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Asymptotics for Minimal Discrete Riesz Energy on Curves in $\mathbb{R}^d$

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Abstract. We consider the s-energy $E(\mathcal{Z}_n\ ;\ s) = \sum_{i\neq j} K\big(\|z_{i,n} - z_{j,n}\|\ ;\ s\big)$ for point sets $\mathcal{Z}_n = \{z_{k,n} : k = 0, \dots, n\}$ on certain compact sets Γ in $\mathbb{R}^d$ having finite one-dimensional Hausdorff measure, where

$$K(t\ ;\ s) = \begin{cases} t^{-s}, & \text{if } s > 0, \\ -\ln t, & \text{if } s = 0, \end{cases}$$

is the Riesz kernel. Asymptotics for the minimum s-energy and the distribution of minimizing sequences of points is studied. In particular, we prove that, for $s \geq 1$, the minimizing nodes for a rectifiable Jordan curve Γ distribute asymptotically uniformly with respect to arclength as $n \to \infty$.

1 Introduction

Assume that $\Gamma \subset \mathbb{R}^d$, $d \geq 1$, is a compact set. For $s \geq 0$ we define the Riesz kernel

$$K(t\ ;\ s) = \begin{cases} t^{-s}, & \text{if } s > 0, \\ -\ln t, & \text{if } s = 0. \end{cases}$$

Given a set of $n + 1$ distinct points $\mathcal{Z}_n = \{z_{k,n}\}_{k=0}^{n}$ on Γ, we consider the (doubled) discrete Riesz energy (or s-energy)

$$E(\mathcal{Z}_n\ ;\ s) = \sum_{i\neq j} K(\|z_{i,n} - z_{j,n}\|\ ;\ s), \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Our aim is to investigate the minimum s-energy

$$\mathcal{E}(n, s, \Gamma) := \min\{E(\mathcal{Z}_n\ ;\ s) : \mathcal{Z}_n \subset \Gamma\} \tag{1.2}$$

and the asymptotic distribution, as $n \to \infty$, of minimizing $(n + 1)$-point configurations. The latter is analyzed in the weak sense, that is, for any $\mathcal{Z}_n \subset \Gamma$ we define the

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unit counting measure

$$\nu(\mathcal{Z}_n) = \frac{1}{n+1} \sum_{k=0}^{n} \delta_{z_{k,n}}, \tag{1.3}$$

and study the convergence in the sense of the weak-* topology:

$$\nu(\mathcal{Z}_n) \xrightarrow{*} \nu \Leftrightarrow \lim_{n\to\infty} \int f\, d\nu(\mathcal{Z}_n) = \int f\, d\nu, \quad \text{for any } f \in C(\Gamma).$$

The expression in (1.1) is a discretization of the continuous energy

$$I(\mu\,;s) = \iint K(\|x-y\|\,;s)\, d\mu(x)\, d\mu(y), \tag{1.4}$$

which is defined, though not necessarily finite, for any positive Borel measure μ supported on Γ. The novelty of the present paper is the investigation of minimum discrete s-energy for rectifiable curves Γ in the case when $s \geq 1$, which is indeed a situation for which $I(\mu\,;s) = +\infty$ for every such measure μ (see, for example, [2, Theorem 6.4]). We remark that the divergence of the continuous energy means that the nearest neighbor interactions are dominating. In fact, for n fixed, in the limit as $s \to +\infty$ we arrive at the best-packing problem on Γ, that is, the problem of maximizing the minimal distance among pairs of the $n+1$ points on Γ.

In the simplest situation when Γ is a line segment, the $n+1$ equally spaced points provide the extremal configuration for best-packing. Such points are obviously asymptotically (as $n \to \infty$) uniformly distributed with respect to arclength. As we shall show, this same asymptotic behavior (as $n \to \infty$) holds for all s-energy extremal configurations whenever $s \geq 1$. (It is easy to verify that equally spaced points on a segment are *not* s-energy minimizing for any $s < \infty$.) More generally, we prove that if Γ is a rectifiable Jordan arc or curve in $\mathbb{R}^d$, then minimizing s-energy point sets for $s \geq 1$ are asymptotically uniformly distributed with respect to arclength on Γ as $n \to \infty$. Furthermore, we give asymptotics for the minimum energy $\mathcal{E}(n, s, \Gamma)$ in this case.

The situation for finite continuous energy ($0 \leq s < 1$) is classical: the picture is governed there by the equilibrium measure which provides the minimum value for the energy (1.4) among all the unit measures supported on Γ. Nevertheless, for completeness we also present the result corresponding to this case.

The paper is organized as follows. In Section 2 we consider the case when $s \in [0,1)$. In Section 3 we present the main results of the paper, namely those dealing with the case $s \geq 1$. Finally, proofs of all results are given in Section 4.

2 Finite Energy: $s \in [0,1)$

Let $\mathcal{M}(\Gamma)$ be the class of all positive unit Borel measures μ supported on a compact set Γ having finite positive one-dimensional Hausdorff measure. It is well-known [5, Chapter II] that for $0 \le s < 1$ there exists a unique measure $\mu_s \in \mathcal{M}(\Gamma)$, called the *minimizing* (*equilibrium*) *measure* on Γ, such that

$$\omega_s := I(\mu_s\,;s) = \min_{\mu\in\mathcal{M}(\Gamma)} I(\mu\,;s) < \infty.$$

It is characterized by the fact that its potential

$$\int K(\|x-y\|\,;s)\,d\mu_s(y) \begin{cases} \le \omega_s, & x \in \operatorname{supp}(\mu_s), \\ \ge \omega_s, & \text{approximately everywhere on } \Gamma. \end{cases}$$

This provides a general approach for computing μ_s by solving the corresponding singular integral equation on $\operatorname{supp}(\mu_s)$. Furthermore, it is known that

$$\lim_{n\to\infty} \frac{\mathcal{E}(n,s,\Gamma)}{n^2} = \omega_s, \quad 0 \le s < 1.$$

We remark that point sets $\mathcal{Z}_n$ that attain this minimum energy $\mathcal{E}(n,s,\Gamma)$ are called *Fekete points* with respect to Riesz energy. For the case $\Gamma = [-1,1] \subset \mathbb{R}$, an explicit expression for the density μ_s' with respect to Lebesgue measure is given in [5, Appendix]:

$$\mu_s'(x) = \frac{\mathbf{\Gamma}(1+s/2)}{\sqrt{\pi}\,\mathbf{\Gamma}\big((1+s)/2\big)}(1-x^2)^{(s-1)/2}, \quad \omega_s = \frac{\sqrt{\pi}\,\mathbf{\Gamma}(1+s/2)}{\cos(\pi s/2)\mathbf{\Gamma}\big((1+s)/2\big)}.$$

Definition 2.1 A sequence of point sets $\{\mathcal{Z}_n\} \subset \Gamma$ is *asymptotically s-energy minimizing on* Γ (briefly, $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\,;s)$) for $0 \le s < 1$ if

$$\lim_{n\to\infty} \frac{E(\mathcal{Z}_n\,;s)}{n^2} = \omega_s.$$

Using standard arguments from potential theory we present, for the convenience of the reader, the proof of the following.

