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THE LAPLACE TRANSFORM OF $\exp(e^t)$

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Abstract

In an earlier paper [4], the author showed how Laplace transforms might be assigned to a class of superexponential functions for which the usual defining integral diverges. The present paper considers the case of the function $\exp(e^t)$, which arises in combinatorial contexts and whose Laplace transform may be assigned by means of an extension of techniques described in the previous paper.

1. Introduction

In a previous paper [4], Deakin, following earlier work by Vignaux [9] and Berg [2], was able to assign Laplace transforms to a number of superexponential functions by means of asymptotic series. Such Laplace transforms possess most of the usual properties of the standard Laplace transform. The purpose of the present paper is to consider the Laplace transform to be assigned to the function $\exp(e^t)$.

Indeed a formal Laplace transform has already been considered for this function by Flajolet and Schott [8] in a combinatorial context.

2. Definitions and terminology

The function $\exp(e^t)$ itself takes the value e when $t = 0$, and it will be more convenient to normalise this to unity. Thus we define

$$F(t) = \exp(e^t - 1) \tag{2.1}$$

and discuss the Laplace transform of $F(t)$.

Deakin's [4] technique for assigning Laplace transforms to analytic but superexponential functions such as $F(t)$ is to expand the function as a Taylor series, take

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term by term Laplace transforms of the result and thus arrive at an asymptotic series in negative powers of the new variable s. One then seeks to assign a sum to this series.

The series in s^{-1} itself may be viewed as the Laplace transform of the original function or, heuristically, we may apply this term to some asymptotic sum of that series. Such sums are not, of course, unique, so that, *par abus de langage*, the term "Laplace transform of $F(t)$" may be applied to various functions $f(s)$.

If we apply this technique to the function defined by (2.1), there results

$$F(t) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{2.2}$$

where B_n are the so-called "Bell numbers". $F(t)$ is thus a generating function for the Bell numbers and one may use it to obtain "Dobinski's formula", an explicit expression

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \tag{2.3}$$

for the nth Bell number.

The Bell numbers are also closely connected with the more familiar Stirling numbers, as

$$B_n = \sum_{m=0}^{n} \mathbb{S}_n^{(m)}, \tag{2.4}$$

where $\mathbb{S}_n^{(m)}$ are Stirling numbers of the second kind. (The notation here used is that of Abramowitz and Stegun [1, page 824].) For more on this background, see [7; 3, page 210].

The Laplace transform of $F(t)$ may therefore be given as

$$f(s) \sim \sum_{n=0}^{\infty} B_n s^{-(n+1)} \tag{2.5}$$

and we wish to sum this series.

3. The incomplete gamma functions

The series (2.5) is, of course, divergent and thus, although it may be assigned sums to which it is asymptotic, such sums will not be unique. Two such sums are readily derived.

If we expand $F(t)$ in terms, not of t, but of e^t, we find

$$F(t) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{e^{nt}}{n!} \tag{3.1}$$

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and formal term by term Laplace transformation yields

$$f(s) \sim \frac{1}{e}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{s-n}, \tag{3.2}$$

which may be summed [1, page 263] to give

$$f(s) \sim \frac{1}{e}(-1)^{s-1}\gamma(-s,-1), \tag{3.3}$$

where $\gamma(a,x)$ is an incomplete gamma function [1, page 260].

This function, which tends to zero as $\mathrm{Re}(s) \to \infty$ for other than integral n, may nonetheless be seen as an unsatisfactory Laplace transform because of the poles at positive integral values of s. It does however lead to other forms for $f(s)$, in particular we have [5, page 135] that

$$f(s) \sim \frac{1}{s} + \frac{1}{s(s-1)} + \frac{1}{s(s-1)(s-2)} + \dots . \tag{3.4}$$

Note also that if we apply the known result [6, page 147]

$$\mathscr{L}\{\exp(-ae^{-t})\} = a^{-s}\gamma(-s,a) \tag{3.5}$$

outside its normal range of validity, and set $a = -1$, we reach (3.3) via a scale theorem. This procedure is justified in the author's previous paper [4].

This does not however exhaust the matter. We also have [6, page 147]

$$\mathscr{L}\{\exp(-ae^{t})\} = a^{s}\Gamma(-s,a), \tag{3.6}$$

where $\Gamma(a,x)$ is also an incomplete gamma function [1, page 260]. In this case, the above procedure produces, when we set $a = -1$ in (3.6),

$$f(s) \sim \frac{1}{e}(-1)^{s}\Gamma(-s,-1), \tag{3.7}$$

that is,

$$f(s) \sim \{(-1)^{s-1}\gamma(-s,-1) + (-1)^{s}\Gamma(-s)\}/e. \tag{3.8}$$

This form of the sum removes the poles already noticed in the sum given in (3.3). Thus if one were to choose between the two functions on the right-hand sides of (3.3), (3.7) as representations of the Laplace transform given by the right-hand side of (2.5), the latter form might be preferred.

