BULL. AUSTRAL. MATH. SOC. 34C10
VOL. 9 (1973), 219-226. (34G05)

# Differential equations in Banach spaces

E.S. Noussair

Let $H$ be a fixed Hilbert space and $B(H, H)$ be the Banach space of bounded linear operators from $H$ to $H$ with the uniform operator topology. Oscillation criteria are obtained for the operator differential equation

$$\frac{d}{dt}\left(A(t)\frac{dY}{dt}\right) + C(t, Y)Y = 0 , \quad t \geq 0 ,$$

where the coefficients $A, C$ are linear operators from $B(H, H)$ to $B(H, H)$, for each $t \geq 0$. A solution $Y : R^+ \to B(H, H)$ is said to be oscillatory if there exists a sequence of points $t_i \in R^+$, so that $t_i \to \infty$ as $i \to \infty$, and $Y(t_i)$ fails to have a bounded inverse. The main theorem states that a solution $Y$ is oscillatory if an associated scalar differential equation is oscillatory.

## 1. Introduction

In this paper we study those aspects of the qualitative behaviour of solutions of second order differential equations in Banach spaces involving the notions of oscillation and non-oscillation. The basic tool in the analysis is a generalization of Picone's identity [7], [8].

Some other papers pertinent to the subject under consideration here are [4], [5], [6].

Let $H$ be a fixed Hilbert space and $B(H, H)$ be the Banach space of

Received 17 April 1973.

bounded linear operators from $H$ to $H$ with the uniform operator topology. Consider the differential equation

$$TY = \frac{d}{dt}\left(A(t)\,\frac{dY}{dt}\right) + C(t,\, Y)Y = 0$$

for $t \in R^+ = \{t : t > 0\}$. Solutions $Y(t)$ are functions from $R^+$ into $B(H,\, H)$. We assume that $C(t,\, Y)$ is a linear map from $B(H,\, H)$ to $B(H,\, H)$, for each $t \in R^+$, and further, that $A(t)$, $C(t,\, Y)$ are continuous in the uniform operator topology as functions from $R^+$ to $B(H,\, H)$, and from $R^+ \times B(H,\, H)$ to $B(H,\, H)$ respectively. We also assume for each $t \geq 0$ and each $Y \in B(H,\, H)$ that $C(t,\, Y)$ is self-adjoint $(C = C^*)$, and $A(t)$ is positive definite, that is, $\left(A(t)e,\, e\right) > 0$ for all $t > 0$ and for all $0 \neq e \in H$, where $(\;,\;)$ is the inner product in $H$.

Derivatives of $Y(t)$ are computed in the uniform operator topology, that is,

$$\lim_{\delta \to 0} \left\| \frac{Y(t+\delta)-Y(t)}{\delta} - Y'(t) \right\| = 0 .$$

Under the above assumptions the initial value problem:

$$TY = 0 \qquad t \in R^+ ,$$

$$Y\left(t_0\right) = Y_0 , \quad \frac{d}{dt}\, Y\left(t_0\right) = Y_1 , \quad t_0 \in R^+$$

has a unique solution [4]. Motivated by the finite dimensional case we introduce the following definitions.

**DEFINITION.** A solution $Y(t)$ is said to be *nonsingular* at $t_0$ if $Y\left(t_0\right)$ has a bounded inverse. $Y(t)$ is said to be *oscillatory* if there exists a sequence $\left\{t_i\right\} \in R^+$ such that $t_i \to \infty$ as $i \to \infty$, and each $t_i$ is a singular point of $Y$. This definition is a natural extension of the finite dimensional case, where it is customary to say that matrix solutions of (1) are nonsingular at $t_0$ if the determinant of $Y\left(t_0\right)$ is not zero.