Theorem 2.2 *If* $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\,;s)$, *then*

$$\nu(\mathcal{Z}_n) \xrightarrow{*} \mu_s \quad \text{as } n \to \infty.$$

3 Infinite Energy: $s \ge 1$

For any Borel set Γ in $\mathbb{R}^d$ we use both $m_1(\Gamma)$ and $|\Gamma|$ to denote its one-dimensional Hausdorff measure. If $0 < |\Gamma| < \infty$, we let λ_Γ be normalized one-dimensional measure supported on Γ, *i.e.*, $\lambda_\Gamma(\cdot) := |\cdot|/|\Gamma|$.

First, we assume that Γ is a rectifiable Jordan arc that includes its endpoints. For such arcs (as well as for other related sets such as their unions, subsets, *etc.*), it is well-known that their one-dimensional Hausdorff measure $m_1(\Gamma)$ is the same as the Lebesgue (arclength) measure inherited from parametrizations (*cf.* [2, Chapter 3]).

For $z_1, z_2 \in \Gamma$, let $\Gamma(z_1, z_2)$ denote the closed subarc of Γ joining these two points, and $\ell(z_1, z_2) := |\Gamma(z_1, z_2)|$. Let $\mathcal{Z}_n^*$ be the set of $n+1$ equally spaced points on Γ, *i.e.*, if τ is an endpoint of Γ, then

$$\mathcal{Z}_n^* = \{z_{k,n}^* \in \Gamma : \ell(\tau, z_{k,n}^*) = k|\Gamma|/n,\ k = 0, \dots, n\}.$$

Obviously, $\nu(\mathcal{Z}_n^*) \xrightarrow{*} \lambda_\Gamma$.

One of our goals is to show that the same asymptotic takes place for every s-energy minimizing sequence, whenever $s \geq 1$ and Γ is the finite union of rectifiable Jordan arcs or closed curves. We remark that standard potential theoretic arguments cannot be applied in this case.

We begin with results on the asymptotic behavior of the minimum energy. Let

$$r_n(s) := \begin{cases} n^{1+s}, & \text{if } s > 1, \\ n^2 \ln n, & \text{if } s = 1, \end{cases} \qquad C(\Gamma\,;s) := 2|\Gamma|^{-s}\tilde{\zeta}(s),$$

where

$$\tilde{\zeta}(s) := \begin{cases} \zeta(s), & \text{if } s > 1, \\ 1, & \text{if } s = 1, \end{cases} \quad \text{and} \quad \zeta(s) = \sum_{k=1}^{\infty} k^{-s}.$$

Theorem 3.1 *If Γ is a rectifiable Jordan arc and $s \geq 1$, then*

$$\lim_{n\to\infty} \frac{\mathcal{E}(n, s, \Gamma)}{r_n(s)} = C(\Gamma\,;s). \tag{3.1}$$

This result is a special case of the following, which in particular applies to closed Jordan curves.

Theorem 3.2 *If $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$, where each Γ_j is a rectifiable Jordan arc and*

$$|\Gamma| = \sum_{j=1}^{m} |\Gamma_j|, \tag{3.2}$$

then, for $s \geq 1$, (3.1) holds.

We remark that Γ in this last theorem need not be connected. Motivated by this result, we introduce the following definition:

Definition 3.3 Let Γ be as in Theorem 3.2. A sequence of point sets $\{\mathcal{Z}_n\} \subset \Gamma$ is *asymptotically s-energy minimizing on Γ for $s \geq 1$* (briefly, $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\,;s)$) if

$$\lim_{n\to\infty} \frac{E(\mathcal{Z}_n\,;s)}{r_n(s)} = C(\Gamma\,;s). \tag{3.3}$$

Regarding the limiting distribution of asymptotically s-energy minimizing points, we show the following:

Theorem 3.4 *Let Γ be as in Theorem 3.2. If $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\ ;\ s)$, for some $s \geq 1$, then*

$$\nu(\mathcal{Z}_n) \xrightarrow{*} \lambda_\Gamma \quad \text{as } n \to \infty. \tag{3.4}$$

Actually, for $s > 1$, we can say even more. For a rectifiable Jordan arc, let

$$d_{k,n} := \ell(z_{k,n}, z_{k-1,n}), \quad k = 1, \dots, n, \tag{3.5}$$

where the $z_{k,n}$'s are successive points on the arc.

Proposition 3.5 *Let Γ be a rectifiable Jordan arc. If $s > 1$ and $\mathcal{Z}_n \in \mathrm{AEM}(\Gamma\ ;\ s)$, then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Big| d_{k,n} - \frac{L}{n} \Big| = 0, \quad L := |\Gamma|. \tag{3.6}$$

Clearly, (3.6) implies that, for any $\varepsilon > 0$,

$$\operatorname{card}\Big\{ k : d_{k,n} \leq \frac{L-\varepsilon}{n} \text{ or } d_{k,n} \geq \frac{L+\varepsilon}{n} \Big\} = o(n) \quad \text{as } n \to \infty.$$

Another property of a sequence $\{\mathcal{Z}_n\} \subset \Gamma$ is the behavior of the minimal distance between elements of $\mathcal{Z}_n$ as $n \to \infty$. Denote

$$\delta(\mathcal{Z}_n) := \min\{\|x - y\| : x, y \in \mathcal{Z}_n, x \neq y\}.$$

Trivially, for $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\ ;\ s)$, it follows from (3.3) that, for $s > 1$, $\delta(\mathcal{Z}_n) \geq c/n^{1+1/s}$ and, for $s = 1$, $\delta(\mathcal{Z}_n) \geq c/(n^2 \ln n)$ for some constant $c > 0$. However, if $\tilde{\mathcal{Z}}_n = \{\tilde{z}_{k,n}\}_{k=0}^{n}$, $n = 1, 2, \dots$, is an *optimal* sequence, *i.e.*, a sequence for which the minimum in (1.2) is attained, the following separation result holds for the class of *regular*[1] curves. Such Jordan curves (arcs) Γ are characterized by the property that there exists a constant $M > 0$ such that, for any point $z \in \Gamma$ and any $r > 0$, we have

$$|B(z, r) \cap \Gamma| \leq Mr, \tag{3.7}$$

where $B(z, r)$ is the ball $\{w \in \mathbb{R}^d : \|w - z\| < r\}$ (*cf.* [1]).

[1] Regular curves are also known as Ahlfors' or Carleson's curves.

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Proposition 3.6 *If Γ is a regular curve, then there exists a constant $c = c(\Gamma, s) > 0$ such that, for every $n \geq 2$,*

$$\delta(\tilde{\mathcal{Z}}_n) \geq \begin{cases} c/n, & \text{if } s > 1, \\ c/(n \ln n), & \text{if } s = 1. \end{cases} \tag{3.8}$$

Next we consider the question of when equally spaced points are asymptotically s-energy minimizing.

Theorem 3.7 *If Γ is a piecewise smooth[2] Jordan arc or closed curve without cusps,* i.e., *satisfying, for some constant $C > 0$ and any x, $y \in \Gamma$,*

$$\frac{\ell(x, y)}{\|x - y\|} \leq C, \quad x \neq y, \tag{3.9}$$

then the equally spaced points $\{\mathcal{Z}_n^*\} \in \mathrm{AEM}(\Gamma\,;\,s)$ for $s \geq 1$.

