4) $f(t, u) \le a + b|u|^p$, *for every* $(t, u) \in [0, 1] \times \mathbb{R}$. *Then, for each*

$$\lambda \in \left[\frac{2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds}{\int_0^1 F(t, t^2)\, dt}, \frac{c^2}{2\int_0^1 \max_{|\xi|\le c} F(t, \xi)\, dt} \right],$$

IBVP (1.1) has at least three classical solutions.

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PROOF. Owing to Lemma 2.2, the critical points of the functional $\Phi - \lambda\Psi$ are exactly the classical solutions of IBVP (1.1). Hence, to prove our assertion it is enough to apply Theorem 2.1.

Obviously, X is a reflexive real Banach space. From the previous section we have seen that Φ is a continuously Gâteau differentiable, sequentially weakly lower semicontinuous and coercive functional whose Gâteau derivative (2.3) admits a continuous inverse on X^* (see Lemma 2.4). Ψ is a sequentially weakly upper semicontinuous and Gâteau differentiable functional whose derivative (2.4) is compact (see Lemma 2.6).

Setting $r = c^2/2$, $u_0(t) = 0$, $\bar{u}(t) = t^2$ for all $t \in [0, 1]$, we have $u_0, \bar{u} \in X$, $\Phi(u_0) = 0$,

$$\Phi(\bar{u}) = 2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds,$$

and $\Psi(\bar{u}) = \int_0^1 F(t, t^2)\, dt$. Therefore

$$\begin{aligned}(r - \Phi(u_0))&\frac{\Psi(\bar{u})}{\Phi(\bar{u}) - \Phi(u_0)} \\ &= \frac{c^2}{2}\,\frac{\int_0^1 F(t, t^2)\, dt}{2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds}\end{aligned} \tag{3.1}$$

and, from (H2), we obtain $\Phi(u_0) < r < \Phi(\bar{u})$.

On the other hand, for all $u \in X$ such that $\Phi(u) \le r$, we have $\|u\|_X \le (2r)^{1/2}$ and, owing to Lemma 2.3, $\|u\| \le \|u\|_X \le (2r)^{1/2} = c$. Therefore,

$$\sup_{\Phi(u) \le r} \Psi(u) \le \int_0^1 \max_{|\xi| \le c} F(t, \xi)\, dt. \tag{3.2}$$

In view of (3.1), (3.2) and (H3), condition (A1) in Theorem 2.1 is satisfied.

For any $u \in X$, by (H1), (H3) and Lemma 2.3,

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{l} \int_0^{u'(t_j)} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{u(t_j)} N_j(s)\, ds \\ &\quad + \int_0^{u(1)} g(s)\, ds - \lambda \int_0^1 F(t, u)\, dt \\ &\ge \frac{1}{2}\|u\|_X^2 - \lambda a\|u\| - \lambda b\|u\|^{p+1} \\ &\ge \frac{1}{2}\|u\|_X^2 - \lambda a\|u\|_X - \lambda b\|u\|_X^{p+1}.\end{aligned}$$

Since $p < 1$, then $\lim_{\|u\|_X \to \infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$ for all $\lambda \ge 0$. So (A2) in Theorem 2.1 is satisfied. Hence, our claim is proved and Theorem 2.1 ensures the conclusion. □

https://doi.org/10.1017/S0004972710001802 Published online by Cambridge University Press

We consider a special case of IBVP (1.1). Let $f(t, u) = h(u) \in C(\mathbb{R})$. Define $H(\xi) = \int_0^{\xi} h(s)\, ds$. We have the following result.

COROLLARY 3.2. *Suppose that (H1) (or (H1)′) and (H2) hold. Moreover, assume that*

(H3)′
$$\frac{2\max_{|\xi|\le c} H(\xi)}{c^2} < \frac{\int_0^1 H(t^2)\, dt}{2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds},$$

(H4)′ $h(u) \le a + b|u|^p$, *for every* $u \in \mathbb{R}$, *hold. Then, for each*

$$\lambda \in \left[\frac{2 + \sum_{j=1}^{l} \int_0^{t_j^2} I_j(s)\, ds - \sum_{j=1}^{l} \int_0^{2t_j} N_j(s)\, ds + \int_0^1 g(s)\, ds}{\int_0^1 H(t^2)\, dt}, \frac{c^2}{2\max_{|\xi|\le c} H(\xi)}\right],$$

IBVP (1.1) has at least three classical solutions.

EXAMPLE 3.3. Consider the following problem:

$$\begin{cases} u^{(4)}(t) = \lambda h(u) & \text{a.e. } t \in [0, 1], \\ \Delta(u''(t_j)) = I_j(u'(t_j)) & j = 1, \\ \Delta(u'''(t_j)) = N_j(u(t_j)) & j = 1, \\ u(0) = u'(0) = u''(1^-) = 0 & \\ u'''(1^-) = g(u(1)) & \end{cases} \tag{3.3}$$

where $g(u) = u$, $I_j(u) = 32u$ and $N_j(u) = -u$. Clearly, condition (H1) is satisfied. Let

$$h(u) = \begin{cases} \dfrac{1}{32}, & u \le \dfrac{1}{2}, \\ 1000u^{1/3} - \dfrac{1000}{\sqrt[3]{2}} + \dfrac{1}{32}, & u > \dfrac{1}{2}. \end{cases}$$

Then

$$H(u) = \begin{cases} \dfrac{1}{32}u, & u \le \dfrac{1}{2}, \\ 750u^{4/3} - \dfrac{1000}{\sqrt[3]{2}}u + \dfrac{1}{32}u + \dfrac{125}{\sqrt[3]{2}}, & u > \dfrac{1}{2}. \end{cases}$$

Taking $t_j = \frac{1}{2}$, $a = \frac{1}{32}$, $b = 1000$, $l = \frac{1}{3}$, $c = \frac{1}{2}$, then g, I_j, N_j, h satisfy (H2), (H3)′ and (H4)′ in Corollary 3.2. Applying Corollary 3.2, for each $\lambda \in \Lambda_{1/8} = [0.0058, 8]$, IBVP (3.3) has at least three classical solutions.