**DEFINITION.** A solution $Y(t)$ of equation (1) is said to be *prepared* if:

(1) $Y^*(t)A(t)\frac{d}{dt}Y(t) = \left(\frac{d}{dt}Y(t)\right)^*A(t)Y(t)$ for all $t \le 0$ ;

(2) there exists a common nonzero vector $e$ which belongs to the ranges of the operators $\{Y(t) : t$ is a nonsingular point$\}$ .

This definition ensures, as pointed out by Noussair and Swanson [6], for example, that every solution of the scalar equation $Y'' + Y = 0$ is oscillatory, as is well known [3]. However, the nonprepared matrix solution

$$Y(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

is obviously non-oscillatory. Accordingly, the prepared hypothesis on $Y$ is needed in order that an analog of the classical theory of oscillation $[H = R^1]$ can be developed for operator equations.

Condition (2) above is satisfied in the finite dimensional case since a nonsingular operator is onto. We could replace condition (2) by requiring that $Y(t)$ has a bounded inverse defined on the whole space for every nonsingular point $t$ . However, this will restrict the class of oscillatory solutions as the following example shows.

EXAMPLE 1. Let $H = l^2$ the Hilbert space of square-summable sequences $x = \{x_i\}_{i=1}^{\infty}$ . Let $A$ be the right shift operator on $l^2$ , that is $A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$ . In (1) take $A(t) = C(t, Y) = I$ , the identity operator, $Y(0) = 0$ and $\frac{d}{dt}Y(0) = A$ . Then the solution to (1) is $Y(t) = (\sin t)A$ . Now if $\sin t$ is not zero then $Y(t)$ has a bounded inverse. But the range of $Y(t)$ is a proper closed subspace of $l^2$ . However, $Y(t)$ is a prepared solution according to the definition above, as can be easily verified.

## 2. Oscillation theorems

Let $I$ be the closed interval $[t_0, t_1]$ , and let $D_0$ denote the set of all vector functions $u \in C^1(I)$, with range in $H$ such that $u(t_0) = u(t_1) = 0$.

THEOREM 1. *If*

(1) $Y(t)$ *is a prepared solution of* $Y^*TY \le 0$, *and* $e \in H$ *is in the range of* $Y(t)$ *for all* $t \in I$; *and*

(2) *there exists a nontrivial function* $\phi(t) \in C^1(I)$ *such that*

$$\int_{t_0}^{t_1} \left[(A(t), e, e)\left(\frac{d\phi}{dt}\right)^2 - (C(t, Y)e, e)\phi^2\right]dt \le 0 ,$$

*then*

*(a)* $Y(t)$ *has a singular point in* $I$; *and*

*(b) either* $Y(t)$ *has a singular point in the open interval* $(t_0, t_1)$, *or* $\phi(t)e = Y(t)c$ *on* $I$ *for some constant vector* $c \ne 0$.

Proof. If $Y(t)$ has no singular point in $I$, there exists a unique vector $\omega \in D_0$ satisfying $\phi(t)e = Y(t)\omega(t)$ identically in $I$. The following identity, a generalization of Picone's identity [8], can be easily verified by differentiation:

$$(2) \quad -\frac{d}{dt}\left(A(t)\left(\frac{dY}{dt}\right)\omega,\ Y\omega\right) = -(A(t)e, e)\left(\frac{d\phi}{dt}\right)^2 - (C(t, Y)e, e)\phi^2 - (Y^*TY\omega, \omega)$$
$$- \left(\left[Y^*A(t)\frac{dY}{dt} - \left(\frac{dY}{dt}\right)^* A(t)Y\right]\frac{d\omega}{dt},\ \omega\right) + \left(A(t)Y\frac{d\omega}{dt},\ Y\frac{d\omega}{dt}\right) .$$