In case Γ is a closed Jordan curve, $\ell(x, y)$ in (3.9) denotes the length of the shortest arc joining x and y.

The condition (3.9) in Theorem 3.7 is not superfluous. The following example shows that the presence of a cusp can prevent $\{\mathcal{Z}_n^*\}$ from being asymptotically s-energy minimizing on Γ.

Example 1 Let $\Gamma_- = \{(x, y) \in \mathbb{R}^2 : \|(x + 1, y)\| = 1, y \geq 0\}$, $\Gamma_+ = \{(x, y) \in \mathbb{R}^2 : \|(x - 1, y)\| = 1, y \geq 0\}$, and $\Gamma = \Gamma_- \cup \Gamma_+$. Observe that Γ has a cusp at 0.

For n odd, $n = 2k + 1$, we have that

$$\begin{aligned} E(\mathcal{Z}_n^*\,;\,s) &\geq K(\|z_{k,n}^* - z_{k+1,n}^*\|\,;\,s) = \big(2[1 - \cos(\pi/n)]\big)^{-s} \\ &= \left(\frac{n}{\pi}\right)^{2s}\big(1 + o(1)\big) \quad \text{as } n \to \infty. \end{aligned}$$

Thus, for $s > 1$,

$$\limsup_{n\to\infty} \frac{E(\mathcal{Z}_n^*\,;\,s)}{r_n(s)} = \infty,$$

and $\{\mathcal{Z}_n^*\} \notin \mathrm{AEM}(\Gamma\,;\,s)$.

Actually, Theorem 3.7 can be extended to certain cases when Γ has a cusp(s). The answer to the question whether $\{\mathcal{Z}_n^*\} \in \mathrm{AEM}(\Gamma\,;\,s)$ depends on the mutual relation between the order ρ of the cusp and s.

For $\rho \geq 1$ we define

$$s^*(\rho) := \begin{cases} 1/(\rho - 1), & \text{if } \rho > 1, \\ +\infty, & \text{if } \rho = 1. \end{cases}$$

The function $s^*(\rho)$ decreases from $+\infty$ to 0 as ρ increases from 1 to $+\infty$. The value $\rho = 2$ is the critical one: $s^*(2) = 1$.

[2]A Jordan arc Γ is *smooth* if there exists a parametrization $\varphi \in C^1([0, 1])$ of Γ with $\varphi'(t) \neq \mathbf{0}$ for all $t \in [0, 1]$, and it is *piecewise smooth* if it consists of a finite number of smooth subarcs.

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Theorem 3.8 *Let Γ be a Jordan arc consisting of two smooth subarcs Γ_1 and Γ_2 with a common endpoint τ. Suppose that, for some constants $c > 0$ and $1 \le \rho < 2$,*

$$\|z - y\| \ge c\|z - \tau\|^{\rho}, \quad \text{for all } z \in \Gamma_j,\ y \in \Gamma_i,\ i \ne j. \tag{3.10}$$

Then $\{\mathcal{Z}_n^*\} \in \mathrm{AEM}(\Gamma\,;s)$ if

$$(1 \le) s < s^*(\rho). \tag{3.11}$$

In addition, $\{\mathcal{Z}_n^*\} \in \mathrm{AEM}(\Gamma\,;1)$ if (3.10) is satisfied with $\rho = 2$.

This statement can be easily generalized to the case when a Jordan arc Γ consists of a finite number of smooth subarcs satisfying (3.10).

Corollary 3.9 *There are piecewise smooth Jordan arcs with cusps such that the equally spaced points are asymptotically s-energy minimizing for any $s \ge 1$. For instance,*

$$\Gamma := \Big\{ (x, y) \in \mathbb{R}^2 : y = \frac{x}{\ln(e/x)}, x \in (0, 1] \Big\} \cup [0, 1].$$

The following example shows that Theorem 3.8 is sharp.

Example 2 For $\rho > 1$, let $\Gamma^{(\rho)} := \{(x, y) \in \mathbb{R}^2 : y = |x|^{1/\rho}, x \in [-1, 1]\}$. Clearly, $\Gamma^{(\rho)}$ satisfies (3.10) ($\tau = 0$). We claim that $\{\mathcal{Z}_n^*\} \notin \mathrm{AEM}(\Gamma^{(\rho)}\,; s)$ for any $s \ge \max\{s^*(\rho), 1\}$, except for the case when $\rho = 2$ and $s = s^*(2) = 1$. The verification is given in Section 4.

4 Proofs

Now, we turn to the proofs of the formulated results.

Proof of Theorem 2.2 We use the standard arguments, well-known from the potential theory (see [5, Chapter II, Section 3]). Set $\nu_n = \nu(\mathcal{Z}_n)$. For an arbitrary $\varepsilon > 0$ define the truncated kernel

$$K_\varepsilon(t\,; s) := \min\{K(t\,; s), K(\varepsilon\,; s)\}\,;$$

in particular,

$$\iint_{x\ne y} K_\varepsilon(\|x - y\|\,; s)\, d\nu_n(x)\, d\nu_n(y) \le \iint_{x\ne y} K(\|x - y\|\,; s)\, d\nu_n(x)\, d\nu_n(y) = \frac{E(\mathcal{Z}_n\,; s)}{(n + 1)^2}.$$

Then,

$$
\begin{aligned}
\iint K_\varepsilon(\|x-y\|\,;s)\,d\nu_n(x)\,d\nu_n(y) &= \iint_{x\neq y} K_\varepsilon(\|x-y\|\,;s)\,d\nu_n(x)\,d\nu_n(y)\\
&\qquad + \iint_{x=y} K_\varepsilon(\|x-y\|\,;s)\,d\nu_n(x)\,d\nu_n(y)\\
&= \iint_{x\neq y} K_\varepsilon(\|x-y\|\,;s)\,d\nu_n(x)\,d\nu_n(y)\\
&\qquad + \frac{K(\varepsilon\,;s)}{n+1}\\
&\leq \frac{E(\mathcal{Z}_n\,;s)}{(n+1)^2} + \frac{K(\varepsilon\,;s)}{n+1}.
\end{aligned}
$$

Therefore, if $\mathcal{Z}_n \in \mathrm{AEM}(\Gamma\,;s)$, then by Definition 2.1,

$$
\limsup_{n\to\infty} \iint K_\varepsilon(\|x-y\|\,;s)\,d\nu_n(x)\,d\nu_n(y) \leq \omega_s. \tag{4.1}
$$

Now, using that the sequence $\nu_n = \nu(\mathcal{Z}_n)$ is weakly compact, we can take a subsequence $\Lambda \subset \mathbb{N}$ such that $\nu(\mathcal{Z}_n) \xrightarrow{*} \nu$, where ν is a unit measure on Γ. By (4.1),

$$
\iint K_\varepsilon(\|x-y\|\,;s)\,d\nu(x)\,d\nu(y) \leq \omega_s,
$$

and since $\varepsilon > 0$ is arbitrary, we conclude on using the monotone convergence theorem that $I(\nu\,;s) \leq \omega_s$. It remains to use the uniqueness of the equilibrium measure μ_s. ■