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4. A recurrence relation

The function $F(t)$ satisfies the differential equation

$$F'(t) = e^t F(t), \tag{4.1}$$

and if we take the Laplace transform of this and recall that $F(0) = 1$, there results

$$sf(s) = 1 + f(s-1). \tag{4.2}$$

It may be readily seen [1, page 262] that both of the (3.3) and (3.7) satisfy this relation, as, naturally, do the expansions (3.2), (3.4). Expression (2.5) also satisfies the recurrence relation (4.2). For, substituting from (2.5) into (4.2), we find the requirement reduces after some tedious and intricate, but otherwise undemanding, algebraic manipulations, to

$$B_{n+1} = \sum_{r=0}^{n} \binom{n}{r} B_r. \tag{4.3}$$

This is a known property of the Bell numbers which follows immediately from a similar property,

$$\sum_{r=m}^{n} \binom{n}{r} \mathfrak{S}_r^{(m)} = \mathfrak{S}_{n+1}^{(m+1)} \tag{4.4}$$

for Stirling numbers of the second kind. (See [3, page 209].)

The general solution to (4.2) is however

$$f(s) = f_0(s) + \phi(s)/\Gamma(s+1), \tag{4.5}$$

where $f_0(s)$ is some particular solution, here chosen to satisfy $f_0(0) = 0$, and $\phi(s)$ is an arbitrary function of period 1. Since $f(s)$ is a Laplace transform and thus analytic in s, then $f_0(s)$, $\phi(s)$ must be chosen also to be analytic in s.

We now turn to a characterisation of $f_0(s)$.

5. The function $f_0(s)$

It is straightforward to discover the values of $f_0(s)$ for $s = n$ (a positive integer). We find

$$f_0(n) = \frac{1}{n!} \sum_{j=0}^{n-1} j! \tag{5.1}$$

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But now [1, page 262] we find

$$f_0(n) = \{(-1)^n \Gamma(-n, -1) - E_1(-1)/n!\}/e, \tag{5.2}$$

where $E_1(z)$ is Abramowitz and Stegun's first exponential integral [1, page 228].

Equation (5.2) is the specialisation to positive integers of the more general formula

$$f_0(s) = \{(-1)^s \Gamma(-s, -1) - E_1(-1)/\Gamma(s+1)\}/e. \tag{5.3}$$

This is the most natural generalisation and will be used in what follows to *define* $f_0(s)$. However it will be seen from (4.5) that $f_0(s)$ is not uniquely defined prior to this point.

Now

$$\begin{aligned} E_1(-1) &= -e\left(0.697175\ldots + i\frac{\pi}{e}\right) \\ &= -be - i\pi \text{ (say)} \end{aligned}$$

[1, page 250], where the defining integral has been indented upwards at the origin. So we may now write (5.3) as

$$f_0(s) = \{(-1)^s[\Gamma(-s) - \gamma(-s, -1)] + (be + i\pi)/\Gamma(s+1)\}/e. \tag{5.4}$$

Thus [1, page 262],

$$f_0(s) = \{(-1)^s \Gamma(-s) + M(-s, 1-s, 1)/s + (be + i\pi)/\Gamma(s+1)\}/e, \tag{5.5}$$

where $M(a, b, z)$ is Kummer's confluent hypergeometric function [1, page 504].

Choose now the principal value

$$(-1)^s = \cos \pi s + i \sin \pi s \tag{5.6}$$

and substitute this into (5.5). The terms with imaginary coefficients combine to give $i(\sin \pi s \Gamma(-s) + \pi/\Gamma(1+s))$, an expression which vanishes as a result of the reflection formula for the gamma function [1, page 256].

Thus write

$$f_0(s) = \{\Gamma(-s) \cos \pi s + M(-s, 1-s, 1)/s + be/\Gamma(s+1)\}/e. \tag{5.7}$$

We now expand $f_0(s)$ as a power series in s.

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6. The power series for $f_0(s)$

The various terms on the right-hand side of (5.7) will now be expanded in power series about $s = 0$. We have [1, page 256]

$$\frac{1}{\Gamma(z)} = \sum_{n=1}^{\infty} c_n z^n, \tag{6.1}$$

where $c_1 = 1$, $c_2 = \gamma$, and $c_2, \ldots, c_{26}$ have been tabulated. In consequence of Weierstrass' form for the gamma function [10, page 236], the coefficients c_n are polynomials in $\gamma, \zeta(2), \zeta(3), \ldots$, that is, the values of Riemann's ζ-function [1, page 807] for integral arguments. Now write

$$\Gamma(-s) = \left[\frac{1}{\Gamma(-s)}\right]^{-1} = -\frac{1}{s}\left[1 + \sum_{n=1}^{\infty} d_n s^n\right]. \tag{6.2}$$

The d_n are likewise polynomials in $\gamma, \zeta(2), \zeta(3), \ldots$. Indeed it may be calculated that

$$\begin{aligned} &d_1 = \gamma, \quad d_2 = \gamma^2 - c_3 = 0.9890559953, \\ &d_3 = \gamma^3 - 2c_3\gamma + c_4 = 0.9074790460, \\ &d_4 = \gamma^4 - 3c_3\gamma^2 + 2c_4\gamma + c_3^2 - c_5 = 0.9817280869, \end{aligned}$$

etc.

