SYMBOLS FOR TRACE CLASS HANKEL OPERATORS WITH GOOD ESTIMATES FOR NORMS

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Introduction. Peller [**4, 5**] has proved that a Hankel operator S on the Hardy space H^2 is in the trace class if and only if $S = S_{\bar{h}}$ with h analytic on the open unit disc D and with its second derivative belonging to the Bergman space L^1_a. This theorem does not include an estimate for the trace class norm $\|S\|_1$ of the operator in terms of the symbol function. In fact it is clear that $\|h''\|_{L^1_a}$ cannot give an estimate for $\|S_{\bar{h}}\|_1$ since the first two terms in the coefficient sequence of the Hankel operator have been removed by differentiation.

We give a slightly modified version of Peller's theorem which eliminates this difficulty and leads to a satisfactory estimate for $\|S\|_1$. The proof uses a modified version of the Coifman–Rochberg decomposition theorem for L^1_a, [**3**]. As a corollary, we obtain a bounded projection of the trace class onto its Hankel operators, again with a good estimate of the norm. For other bounded projections with the same domain and range, see [**4**].

Notation. Let $L^p = L^p(\partial D)$ with normalized Lebesgue measure, let H^p denote the usual Hardy space of functions on ∂D, and let L^p_a denote the Bergman space of analytic functions on D for which

$$\|f\|_{L^p_a} = \left(\frac{1}{\pi}\int\int_D |f(z)|^p \, dx \, dy\right)^{1/p} < \infty.$$

With $\phi \in L^2$, S_ϕ denotes the Hankel operator with symbol ϕ, that is the operator with domain and range in H^2 and with matrix (a_{i+j}), where the *coefficient sequence* $\{a_n\}$ is given by

$$a_n = \hat{\phi}(-n) \quad (n \geq 0).$$

In particular, when $\phi \in L^\infty$,

$$S_\phi = PJM_\phi \mid H^2,$$

where P is the orthogonal projection of L^2 onto H^2, $(Jf)(\zeta) = f(\bar{\zeta})(f \in L^2, \zeta \in \partial D)$, and M_ϕ is multiplication by ϕ (see Power [**8**]).

Several elementary functions will be needed, and it is convenient to list them here. With $w \in D$, $z \in D \cup \partial D$, write

$$f_w(z) = (1 - \bar{w}z)^{-3/2}, \qquad v_w(z) = (1 - |w|^2)^{1/2}/(1 - \bar{w}z),$$
$$g_w(z) = (1 - |w|^2)/(1 - \bar{w}z), \qquad h_w(z) = z^2 g_w(z),$$
$$b_w(z) = 2(1 - |w|^2)/(1 - \bar{w}z)^3.$$

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THEOREM 1. *Let $g \in H^2$ and let $h(z) = z^2 g(z)$ $(z \in D)$, where $g(z)$ is the usual analytic extension of g to D. Then $S_{\bar{g}}$ is trace class if and only if $h'' \in L^1_a$. Also*

$$\frac{\pi}{8}\|h''\|_{L^1_a} \le \|S_{\bar{g}}\|_1 \le \|h''\|_{L^1_a}. \tag{1}$$

The constant $\pi/8$ is best possible.

Proof. Let $S_{\bar{g}}$ belong to the trace class. Then

$$S_{\bar{g}} = \sum_{k=1}^{\infty} \lambda_k (u_k \otimes v_k)$$

with $u_k, v_k \in H^2$, $\|u_k\|_2 = \|v_k\|_2 = 1$ for all k and with $\sum_{k=1}^{\infty} |\lambda_k| = \|S_{\bar{g}}\|_1$. For $w \in D$,

$$h''(w) = \frac{1}{\pi}\int_0^{2\pi} g(e^{i\theta}) \frac{e^{3i\theta}}{(e^{i\theta} - w)^3}\, d\theta,$$

and so

$$\begin{aligned}\overline{h''(w)} &= \frac{1}{\pi}\int_0^{2\pi} \overline{g(e^{i\theta})}(f_w(e^{i\theta}))^2\, d\theta \\ &= 2(\bar{g}f_w, \bar{f}_w) \\ &= 2(\bar{g}f_w, Jf_{\bar{w}}) \\ &= 2(PJ\bar{g}f_w, f_{\bar{w}}) = 2(S_{\bar{g}}f_w, f_{\bar{w}}).\end{aligned}$$