Proof of Theorem 3.1 We will need the following elementary fact:

Lemma 4.1 *For $s \geq 1$ and $r_1,\dots,r_n > 0$,*

$$
\frac{1}{n}\sum_{k=1}^{n} r_k^s \geq \Big(\frac{1}{n}\sum_{k=1}^{n} r_k\Big)^s, \tag{4.2}
$$

and

$$
\Big(\sum_{k=1}^{n} r_k\Big)\Big(\sum_{k=1}^{n}\frac{1}{r_k}\Big) = n^2\Big(1+\frac{1}{2n^2}\sum_{i\neq j}\frac{(r_i-r_j)^2}{r_ir_j}\Big) \geq n^2. \tag{4.3}
$$

Proof The inequality (4.2) is an immediate consequence of the convexity of the function x^s. Further, we have

$$
\begin{aligned}
\Big(\sum_{k=1}^{n} r_k\Big)\Big(\sum_{k=1}^{n}\frac{1}{r_k}\Big) &= \sum_{i,j=1}^{n}\frac{r_i}{r_j} = n + \sum_{i\neq j}\frac{r_i}{r_j} = n + \frac{1}{2}\sum_{i\neq j}\Big(\frac{r_i}{r_j}+\frac{r_j}{r_i}\Big)\\
&= n + \frac{1}{2}\sum_{i\neq j}\Big[\Big(\frac{r_i}{r_j}-2+\frac{r_j}{r_i}\Big)+2\Big] = n^2 + \frac{1}{2}\sum_{i\neq j}\frac{(r_i-r_j)^2}{r_ir_j},
\end{aligned}
$$

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and (4.3) follows. Observe that this inequality is a refinement of the well-known inequality between the arithmetic and the harmonic means. ■

First we show the following.

Lemma 4.2 *Let $s \geq 1$. Then, for any rectifiable Jordan arc Γ and for any sequence of point sets $\{\mathcal{Z}_n\} \subset \Gamma$,*

$$\liminf_{n\to\infty} \frac{E(\mathcal{Z}_n\,;s)}{r_n(s)} \geq C(\Gamma\,;s). \tag{4.4}$$

Proof Let $\mathcal{Z}_n = \{z_{k,n}\}_{k=0}^{n}$, where the points $z_{k,n}$, $k = 0, 1, \dots, n$, are located on Γ in successive order. Since, for every $z, z' \in \Gamma$,

$$\|z - z'\| \leq \ell(z, z'),$$

we have

$$E(\mathcal{Z}_n\,;s) = \sum_{i\neq j} \frac{1}{\|z_{i,n} - z_{j,n}\|^s} \geq \sum_{i\neq j} \frac{1}{\ell(z_{i,n}, z_{j,n})^s} = \sum_{k=1}^{n} \hat{E}_k(\mathcal{Z}_n\,;s) =: \hat{E}(\mathcal{Z}_n\,;s),$$

where

$$\hat{E}_k(\mathcal{Z}_n\,;s) := \sum_{|i-j|=k} \frac{1}{\ell(z_{i,n}, z_{j,n})^s}.$$

In particular, using the notation (3.5), we have

$$\hat{E}_1(\mathcal{Z}_n\,;s) = 2\sum_{k=1}^{n} d_{k,n}^{-s}.$$

Inequality (4.2), applied to $\hat{E}_1(\mathcal{Z}_n\,;s)$, gives

$$\hat{E}_1(\mathcal{Z}_n\,;s) \geq 2n\Big(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{d_{k,n}}\Big)^s = 2n^{1-s}\Big(\sum_{k=1}^{n}\frac{1}{d_{k,n}}\Big)^s, \tag{4.5}$$

which is indeed trivial for $s = 1$. Now using (4.3) and taking into account that $\sum_{k=1}^{n} d_{k,n} \leq L := |\Gamma|$, we obtain

$$\hat{E}_1(\mathcal{Z}_n\,;s) \geq 2n^{1+s}\Big(\sum_{k=1}^{n} d_{k,n}\Big)^{-s} \geq 2L^{-s}n^{1+s}. \tag{4.6}$$

Analogously,

$$\hat{E}_2(\mathcal{Z}_n\,;s) = 2\sum_{k=1}^{n-1}\frac{1}{(d_{k,n} + d_{k+1,n})^s},$$

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and reasoning as above, we obtain that

$$\hat{E}_2(\mathcal{Z}_n\,;s) \geq 2(n-1)^{1+s}\Big(\sum_{k=1}^{n-1}(d_{k,n}+d_{k+1,n})\Big)^{-s}.$$

But

$$\sum_{k=1}^{n-1}(d_{k,n}+d_{k+1,n}) = 2\sum_{k=1}^{n} d_{k,n} - (d_{1,n}+d_{n,n}) \leq 2L,$$

and so

$$\hat{E}_2(\mathcal{Z}_n\,;s) \geq 2(n-1)^{1+s}(2L)^{-s}.$$

Continuing in the same fashion, we obtain that

$$\hat{E}_k(\mathcal{Z}_n\,;s) \geq 2(n-k+1)^{1+s}(kL)^{-s}, \quad k=1,\dots,n. \tag{4.7}$$

Consequently,

$$\hat{E}(\mathcal{Z}_n\,;s) \geq 2n^{1+s}\sum_{k=1}^{n}\Big(1-\frac{k-1}{n}\Big)^{1+s}(kL)^{-s}. \tag{4.8}$$

For $s > 1$, since $0 \leq (1-k/n)^{1+s} \leq 1$, we can apply the Lebesgue dominated convergence theorem to get that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Big(1-\frac{k-1}{n}\Big)^{1+s}k^{-s} = \sum_{k=1}^{\infty}k^{-s} = \zeta(s),$$

and (4.4) follows for $s > 1$.

For $s = 1$, by (4.8),

$$\frac{L}{2n^2}\hat{E}(\mathcal{Z}_n\,;1) - \sum_{k=1}^{n}\frac{1}{k} \geq \sum_{k=1}^{n}\Big(-\frac{2}{n}+\frac{k}{n^2}\Big) = -2+\frac{n-1}{2n},$$

so that

$$\liminf_{n\to\infty}\Big(\frac{L\,E(\mathcal{Z}_n\,;1)}{2n^2}-\ln n\Big) \geq \gamma - 3/2, \quad \text{where } \gamma := \lim_{n\to\infty}\Big(\sum_{k=1}^{n}\frac{1}{k}-\ln n\Big)$$

is the Euler's constant. This implies

$$\liminf_{n\to\infty}\frac{E(\mathcal{Z}_n\,;1)}{n^2\ln n} \geq \frac{2}{L},$$

which proves (4.4) for $s = 1$. ■

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Now we show that

$$\limsup_{n\to\infty}\frac{\mathcal{E}(n,s,\Gamma)}{r_n(s)}\le C(\Gamma\,;s) \tag{4.9}$$

by constructing an "almost optimal" sequence $\{\tilde{\mathcal{Z}}_n\}$.