Since $Y$ is a prepared solution of $Y^*TY \le 0$ and $\omega(t_0) = \omega(t_1) = 0$, integration of (2) over $I$ and use of Green's formula yields

$$(3) \quad \int_{t_0}^{t_1} \left[(A(t)e, e)\left(\frac{d\phi}{dt}\right)^2 - (C(t, Y)e, e)\phi^2\right]dt \ge \int_{t_0}^{t_1} \left(A(t)Y\frac{d\omega}{dt},\ Y\frac{d\omega}{dt}\right)dt .$$

Since $A(t)$ is positive-definite, by the assumptions made on the coefficients of the operator $T$, inequality (3) and hypothesis (2) imply that $\frac{d\omega}{dt} = 0$ identically in $I$, that is, $\phi(t)e = Y(t)\omega(t) = Y(t)c$

https://doi.org/10.1017/S0004972700043112 Published online by Cambridge University Press

identically in $I$, for some constant vector $c$, with $c \neq 0$ since $\phi(t)$ is nontrivial by the hypothesis (2). Since $\phi(t)$ and $Y$ are continuous in $I$ and $\phi(t_0) = \phi(t_1) = 0$, the equality $\frac{d\omega}{dt} = 0$ implies that $t_0, t_1$ are singular points of $Y$. This proves conclusion *(a)* of the theorem. The proof of the "strong conclusion" *(b)* is similar to the proof in the finite dimensional case [5].

THEOREM 2. *Every prepared solution of* $Y^*TY \leq 0$ *is oscillatory in* $R^+$ *if the scalar equation*

$$lu = \frac{d}{dt}\left[(A(t)e,\ e)\frac{du}{dt}\right] + (C(t,\ Y)e,\ e)u = 0 \tag{4}$$

*is oscillatory in* $R^+$ *for every unit vector* $e \in H$, *and for every nonsingular operator* $Y \in B(H,\ H)$.

Proof. Let $Y(t)$ be a prepared solution of $Y^*TY \leq 0$. If $Y(t)$ is not oscillatory then there exists a number $r > 0$ such that $Y(t)$ has no singular points for $t \geq r$. Since $Y$ is prepared, there exists a vector $e$ in the range of $Y(t)$ for all $t \geq 0$.

By hypothesis, equation (4) is oscillatory in $[r, \infty)$. Hence there exist points $r < t_0 < t_1$, a function $\phi(t)$ defined for $t_0 \leq t \leq t_1$ such that $\phi(t_0) = \phi(t_1) = 0$ and $l\phi = 0$ in $[t_0, t_1]$. Then

$$\int_{t_0}^{t_1} (A(t)e,\ e)\left(\frac{d\phi}{dt}\right)^2 - (C(t,\ Y)e,\ e)\phi^2 dt = 0 . \tag{5}$$

Theorem 2 is therefore a consequence of Theorem 1.

Theorem 2 extends all oscillation criteria for ordinary differential equations to equation (1).

COROLLARY 3. *Every prepared solution of equation* (1) *is oscillatory in* $[a, \infty)$ *if, for every unit vector* $e \in H$ *and for all operators* $Y$ *such that* $Y(t)$ *has a bounded inverse for sufficiently large* $t$, *one of the following criteria is satisfied:*

(1) $\displaystyle\int_0^\infty \frac{1}{(A(t)e,e)}\, dt = \infty$, $\displaystyle\int_0^\infty (C(t,\ Y(t))e,\ e)dt = \infty$,

(2) $(A(t)e,\ e) \le K$ , $(C(t,\ Y(t))e,\ e) \ge 0$ *for large* $t$ , *and*

$$\limsup_{t\to\infty} t \int_t^\infty (C(t,\ Y(t))e,\ e)dt > 1 \ ,$$

(3) $(A(t)e,\ e) \le K$ , $4t^2(C(t,\ Y(t))e,\ e) > (A(t)e,\ e)$ *for sufficiently large* $t$ , *and*

$$\int_0^\infty t\left|(C(t,\ Y(t))e,\ e) - \frac{(A(t)e,e)}{4t^2}\right|dt = \infty \ .$$

Proof. The proof follows from Theorem 2 by applying known oscillation criteria, [2], [3], [5], for the ordinary equation (4).