Since

$$\begin{aligned}(S_{\bar{g}}f_w, f_{\bar{w}}) &= \sum_{k=1}^{\infty} \lambda_k((u_k \otimes v_k)f_w, f_{\bar{w}}) \\ &= \sum_{k=1}^{\infty} \lambda_k (f_w, v_k)(u_k, f_{\bar{w}}),\end{aligned}$$

$$h''(w) = 2\sum_{k=1}^{\infty} \bar{\lambda}_k (v_k, f_w)(f_{\bar{w}}, u_k).$$

Given $v \in H^2$, let $(Tv)(w) = (v, f_w)(w \in D)$. Then

$$\begin{aligned}(Tv)(w) &= \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\theta})(1 - we^{-i\theta})^{-3/2}\, d\theta \\ &= \sum_{n=0}^{\infty} \beta_n w^n,\end{aligned}$$

with

$$\beta_n = (-1)^n \binom{-3/2}{n} \hat{v}(n) = \frac{1}{n!}\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n+1)}{2}\,\hat{v}(n).$$

Therefore

$$\|Tv\|^2_{L^2_a} = \frac{1}{\pi}\int_0^1\int_0^{2\pi} |Tv(re^{i\theta})|^2\, r\, d\theta\, dr$$
$$= 2\int_0^1 \sum_{n=0}^{\infty} |\beta_n|^2 r^{2n+1}\, dr = \sum_{n=0}^{\infty} \frac{|\beta_n|^2}{n+1}.$$

Since $\alpha_n = \binom{-3/2}{n}^2 \Big/ (n+1)$ increases with n and, by Stirling's formula, converges to $4/\pi$, we have

$$\|Tv\|^2_{L^2_a} \leq \frac{4}{\pi}\|v\|_2^2.$$

Since also

$$\overline{(f_{\bar{w}}, u)} = (u, f_{\bar{w}}) = (Tu)(\bar{w}),$$

the Cauchy–Schwarz inequality now gives

$$\|h''\|_{L^1_a} \leq 2\sum_{k=1}^{\infty} |\lambda_k|\, \|Tv_k\|_{L^2_a}\, \|Tu_k\|_{L^2_a}$$
$$\leq \frac{8}{\pi}\sum_{k=1}^{\infty} |\lambda_k|\, \|v_k\|_2\, \|u_k\|_2$$
$$= \frac{8}{\pi}\|S_{\bar{g}}\|_1. \tag{2}$$

This proves that $h'' \in L^1_a$ and establishes the first inequality. To complete the proof we need two lemmas. First however we note some properties of the elementary functions g_w, h_w, b_w.

A routine calculation gives

$$S_{\bar{g}_w} = v_{\bar{w}} \otimes v_w,$$

and, since $\|v_w\|_2 = 1$, this shows that $S_{\bar{g}_w}$ is a rank one Hankel operator and

$$\|S_{\bar{g}_w}\|_1 = 1 \quad (w \in D). \tag{3}$$

Since $h_w(z) = z^2 g_w(z)$ and $h''_w = b_w$, inequality (2) shows that

$$\|b_w\|_{L^1_a} \leq \frac{8}{\pi} \quad (w \in D). \tag{4}$$

Let B denote the space of Bloch functions that vanish at 0, that is functions f analytic on D with $f(0) = 0$ and with $(1-|z|^2)f'(z)$ bounded on D, and let

$$\|f\|_B = \sup\{(1-|z|^2)\, |f'(z)| : z \in D\}.$$

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Given $f \in L_a^1$ and $g \in B$, let

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} (1 - r^2) g'(re^{-i\theta}) f(re^{i\theta}) r \, d\theta \, dr,$$

which plainly satisfies

$$|\langle f, g \rangle| \leq \|f\|_{L_a^1} \|g\|_B. \tag{5}$$

It is known that B represents the dual space of L_a^1 through $\langle , \rangle$. (See [**1**], where a slightly different space is used in place of L_a^1.) In Lemma 2 we state this result with estimates for norms.

Let $c_w(z) = 2(1 - wz)^{-3} (w, z \in D)$, and given $\psi \in (L_a^1)^*$, let

$$g_\psi(z) = \int_{[0,z]} \psi(c_w) \, dw,$$

where the integration is along the line segment joining 0 to z. Then g_ψ is analytic in D and

$$(1 - |z|^2) g'_\psi(z) = \psi(b_{\bar{z}}) \quad (z \in D). \tag{6}$$

With (4), (6) shows that $g_\psi \in B$ and $\|g_\psi\|_B \leq \frac{8}{\pi} \|\psi\|$. Standard arguments now complete the proof of the following lemma.