First we note that, for any rectifiable arc Γ,

$$\lim_{\zeta\to z}\frac{\ell(z,\zeta)}{\|z-\zeta\|}=1 \quad \text{a.e. on } \Gamma. \tag{4.10}$$

Indeed, if $\varphi(t)\colon [0,|\Gamma|]\to\Gamma$ denotes the natural parametrization of Γ, then $\varphi\in$ Lip 1 and, moreover, $\|\varphi'(t)\|=1$ a.e. on $[0,|\Gamma|]$.

For $\delta>0$ and $\varepsilon>0$, we define the sets

$$\Gamma_{\delta,\varepsilon}:=\left\{z\in\Gamma:\frac{\ell(z,\zeta)}{\|z-\zeta\|}\le 1+\delta \text{ if } \|z-\zeta\|<\varepsilon\right\}.$$

Clearly, each $\Gamma_{\delta,\varepsilon}$ is a closed subset of Γ, and $\Gamma_{\delta,\varepsilon_1}\subseteq\Gamma_{\delta,\varepsilon_2}$ if $\varepsilon_1>\varepsilon_2$. It follows from (4.10) that, for any fixed $\delta>0$,

$$\Big|\bigcup_{\varepsilon>0}\Gamma_{\delta,\varepsilon}\Big|=|\Gamma|. \tag{4.11}$$

Since the sets $\Gamma_{\delta,\varepsilon}$ increase as $\varepsilon\searrow 0$, (4.11) implies that, for fixed $\delta>0$ and $\sigma>0$, one can find $\varepsilon=\varepsilon(\delta,\sigma)>0$ such that

$$|\Gamma_{\delta,\varepsilon}|\ge|\Gamma|-\sigma. \tag{4.12}$$

For an arbitrary $n\in\mathbb{N}$, we form $\tilde{\mathcal{Z}}_n=\{z_{k,n}\}_{k=0}^{n}\subset\Gamma_{\delta,\varepsilon}$ as follows. Let τ be an endpoint of Γ, and we choose $z_{k,n}$ such that

$$|\Gamma(\tau,z_{k,n})\cap\Gamma_{\delta,\varepsilon}|=\frac{|\Gamma_{\delta,\varepsilon}|}{n}k,\quad k=\overline{0,n}. \tag{4.13}$$

Then

$$\begin{aligned}
E(\tilde{\mathcal{Z}}_n\,;s)&=2\sum_{k=0}^{n-1}\sum_{j>k}K(\|z_{k,n}-z_{j,n}\|\,;s)\\
&=2\sum_{k=0}^{n-1}\Bigg(\sum_{\substack{j>k\\ \|z_{k,n}-z_{j,n}\|\ge\varepsilon}}+\sum_{\substack{j>k\\ \|z_{k,n}-z_{j,n}\|<\varepsilon}}\Bigg)K(\|z_{k,n}-z_{j,n}\|\,;s)\\
&=:2\sum_{k=0}^{n-1}\Big(\sum\nolimits_1+\sum\nolimits_2\Big).
\end{aligned} \tag{4.14}$$

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The sum $\sum_1$ can be trivially estimated:

$$\sum\nolimits_1 \le \varepsilon^{-s}(n-k). \tag{4.15}$$

For the sum $\sum_2$, for n large enough, using the definition of $\Gamma_{\delta,\varepsilon}$, (4.12), and (4.13) we have

$$\begin{aligned}\sum\nolimits_2 &\le (1+\delta)^s \sum_{\substack{j>k\\ \|z_{k,n}-z_{j,n}\|<\varepsilon}} K\big(\ell(z_{k,n}, z_{j,n})\,;s\big)\\
&\le (1+\delta)^s \sum_{\substack{j>k\\ \|z_{k,n}-z_{j,n}\|<\varepsilon}} K\left(\frac{|\Gamma_{\delta,\varepsilon}|}{n}(j-k)\,;s\right)\\
&\le (1+\delta)^s \frac{n^s}{|\Gamma_{\delta,\varepsilon}|^s} \sum_{i=1}^{[(1+\delta)\varepsilon n/|\Gamma_{\delta,\varepsilon}|]} i^{-s} \le (1+\delta)^s n^s (|\Gamma|-\sigma)^{-s} \sum_{i=1}^{[(1+\delta)\varepsilon n/|\Gamma_{\delta,\varepsilon}|]} i^{-s}.\end{aligned}$$

We continue the estimate for cases $s>1$ and $s=1$ separately. If $s>1$, then

$$\sum\nolimits_2 < (1+\delta)^s(|\Gamma|-\sigma)^{-s} n^s \sum_{i=1}^{\infty} i^{-s} = (1+\delta)^s(|\Gamma|-\sigma)^{-s}\zeta(s) n^s. \tag{4.16}$$

In the case when $s=1$,

(4.17)

$$\begin{aligned}\sum\nolimits_2 &\le \frac{(1+\delta)n}{|\Gamma|-\sigma}\left(\left(\sum_{i=1}^{[(1+\delta)\varepsilon n/|\Gamma_{\delta,\varepsilon}|]} \frac{1}{i} - \ln\left[\frac{(1+\delta)\varepsilon n}{|\Gamma_{\delta,\varepsilon}|}+1\right]\right) + \ln\left[\frac{(1+\delta)\varepsilon n}{|\Gamma_{\delta,\varepsilon}|}+1\right]\right)\\
&\le \frac{(1+\delta)n}{|\Gamma|-\sigma}\left(\gamma + \ln\left(\frac{(1+\delta)\varepsilon}{|\Gamma_{\delta,\varepsilon}|}+\frac{1}{n}\right) + \ln n\right)\\
&< (1+\delta)(|\Gamma|-\sigma)^{-1}(\gamma+\ln n)n = (1+\delta)(|\Gamma|-\sigma)^{-1} n\ln n + O(n),\end{aligned}$$

provided $\varepsilon < (|\Gamma|-\sigma)/(1+\delta)$ and n is large enough.

Thus, for any $s \ge 1$, substituting (4.15) and either (4.16), if $s>1$, or (4.17), if $s=1$, into (4.14) yields[3]

$$E(\tilde{\mathcal{Z}}_n\,;s) \le 2(1+\delta)^s(|\Gamma|-\sigma)^{-s}\tilde{\zeta}(s) r_n(s) + O(n^2)\varepsilon^{-s}. \tag{4.18}$$

This implies that

$$\limsup_{n\to\infty} \frac{E(\tilde{\mathcal{Z}}_n\,;1)}{r_n(s)} \le 2(1+\delta)^s(|\Gamma|-\sigma)^{-s}\tilde{\zeta}(s).$$

[3] We indicate the dependence on ε explicitly for future reference.

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Therefore, for the minimal s-energy, $\mathcal{E}(n,s,\Gamma)$, we obtain

$$\limsup_{n\to\infty} \frac{\mathcal{E}(n,s,\Gamma)}{r_n(s)} \le 2(1+\delta)^s(|\Gamma|-\sigma)^{-s}\tilde{\zeta}(s).$$

Since $\delta > 0$ and $\sigma > 0$ are arbitrary, we get the required upper estimate (4.9) which together with (4.4) gives (3.1). ∎

Proof of Theorem 3.2 The proof utilizes arguments of Hardin and Saff in [3], appearing in the two following auxiliary results, from which the conclusion of the theorem will follow.