A recent result of Hayden and Howard, [1], is criterion (1) of Corollary 3 when $A(t) = I$ , the identity operator.

When $H$ is finite dimensional, the following stronger version of Theorem 2 is valid, and gives a new oscillation theorem for matrix differential inequalities.

THEOREM 4. *Every prepared solution of the matrix differential inequality* $Y^*TY \le 0$ *is oscillatory in* $R^+$ *if there exists a non-zero vector* $e \in H$ *such that*

$$lu = \frac{d}{dt}\left[(A(t)e,\ e)\frac{du}{dt}\right] + (C(t,\ Y)e,\ e) = 0 \tag{4}$$

*is oscillatory in* $R^+$ , *for every nonsingular matrix* $Y$ *in* $B(H,\ H)$ .

Proof. If $Y(t)$ is a prepared solution of $Y^*TY \le 0$ which has no singular point for $t \ge r$ , for some $r$ , then $Y(t)$ has a bounded inverse defined on the whole space $H$ for $t \ge r$ . In particular $e$ belongs to the range of $Y$ for all $t \ge r$ .

The rest of the proof is identical to the proof of Theorem 2.

We shall give now a simple application of Theorem 1. Consider the equation

$$TY = \frac{d^2Y}{dt^2} + C(t)Y(t) = 0 \ , \tag{5}$$

where $C(t)$ is a self-adjoint operator for each $t \ge 0$ . Let $P$ and $Q$ be two bounded operators on $H$ such that $P^*Q$ is self-adjoint. Let $Y$

https://doi.org/10.1017/S0004972700043112 Published online by Cambridge University Press

be the solution of the initial value problem

$$(6) \qquad TY = 0 \text{ in } R^+ ,$$
$$Y(t_0) = P , \quad \frac{dY}{dt}(t_0) = Q , \quad t_0 \in R^+ .$$

Then $Y^*(t_0) \frac{d}{dt} Y(t_0) = \left[\frac{d}{dt} Y^*(t_0)\right] Y(t_0)$ . But, from equation (5), it is easy to see that

$$Y^*(t) \frac{d}{dt} Y(t) - \left[\frac{d}{dt} Y^*(t)\right] Y(t) = K ,$$

where $K$ is a constant operator. Hence

$$Y^*(t) \frac{d}{dt} Y(t) - \left[\frac{d}{dt} Y^*(t)\right] Y(t) \equiv 0 \text{ for all } t \geq t_0 .$$

THEOREM 5. *If there exists a nonzero vector* $e \in H$ *such that the scalar equation*

$$lu = \frac{d^2u}{dt^2} + (C(t)e, e)u = 0$$

*is oscillatory in* $R^+$ *, then the solution* $Y$ *of the initial value problem* (6) *has the property that there exists a sequence* $\{t_i\}$ *,* $t_i \to \infty$ *, such that either:*

*(a)* $Y(t_i)$ *has no bounded inverse, or*

*(b) the range of* $Y(t_i)$ *is not the whole space.*

Proof. If such sequence doesn't exist, then there is a $t_1 > 0$ such that $Y(t)$ has a bounded inverse which is defined on the whole space for $t \geq t_1$ . Hence $e \in$ range of $Y(t)$ for all $t \geq t_1$ . Since $lu$ is oscillatory in $[t_1, \infty)$ by hypothesis, there exist points $t'' > t' \geq t_1$ and a function $\phi \in C'(t', t'')$ such that $l\phi = 0$ and $\phi(t') = \phi(t'') = 0$ . Hence hypotheses (1) and (2) of Theorem 1 are satisfied and we obtain the contradiction that $Y(t)$ has a singular point in $[t_1, \infty)$ . This completes the proof.

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School of Mathematics,
University of New South Wales,
Kensington,
New South Wales.