LEMMA 2. *The mapping $\psi \to g_\psi$ is a linear bijection of $(L_a^1)^*$ onto B,*

$$\psi(f) = \langle f, g_\psi \rangle \quad (f \in L_a^1, \psi \in (L_a^1)^*), \tag{7}$$

and

$$\frac{\pi}{8} \|g_\psi\|_B \leq \|\psi\| \leq \|g_\psi\|_B \quad (\psi \in (L_a^1)^*) \tag{8}$$

The following lemma is a modified version of the Coifman–Rochberg decomposition theorem [**3**] for L_a^1 with estimates of norms.

LEMMA 3. *L_a^1 is the set of functions f of the form*

$$f = \sum_{k=1}^{\infty} \lambda_k b_{w_k} \tag{9}$$

with $w_k \in D$, $\lambda_k \in \mathbb{C}$ and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. Also

$$\frac{\pi}{8} \|f\|_{L_a^1} \leq \inf \sum_{k=1}^{\infty} |\lambda_k| \leq \|f\|_{L_a^1}, \tag{10}$$

where the infimum is taken over all decompositions (9) *of f.*

Proof. By Lemma 2 and (6),

$$\|\psi\| \leq \|g_\psi\|_B \leq \sup\{|\psi(b_z)| : z \in D\}.$$

Using also (4), the lemma now follows at once from ([**2**], Theorem 1).

Proof of Theorem 1 *continued*. Suppose that $h'' \in L_a^1$ and $\varepsilon > 0$. By Lemma 3, there exist $w_k \in D$, $\lambda_k \in \mathbb{C}$ with $\sum_{k=1}^{\infty} |\lambda_k| < \|h''\|_{L_a^1} + \varepsilon$ and (since $h''_w = b_w$),

$$h'' = \sum_{k=1}^{\infty} \lambda_k h''_{w_k}.$$

This series converges uniformly on compact subsets of D since $|b_w(z)| \le 2(1-|z|)^{-3}$, and so integration twice gives

$$h = \sum_{k=1}^{\infty} \lambda_k h_{w_k},$$

where we have used the fact that both sides have zero constant term and first degree term. Thus $g = \sum_{k=1}^{\infty} \lambda_k g_{w_k}$, $S_{\bar g} = \sum_{k=1}^{\infty} \bar\lambda_k S_{\bar g_{w_k}}$, and

$$\|S_{\bar g}\|_1 \le \sum_{k=1}^{\infty} |\lambda_k| < \|h''\|_{L_a^1} + \varepsilon.$$

It remains to prove that the constant $\pi/8$ is best possible. Let $w \in D$ and $\phi_w(z) = \sqrt{2}\,(1-|w|^2)^{1/2}/(1-\bar w z)^{3/2}$, so that $b_w = \phi_w^2$. Then

$$\|h''_w\|_{L_a^1} = \|\phi_w\|_{L_a^2}^2 = 2(1-|w|^2) \sum_{n=0}^{\infty} \alpha_n |w|^{2n},$$

with $\alpha_n = \binom{-3/2}{n}^2 \Big/ (n+1)$ as before. Since the sequence $\{\alpha_n\}$ increases, Abel's theorem gives

$$\begin{aligned}
\sup_{w\in D} \|h''_w\|_{L_a^1} &= \sup_{0\le t<1} 2(1-t) \sum_{n=0}^{\infty} \alpha_n t^n \\
&= 2 \lim_{t\to 1-0} \left\{ \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n - \alpha_{n-1}) t^n \right\} \\
&= 2\left\{ \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n - \alpha_{n-1}) \right\} \\
&= 2 \lim_{n\to\infty} \alpha_n = 8/\pi. \qquad (11)
\end{aligned}$$

Since $\|S_{\bar g_w}\|_1 = 1$, this completes the proof of the theorem.

Remarks. (1) We have proved that

$$\|S_{\bar g}\|_1 \le C \,\|h''\|_{L_a^1} \quad (h'' \in L_a^1),$$

for some constant $C \le 1$, but we do not know whether 1 is the best value of C. It is easy to prove that $C \ge \frac{1}{2}$, by taking $g = g_0 (=1)$. We have $\|S_{\bar g_0}\| = 1$ and $\|h''_0\|_{L_a^1} = 2$.

(2) The same constants $\pi/8$ and 1 occur also in Lemmas 2 and 3 and it is natural to ask whether they are best possible.