Next, from the definition of $M(a, b, z)$ [1, page 504], we have

$$\frac{1}{s} M(-s, 1-s, 1) = \frac{1}{s} - \sum_{n=1}^{\infty} e_n s^{n-1}, \tag{6.3}$$

where

$$e_n = \sum_{r=1}^{\infty} \frac{1}{r!\, r^n}. \tag{6.4}$$

(Formally, $e_n = eB_{-n}$.)

Clearly $e_0 = e - 1$, and we also have [1, page 229]

$$e_1 = be - \gamma. \tag{6.5}$$

Then $e_2 = 1.1464990725$, $e_3 = 1.0693976089$, $e_4 = 1.0334848677$, etc.

Finally

$$\frac{1}{\Gamma(s+1)} = \frac{1}{s} \sum_{n=1}^{\infty} c_n s^n = 1 + \gamma s + c_3 s^2 + c_4 s^3 + \ldots. \tag{6.6}$$

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Thus

$$f_0(s) = \frac{1}{e}\left\{ -\frac{1}{s}\left(1 - \frac{\pi^2 s^2}{2} + \dots\right)\left(1 + \gamma s + d_2 s^3 + d_3 s^3 + \dots\right) \right. \tag{6.7}$$
$$\left. + \frac{1}{s} - e_1 - e_2 s - e_3 s^2 - \dots + be\left(1 + \gamma s + c_3 s^2 + \dots\right) \right\}.$$

The terms in s^{-1} clearly vanish, as do the constant terms – in view of (6.5). We thus have

$$f_0(s) = \sum_{n=1}^{\infty} a_n s^n = a_1 s + a_2 s^2 + \dots , \tag{6.8}$$

where

$$a_1 = \left(\frac{\pi^2}{2} + be\gamma - d_2 - e_2\right)/e = 1.4322057347,$$
$$a_2 = \left(\gamma\frac{\pi^2}{2} + c_3 be - d_3 - e_3\right)/e = -0.136629604,$$

etc.

7. An infinite matrix

Although the separate series (6.2) and (6.3) are each convergent (as may readily be seen) only in the domain $|s| < 1$, the series (6.8) is everywhere convergent in view of (5.3) and the fact that $\Gamma(-s, -1)$ is an entire function [1, pages 504, 510]. We may thus use the series (6.8) to evaluate the function $f_0(s)$ and its derivatives at the point $s = 1$.

The function $f_0(s)$ is to have all its derivatives continuous at the point $s = 1$. Write (4.2) in the form

$$(s+1)f_0(s+1) = 1 + f_0(s) \tag{7.1}$$

and expand in powers of s, equating like powers from both sides. Following some tedious though routine algebra we find

$$\sum_{r=1}^{\infty} a_r = 1 \tag{7.2}$$

and

$$a_{n-1} + na_n + \sum_{r=n+1}^{\infty} \binom{r+1}{n} a_r = 0 \tag{7.3}$$

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for $n \geq 1$. Equations (7.2), (7.3) together make up an infinite array of linear equations

$$\mathbf{Aa} = \mathbf{b}, \tag{7.4}$$

where $\mathbf{a} = \begin{pmatrix} a_1, & a_2, & a_3, & \ldots \end{pmatrix}^{\mathsf{T}}$, $\mathbf{b} = \begin{pmatrix} 1, & 0, & 0, & \ldots \end{pmatrix}^{\mathsf{T}}$ and $\mathbf{A}$ is the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ 1 & 3 & 4 & 5 & \ldots \\ 1 & 2 & 6 & 10 & \ldots \\ 0 & 1 & 3 & 10 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

(a slightly modified Pascal's triangle).

Thus the coefficients a_n give the solution of the equation $\mathbf{Ax} = \mathbf{b}$.

It is of interest to note that successive truncation of the system appears to produce good approximation to the value of the first few a_n. The 9th approximation gives

$$a_1 \simeq 1.4322055$$

and the 10th

$$a_1 \simeq 1.4322094.$$

(*Cf.* the correct value in Section 7. For a_2 the corresponding values are -0.1365674 and -0.1366221 respectively.)

8. Other approaches

The asymptotic series (2.5) has also been considered by Flajolet [7] and more recently by Flajolet and Schott [8]. These authors give a continued fraction expansion for $f(s)$ and they relate this to classical results on the Poisson-Charlier polynomials. For details, see their original papers, but note that the connection of $f(s)$ to the incomplete gamma function was already indicated by Flajolet and Schott.

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