References

[1] R. P. Agarwal, D. Franco and D. O'Regan, 'Singular boundary value problems for first and second order impulsive differential equations', *Aequationes Math.* **69** (2005), 83–96.

[2] B. Ahmad and J. J. Nieto, 'Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions', *Nonlinear Anal.* **69** (2008), 3291–3298.

[3] G. Bonanno and B. Di. Bella, 'A boundary value problem for fourth-order elastic beam equations', *J. Math. Anal. Appl.* **343** (2008), 1166–1176.

[4] G. Bonanno and P. Candito, 'Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities', *J. Differential Equations* **244** (2008), 3031–3059.

[5] G. Bonanno and S. A. Marano, 'On the structure of the critical set of non-differentiable functions with a weak compactness condition', *Appl. Anal.* **89** (2010), 1–10.

[6] G. Bonanno and G. Riccobono, 'Multiplicity results for Sturm–Liouville boundary value problems', *Appl. Math. Comput.* **210** (2009), 294–297.

[7] L. Chen and J. Sun, 'Nonlinear boundary value problem for first order impulsive functional differential equations', *J. Math. Anal. Appl.* **318** (2006), 726–741.

[8] L. Chen, C. C. Tisdel and R. Yuan, 'On the solvability of periodic boundary value problems with impulse', *J. Math. Anal. Appl.* **331**(2) (2007), 233–244.

[9] J. Chu and J. J. Nieto, 'Impulsive periodic solution of first-order singular differential equations', *Bull. London Math. Soc.* **40** (2008), 143–150.

[10] Z. Du, X. Lin and C. C. Tisdel, 'A multiplicity result for p-Laplacian boundary value problems via critical points theorem', *Appl. Math. Comput.* **205** (2008), 231–237.

[11] E. Hernandez, H. R. Henriquez and M. A. McKibben, 'Existence results for abstract impulsive second-order neutral functional differential equations', *Nonlinear Anal.* **70** (2009), 2736–2751.

[12] I. Karaca, 'On positive solutions for fourth-order boundary value problem with impulse', *J. Comput. Appl. Math.* **225** (2009), 356–364.

[13] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations* (World Scientific, Singapore, 1989).

[14] J. Li, J. J. Nieto and J. Shen, 'Impulsive periodic boundary value problems of first-order differential equations', *J. Math. Anal. Appl.* **325** (2007), 226–299.

[15] T. Ma, 'Positive solutions for a beam equation on a nonlinear elastic foundation', *Math. Comput. Modelling* **39** (2004), 1195–1201.

[16] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems* (Springer, Berlin, 1989).

[17] J. J. Nieto and D. O'Regan, 'Variational approach to impulsive differential equations', *Nonlinear Anal.: Real World Appl.* **10** (2009), 680–690.

[18] J. J. Nieto and R. Rodriguez-Lopez, 'New comparison results for impulsive integro-differential equations and applications', *J. Math. Anal. Appl.* **328** (2007), 1343–1368.

[19] D. Qian and X. Li, 'Periodic solutions for ordinary differential equations with sublinear impulsive effects', *J. Math. Anal. Appl.* **303** (2005), 288–303.

[20] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, 65 (American Mathematical Society, Providence, RI, 1986).

[21] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations* (World Scientific, Singapore, 1995).

[22] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, 3rd edn (Springer, Berlin, 2000).

[23] J. Sun, H. Chen, J. J. Nieto and M. Otero-Novoa, 'Multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects', *Nonlinear Anal.: TMA* **72** (2010), 4575–4586.

[24] J. Sun, H. Chen and L. Yang, 'The existence and multiplicity of solutions for an impulsive differential equation with two parameters via variational method', *Nonlinear Anal.: TMA* **73** (2010), 440–449.

[25] Y. Tian and W. Ge, 'Second-order Sturm–Liouville boundary value problem involving the one-dimensional p-Laplacian', *Rocky Mountain J. Math.* **38** (2008), 309–327.
[26] S. Timoshenko, W. Weaver Jr and D. H. Young, *Vibrations Problems in Engineering*, 5th edn (John Wiley, New York, 1990).
[27] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. 2 (Springer, Berlin, 1990).
[28] L. Zhang and W. Ge, 'Solvability of a kind of Sturm–Liouville boundary value problems with impulses via variational methods', *Acta Appl. Math.* **110** (2010), 1237–1248.
[29] H. Zhang and Z. Li, 'Variational approach to impulsive differential equations with periodic boundary conditions', *Nonlinear Anal.: Real World Appl.* **11** (2010), 67–78.

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https://doi.org/10.1017/S0004972710001802 Published online by Cambridge University Press