First, we generalize Lemma 4.2 for given sets Γ.

Lemma 4.3 *Let Γ be as in Theorem 3.2. Then, for any sequence of point sets $\{\mathcal{Z}_n\} \subset \Gamma$, (4.4) holds.*

Proof Let $\{\mathcal{Z}_n\} \subset \Gamma$ be *any* sequence of point sets, and let $\mathcal{N} \subseteq \mathbb{N}$ be such a sequence that

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \frac{E(\mathcal{Z}_n\,;\,s)}{r_n(s)} = \liminf_{n\to\infty} \frac{E(\mathcal{Z}_n\,;\,s)}{r_n(s)}.$$

Denote $\mathcal{Z}_{n,1} := \mathcal{Z}_n \cap \Gamma_1$, $\mathcal{Z}_{n,j} := (\mathcal{Z}_n \cap \Gamma_j) \setminus (\bigcup_{k=1}^{j-1} \mathcal{Z}_{n,k})$ for $j = \overline{2,m}$, and let $p(j,n) := \operatorname{card} \mathcal{Z}_{n,j}$. We choose a subsequence $\mathcal{N}_1$ of $\mathcal{N}$ such that all the limits

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}_1}} \frac{p(j,n)}{n+1} =: \alpha_j \tag{4.19}$$

exist. Then, clearly, $0 \le \alpha_j \le 1$ for all j, and

$$\sum_{j=1}^{m} \alpha_j = 1. \tag{4.20}$$

We also define

$$\beta_j := |\Gamma_j|/|\Gamma| = \lambda_\Gamma(\Gamma_j), \quad j = 1,\dots,m. \tag{4.21}$$

The condition (3.2) implies that

$$\sum_{j=1}^{m} \beta_j = 1. \tag{4.22}$$

By Theorem 3.1, for each Γ_j,

$$\lim_{n\to\infty} \frac{\mathcal{E}(n,s,\Gamma_j)}{r_n(s)} = 2\tilde{\zeta}(s)|\Gamma_j|^{-s}, \quad j = 1,\dots,m.$$

Thus, using (4.19) we obtain

$$
\begin{aligned}
\liminf_{n\to\infty}\frac{E(\mathcal{Z}_n\,;\,s)}{r_n(s)} &= \lim_{\substack{n\to\infty\\ n\in\mathcal{N}_1}}\frac{E(\mathcal{Z}_n\,;\,s)}{r_n(s)}\\
&\geq \liminf_{\substack{n\to\infty\\ n\in\mathcal{N}_1}}\frac{\sum_{j=1}^m E(\mathcal{Z}_{n,j}\,;\,s)}{r_n(s)} \geq \sum_{j=1}^m \liminf_{\substack{n\to\infty\\ n\in\mathcal{N}_1}}\left(\frac{r_{p(j,n)}(s)}{r_n(s)}\,\frac{E(\mathcal{Z}_{n,j}\,;\,s)}{r_{p(j,n)}(s)}\right)\\
&\geq {\sum_{j=1}^m}' \alpha_j^{1+s}\liminf_{\substack{n\to\infty\\ n\in\mathcal{N}_1}}\frac{E(\mathcal{Z}_{n,j}\,;\,s)}{r_{p(j,n)}(s)} \geq 2\tilde{\zeta}(s){\sum_{j=1}^m}'\alpha_j^{1+s}|\Gamma_j|^{-s}\\
&= 2\tilde{\zeta}(s)\sum_{j=1}^m \alpha_j^{1+s}|\Gamma_j|^{-s} = C(\Gamma\,;\,s)\sum_{j=1}^m\alpha_j^{1+s}\beta_j^{-s}, \qquad (4.23)
\end{aligned}
$$

where $\sum'$ means the sum over such j's that $p(j,n)\to\infty$; note that $\alpha_j = 0$ otherwise. Taking into account the convexity of x^ρ, $\rho > 1$, (4.22), and (4.20), we further conclude that

$$
\sum_{j=1}^m \alpha_j^{1+s}\beta_j^{-s} = \sum_{j=1}^m \beta_j\left(\frac{\alpha_j}{\beta_j}\right)^{1+s} \geq \left(\sum_{j=1}^m \beta_j\frac{\alpha_j}{\beta_j}\right)^{1+s} = 1, \qquad (4.24)
$$

with equality if and only if $\alpha_j = \beta_j$ for all j, and so

$$
\liminf_{n\to\infty}\frac{E(\mathcal{Z}_n\,;\,s)}{r_n(s)} \geq C(\Gamma\,;\,s). \qquad (4.25)
$$

■

We now continue with the proof of Theorem 3.2. Our next step is to show that (3.1) remains valid for unions of rectifiable curves.

Lemma 4.4 *If $\Gamma := \bigcup_{j=1}^m \Gamma_j$, where Γ_j, $j = \overline{1,m}$, are rectifiable Jordan arcs, then*

$$
\limsup_{n\to\infty}\frac{\mathcal{E}(n,s,\Gamma)}{r_n(s)} \leq C(\Gamma\,;\,s).
$$

Remark 4.5 The conclusion of Lemma 4.4, as we will see from its proof, holds true under more general assumptions on Γ; namely: for any $\varepsilon > 0$ small enough, there exists a set $\Gamma_\varepsilon \subset \Gamma$ such that

(i) Γ_ε is a finite union of pairwise disjoint rectifiable Jordan arcs; and

(ii) $|\Gamma_\varepsilon| > |\Gamma| - \varepsilon$.

Proof First, we show that, for any $\varepsilon > 0$ small enough, there exists a set $\Gamma_\varepsilon \subset \Gamma$ such that the conditions (i) and (ii) of Remark 4.5 are satisfied. It is sufficient to show this for $m = 2$, *i.e.*, for a union of two arcs Γ_1 and Γ_2; the general case can be easily proved then by induction.

Assume that $S := \Gamma_1 \cap \Gamma_2 \neq \varnothing$. Since $S \subset \Gamma_2$ is compact, one can find a finite cover of S by open (in topology on Γ_2) disjoint subarcs γ_j's, so that $|\bigcup_j \gamma_j| < |S| + \varepsilon$ and $\Gamma_2 \setminus (\bigcup_j \gamma_j)$ consists of finitely many (closed) subarcs $\Gamma_\varepsilon^{(j)}$, $j = \overline{1,k}$. Denoting, for convenience, $\Gamma_\varepsilon^{(k+1)} := \Gamma_1$, we see that the set

$$\Gamma_\varepsilon := \bigcup_{j=1}^{k+1} \Gamma_\varepsilon^{(j)}$$

satisfies (i) and (ii).