We have already proved in (11) that

$$\sup_{w\in D} \|b_w\|_{L_a^1} = 8/\pi.$$

This shows at once that the constant $\pi/8$ is best possible in Lemma 3, for with $f = b_w$ we have $\inf \sum_{k=1}^{\infty} |\lambda_k| \leq 1$. It follows that $\pi/8$ is also best possible in Lemma 2. For suppose that $C > 0$ with

$$C \, \|g_\psi\|_B \leq \|\psi\| \quad (\psi \in (L_a^1)^*).$$

Given $f \in L_a^1$, there exists $\psi \in (L_a^1)^*$ with $\|\psi\| = 1$ and $\psi(f) = \|f\|_{L_a^1}$. If $f = \sum_{k=1}^{\infty} \lambda_k b_{w_k}$, it follows that

$$\begin{aligned} \|f\|_{L_a^1} &\leq \sum_{k=1}^{\infty} |\lambda_k| \, |\psi(b_{w_k})| \\ &\leq \sum_{k=1}^{\infty} |\lambda_k| \sup_{w \in D} |\psi(b_w)| \\ &\leq \sum_{k=1}^{\infty} |\lambda_k| \, \|g_\psi\|_B \leq C^{-1} \sum_{k=1}^{\infty} |\lambda_k|. \end{aligned}$$

This gives $C \, \|f\|_{L_a^1} \leq \inf \sum_{k=1}^{\infty} |\lambda_k|$, and so $C \leq \pi/8$ by the result just proved for Lemma 3.

Next we prove that if $\|\psi\| \leq C \, \|g_\psi\|_B$ for all $\psi \in (L_a^1)^*$, then $C \geq e/4$. For this we take $f(z) = z^{n-1}$, $g(z) = z^n$. Then $\|f\|_{L_a^1} = 2/(n+1)$, $\|g\|_B = \dfrac{2n}{n+1}\left(1 - \dfrac{2}{n+1}\right)^{\frac{n-1}{2}}$ so that $\lim_{n\to\infty} \|g\|_B = 2/e$, while $\langle f, g \rangle = 1/(n+1)$.

It now follows also that if $\inf \sum_{k=1}^{\infty} |\lambda_k| \leq C \, \|f\|_{L_a^1}$ for all $f \in L_a^1$, then $C \geq e/4$. For given $f \in L_a^1$ with $\|f\|_{L_a^1} = 1$ and $\varepsilon > 0$, we choose w_k, λ_k with $f = \sum_{k=1}^{\infty} \lambda_k b_{w_k}$ and $\sum_{k=1}^{\infty} |\lambda_k| < C + \varepsilon$. Given $\psi \in (L_a^1)^*$, we have

$$|\psi(f)| = \left| \sum_{k=1}^{\infty} \lambda_k \psi(b_{w_k}) \right| \leq (C + \varepsilon) \, \|g_\psi\|_B,$$

$$\|\psi\| \leq C \, \|g_\psi\|_B,$$

and so $C \geq e/4$ by the corresponding result for Lemma 2.

A Hankel operator valued projection on the trace class. Let $\mathscr{C}_1$ denote the Banach space of trace class operators on H^2, $\mathscr{S} \cap \mathscr{C}_1$ the closed subspace of Hankel operators in $\mathscr{C}_1$. As a corollary of Theorem 1 we obtain a bounded projection $P_{\mathscr{S}}$ on the space $\mathscr{C}_1$ with range $\mathscr{S} \cap \mathscr{C}_1$, together with a satisfactory estimate of its norm. $P_{\mathscr{S}}$ is the special case $P_{1/2,1/2}$ of a family of projections $P_{\alpha,\beta}$ of this kind found by A. B. Aleksandrov, and it is known that the natural averaging projection is not bounded on $\mathscr{C}_1$, though it is bounded on the Schatten-von Neumann spaces with $1 < p < \infty$. (See Peller [**4**, **6**, **7**]).

Given an operator $A \in \mathscr{C}_1$ with matrix (a_{ij}) relative to the natural basis, we define $P_{\mathscr{S}}A$ to be the Hankel operator with coefficient sequence $\{b_n\}$ given by

$$b_n = \frac{2(-1)^n}{(n+2)(n+1)} \sum_{i+j=n} \binom{-3/2}{i} \binom{-3/2}{j} a_{ij}. \tag{12}$$

Comparison of the coefficient of z^n in the expansions of $((1-z)^{-3/2})^2$ and $(1-z)^{-3}$ shows that $b_n = c_n$ if $a_{ij} = c_{i+j}$, and so $P_{\mathcal{S}}$ is a projection.