Now, for fixed $\varepsilon > 0$, let $\{\mathcal{Z}_{n,\varepsilon}^{(j)}\} \in \mathrm{AEM}(\Gamma_\varepsilon^{(j)})$, $j = \overline{1,k+1}$, where the $\Gamma_\varepsilon^{(j)}$ as above. We define

$$\beta_j := |\Gamma_\varepsilon^{(j)}|/|\Gamma_\varepsilon|, \quad j = 1, \dots, k+1,$$

and choose nondecreasing integer sequences $\{p(j,n)\}_{n=1}^\infty$, $j = \overline{1,k+1}$, so that

$$\sum_{j=1}^{k+1} p(j,n) = n \quad \text{for all } n \geq k+1 \text{ and } \lim_{n\to\infty} \frac{p(j,n)}{n} = \beta_j, \quad j = 1, \dots, k+1. \tag{4.26}$$

Let

$$\mathcal{Z}_{n,\varepsilon} := \bigcup_{j=1}^{k+1} \mathcal{Z}_{p(j,n),\varepsilon}^{(j)} \subset \Gamma_\varepsilon.$$

We claim that $\mathcal{Z}_{n,\varepsilon} \in \mathrm{AEM}(\Gamma_\varepsilon\,;s)$. Indeed, if we denote, for $i \neq j$,

$$E_\varepsilon^{(i,j)}(n\,;s) := \sum_{\substack{z \in \mathcal{Z}_{p(i,n),\varepsilon}^{(i)} \\ \zeta \in \mathcal{Z}_{p(j,n),\varepsilon}^{(j)}}} K(\|z - \zeta\|\,;s),$$

and take into account that

$$\operatorname{dist}(\Gamma_\varepsilon^{(i)}, \Gamma_\varepsilon^{(j)}) > 0 \quad \text{for } i \neq j,$$

we get

$$E_\varepsilon^{(i,j)}(n\,;s) = O(n^2) = o\big(r_n(s)\big) \quad \text{as } n \to \infty.$$

Furthermore, thanks to (4.26),

$$\lim_{n\to\infty} \frac{r_{p(j,n)}(s)}{r_n(s)} = \beta_j^{1+s}.$$

Thus there exists the limit

$$\lim_{n\to\infty} \frac{E(\mathcal{Z}_{n,\varepsilon}\,;s)}{r_n(s)} = \sum_{j=1}^{k+1} \lim_{n\to\infty} \frac{E(\mathcal{Z}_{p(j,n),\varepsilon}^{(j)}\,;s)}{r_n(s)} = \sum_{j=1}^{k+1} \lim_{n\to\infty} \frac{r_{p(j,n)}(s)}{r_n(s)} \frac{E(\mathcal{Z}_{p(j,n),\varepsilon}^{(j)}\,;s)}{r_{p(j,n)}(s)}$$

$$= \sum_{j=1}^{k+1} \beta_j^{1+s} \cdot 2\tilde{\zeta}(s) |\Gamma_\varepsilon^{(j)}|^{-s} = 2\tilde{\zeta}(s) \sum_{j=1}^{k+1} |\Gamma_\varepsilon^{(j)}| \cdot |\Gamma_\varepsilon|^{-s-1} = C(\Gamma_\varepsilon\,;s).$$

Finally, if $\mathcal{Z}_n$ is s-energy minimizing set on Γ, then $E(\mathcal{Z}_n\ ;\ s) \le E(\mathcal{Z}_{n,\varepsilon}\ ;\ s)$, for each n, and so

$$\limsup_{n\to\infty} \frac{E(\mathcal{Z}_n\ ;\ s)}{r_n(s)} \le C(\Gamma_\varepsilon\ ;\ s) < 2\tilde{\zeta}(s)(|\Gamma| - \varepsilon)^{-s}.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of the lemma as well as the proof of Theorem 3.2. ■

Proof of Theorem 3.4 First we find how asymptotically s-energy minimizing sequences on Γ are distributed with respect to the Γ_j's.

Lemma 4.6 *For sets Γ in Theorem 3.2, if $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\ ;\ s)$, then $\{\mathcal{Z}_{n,j}\} := \{\mathcal{Z}_n\} \cap \Gamma_j \in \mathrm{AEM}(\Gamma_j\ ;\ s)$, $j = \overline{1,m}$.*

Proof Essentially, we will use notations and arguments of the proof of Lemma 4.3, but this time, $\{\mathcal{Z}_n\}$ is an asymptotically s-energy minimizing sequence, and we start with *any* subsequence $\mathcal{N}_1 \subseteq \mathbb{N}$ such that *all* the limits in (4.19) and (4.23) do exist (*i.e.*, an analog of (4.23) holds with lower limits replaced by ordinary limits). Then we can rewrite (4.25) in the form

$$C(\Gamma\ ;\ s) = \lim_{n\to\infty} \frac{E(\mathcal{Z}_n\ ;\ s)}{r_n(s)} \ge C(\Gamma\ ;\ s),$$

with the equality if and only if everywhere in the modified (4.23) equalities hold. It follows then that, for all $j = 1, \dots, m$,

(i) $\alpha_j = \beta_j$;

(ii) $\lim_{\substack{n\to\infty\\ n\in\mathcal{N}_1}} E(\mathcal{Z}_{n,j}\ ;\ s)/r_{p(j,n)}(s) = C(\Gamma_j\ ;\ s)$.

Since $\mathcal{N}_1$ is an arbitrary subsequence of $\mathbb{N}$, we conclude that $\{\mathcal{Z}_{n,j}\} \in \mathrm{AEM}(\Gamma_j\ ;\ s)$ and, additionally, there exists the limit

$$\lim_{n\to\infty} \frac{p(j,n)}{n+1} = \beta_j, \quad j = \overline{1,m}, \tag{4.27}$$

where the β_j's are defined in (4.21). ■

We now continue the proof of Theorem 3.4. If Γ is a rectifiable Jordan arc, $\{\mathcal{Z}_n\} \in \mathrm{AEM}(\Gamma\ ;\ s)$, and $\gamma \subset \Gamma$ is a closed subarc, then representing $\Gamma = \gamma \cup (\overline{\Gamma \setminus \gamma})$ and applying Lemma 4.6, we conclude from (4.27) that

$$\lim_{n\to\infty} \nu_n(\gamma) = \lambda_\Gamma(\gamma). \tag{4.28}$$

We remark that (4.28) trivially holds for open subarcs as well.

Let $K \subset \Gamma$ be a compact set. Then each $K_j := \Gamma_j \cap K$ is compact, and, for any $\varepsilon > 0$, we can find a cover $\mathcal{O}_j \subset \Gamma_j$ of K_j, consisting of finitely many disjoint open subarcs, such that $|\mathcal{O}_j| < |K_j| + \varepsilon$. By (3.2),

$$\sum_{j=1}^m |\mathcal{O}_j| < \sum_{j=1}^m |K_j| + m\varepsilon = |K| + m\varepsilon.$$

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Thus, using (4.28) for each subarc in $\mathcal{O}_j$, $j = \overline{1, m}$, we get

$$\limsup_{n\to\infty} \nu_n(K) \le \limsup_{n\to\infty} \nu_n\Bigl(\bigcup_{j=1}^m \mathcal{O}_j\Bigr) \le \sum_{j=1}^m \limsup_{n\to\infty} \nu_n(\mathcal{O}_j)$$
$$= \sum_{j=1}^m \limsup_{n\to\infty} \Bigl(\frac{p(j,n)}{n+1}\frac{\operatorname{card} \mathcal{O}_j}{p(j,n)}\Bigr)$$
$$= \sum_{j=1}^m \lambda_\Gamma(\Gamma_j)\lambda_{\Gamma_j}(\mathcal{O}_j) < \lambda_\Gamma(K) + m\varepsilon/|\Gamma|.$$