COROLLARY 4. *The projection $P_{\mathcal{S}}$ defined by* (12) *is a bounded projection on $\mathcal{C}_1$ with range $\mathcal{C}_1 \cap \mathcal{S}$, and*

$$\|P_{\mathcal{S}}\| \leq 8/\pi.$$

Proof. Let $A = u \otimes v$ with $u, v \in H^2$. The matrix (a_{ij}) of A is given by $a_{ij} = \hat{u}_i \bar{\hat{v}}_j$, where $\hat{u}_n$, $\hat{v}_n$ are the Fourier coefficients of u, v. Therefore the coefficient sequence $\{b_n\}$ of the Hankel operator $P_{\mathcal{S}}A$ is given by

$$b_n = \frac{2(-1)^n}{(n+2)(n+1)} \sum_{i+j=n} \binom{-3/2}{i}\binom{-3/2}{j} \hat{u}_i \bar{\hat{v}}_j.$$

Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $h(z) = z^2 g(z)$. Then

$$h''(z) = \sum_{n=0}^{\infty} (n+2)(n+1) b_n z^n = 2\psi_1(z)\psi_2(z),$$

with

$$\psi_1(z) = \sum_{k=0}^{\infty} (-1)^k \binom{-3/2}{k} \hat{u}_k z^k, \qquad \psi_2(z) = \sum_{k=0}^{\infty} (-1)^k \binom{-3/2}{k} \bar{\hat{v}}_k z^k.$$

By Theorem 1,

$$\|P_{\mathcal{S}}A\|_1 = \|S_{\bar{g}}\|_1 \leq \|h''\|_{L_a^1} \leq 2\, \|\psi_1\|_{L_a^2}\, \|\psi_2\|_{L_a^2}.$$

By a calculation in the proof of Theorem 1, $\|\psi_1\|_{L_a^2} \leq \left(\frac{4}{\pi}\right)^{1/2} \|u\|_2$, and similarly for $\|\psi_2\|_{L_a^2}$. Thus

$$\|P_{\mathcal{S}}A\|_1 \leq 8/\pi\, \|u\|_2\, \|v\|_2 = 8/\pi\, \|A\|_1.$$

Given arbitrary $A \in \mathcal{C}_1$, we have $A = \sum_{k=1}^{\infty} \lambda_k A_k$, with $A_k = u_k \otimes v_k$, $\|A_k\|_1 = 1$, and $\sum_{k=1}^{\infty} |\lambda_k| = \|A\|_1$. The series $\sum_{k=1}^{\infty} \lambda_k P_{\mathcal{S}} A_k$ converges in the Banach space $\mathcal{C}_1$ to a Hankel operator S and $\|S\|_1 \leq \frac{8}{\pi} \sum_{k=1}^{\infty} |\lambda_k| = \frac{8}{\pi} \|A\|_1$. If $\{b_n^{(k)}\}$ is the coefficient sequence of $P_{\mathcal{S}}A_k$, then the coefficient sequence of S is $\{b_n\}$ given by

$$b_n = \sum_{k=1}^{\infty} \lambda_k b_n^{(k)} = \sum_{k=1}^{\infty} \lambda_k \frac{2(-1)^n}{(n+2)(n+1)} \sum_{i+j=n} \binom{-3/2}{i}\binom{-3/2}{j} a_{ij}^{(k)}$$

where $(a_{ij}^{(k)})$ is the matrix of A_k. Since the matrix (a_{ij}) of A is given by $a_{ij} = \sum_{k=1}^{\infty} \lambda_k a_{ij}^{(k)}$, it follows that

$$b_n = \frac{2(-1)^n}{(n+2)(n+1)} \sum_{i+j=n} \binom{-3/2}{i}\binom{-3/2}{j} a_{ij},$$

and we have proved that $S = P_{\mathcal{S}}A$.

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REMARK. The proof of Corollary 4 has used the inequality $\|S_{\bar{g}}\|_1 \le \|h''\|_{L_a^1}$ from Theorem 1, which we do not know to be best possible. If this can be improved to $\|S_{\bar{g}}\|_1 \le c\,\|h''\|_{L_a^1}$ for some $c<1$, we obtain the improved estimate $\|P_{\mathcal{S}}\| \le c8/\pi$ for the norm of $P_{\mathcal{S}}$.

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