Therefore,

$$\limsup_{n\to\infty} \nu_n(K) \le \lambda_\Gamma(K). \tag{4.29}$$

Next, let $S \subset \Gamma$ be a set satisfying $\lambda_\Gamma(\bar{S} \setminus S^\circ) = 0$, where S° is the interior of S. Then by (4.29)

$$\limsup_{n\to\infty} \nu_n(S) \le \limsup_{n\to\infty} \nu_n(\bar{S}) \le \lambda_\Gamma(\bar{S}) = \lambda_\Gamma(S).$$

These same arguments applied to $\Gamma \setminus S$ yield

$$\liminf_{n\to\infty} \nu_n(S) \ge \lambda_\Gamma(S),$$

and so

$$\lim_{n\to\infty} \nu_n(S) = \lambda_\Gamma(S).$$

By [5, Theorem 0.5], this implies (3.4). ∎

Proof of Proposition 3.5 Applying the identity in (4.3) to (4.5), we get that

$$\hat{E}_1(\mathcal{Z}_n\,;s) \ge 2n^{1+s}\Bigl(\sum_{k=1}^n d_{k,n}\Bigr)^{-s}\bigl(1+\psi(\mathcal{Z}_n)\bigr)^s, \quad \psi(\mathcal{Z}_n) = \frac{1}{2n^2}\sum_{i,j=1}^n \frac{(d_{i,n}-d_{j,n})^2}{d_{i,n}d_{j,n}}. \tag{4.30}$$

By assumption,

$$\lim_{n\to\infty} \frac{E(\mathcal{Z}_n\,;s)}{n^{1+s}} = C(\Gamma\,;s).$$

Thus, taking into account the lower bound (4.7) we see that then necessarily

$$\lim_{n\to\infty} \frac{\hat{E}_1(\mathcal{Z}_n\,;s)}{n^{1+s}} = 2L^{-s},$$

so that by (4.30),

$$\lim_{n\to\infty} \sum_{k=1}^n d_{k,n} = L \quad \text{and} \quad \lim_{n\to\infty} \psi(\mathcal{Z}_n) = 0. \tag{4.31}$$

Let

$$\eta_n := \frac{1}{2}\sum_{k=1}^{n}\Big| d_{k,n} - \frac{L}{n}\Big|, \quad \alpha_n := \frac{1}{2}\Big(L - \sum_{k=1}^{n} d_{k,n}\Big),$$

and

$$K_n^+ = \Big\{1 \le k \le n : d_{k,n} \ge \frac{L}{n}\Big\}, \quad K_n^- = \Big\{1 \le k \le n : d_{k,n} < \frac{L}{n}\Big\}.$$

Obviously, $K_n^+ \neq \varnothing$. Moreover,

$$2\eta_n = \sum_{k=1}^{n}\Big| d_{k,n} - \frac{L}{n}\Big| = \sum_{k\in K_n^+}\Big(d_{k,n} - \frac{L}{n}\Big) - \sum_{k\in K_n^-}\Big(d_{k,n} - \frac{L}{n}\Big),$$

and

$$\sum_{k\in K_n^+}\Big(d_{k,n} - \frac{L}{n}\Big) + \sum_{k\in K_n^-}\Big(d_{k,n} - \frac{L}{n}\Big) = -2\alpha_n.$$

Thus,

$$\sum_{k\in K_n^+}\Big(d_{k,n} - \frac{L}{n}\Big) = \eta_n - \alpha_n, \quad -\sum_{k\in K_n^-}\Big(d_{k,n} - \frac{L}{n}\Big) = \eta_n + \alpha_n.$$

For an arbitrary $\varepsilon > 0$, set

$$K_{\varepsilon,n} := \Big\{k \in K_n^- : d_{k,n} < (1-\varepsilon)\frac{L}{n}\Big\},$$

and let $|K_{\varepsilon,n}|$ denote the number of elements of $K_{\varepsilon,n}$. Then

$$\begin{aligned}\eta_n + \alpha_n &= \sum_{k\in K_n^-}\Big(\frac{L}{n} - d_{k,n}\Big) = \sum_{k\in K_{\varepsilon,n}}\Big(\frac{L}{n} - d_{k,n}\Big) + \sum_{k\in K_n^-\setminus K_{\varepsilon,n}}\Big(\frac{L}{n} - d_{k,n}\Big)\\ &\le |K_{\varepsilon,n}|\frac{L}{n} + n\varepsilon\frac{L}{n} = (\varepsilon + |K_{\varepsilon,n}|/n)L.\end{aligned}$$

Hence,

$$|K_{\varepsilon,n}| \ge n\Big(\frac{\eta_n + \alpha_n}{L} - \varepsilon\Big).$$

Now, we have

$$\begin{aligned}\psi(\mathcal{Z}_n) &= \frac{1}{n^2}\sum_{i,j=1}^{n}\frac{(d_{i,n} - d_{j,n})^2}{d_{i,n}d_{j,n}} \ge \frac{1}{n^2}\sum_{i\in K_n^+}\sum_{j\in K_{\varepsilon,n}}\frac{(d_{i,n} - d_{j,n})^2}{d_{i,n}d_{j,n}}\\ &\ge \frac{1}{n^2}\sum_{i\in K_n^+}\sum_{j\in K_{\varepsilon,n}}\frac{\big(d_{i,n} - (1-\varepsilon)L/n\big)^2}{d_{i,n}L/n}\\ &= \frac{|K_{\varepsilon,n}|}{nL}\sum_{i\in K_n^+}\frac{\big(d_{i,n} - (1-\varepsilon)L/n\big)^2}{d_{i,n}}\\ &\ge \frac{1}{L}\Big(\frac{\eta_n + \alpha_n}{L} - \varepsilon\Big)\sum_{i\in K_n^+}\frac{\big(d_{i,n} - (1-\varepsilon)L/n\big)^2}{d_{i,n}}.\end{aligned}$$

Since, by definition of K_n^+, for $i \in K_n^+$

$$\frac{d_{i,n} - (1-\varepsilon)L/n}{d_{i,n}} = 1 - (1-\varepsilon)\frac{L}{nd_{i,n}} \ge \varepsilon,$$

we continue our chain of the above inequalities as follows:

$$\psi(\mathcal{Z}_n) \ge \frac{\varepsilon}{L}\Big(\frac{\eta_n + \alpha_n}{L} - \varepsilon\Big) \sum_{i \in K_n^+} \Big(d_{i,n} - (1-\varepsilon)\frac{L}{n}\Big)$$

$$\ge \frac{\varepsilon}{L}\Big(\frac{\eta_n + \alpha_n}{L} - \varepsilon\Big) \sum_{i \in K_n^+} \Big(d_{i,n} - \frac{L}{n}\Big) = \frac{\varepsilon}{L}(\eta_n - \alpha_n)\Big(\frac{\eta_n + \alpha_n}{L} - \varepsilon\Big).$$

With $\varepsilon = (\eta_n + \alpha_n)/(2L)$ this yields

$$\psi(\mathcal{Z}_n) \ge \Big(\frac{\eta_n + \alpha_n}{2L}\Big)^2 \frac{\eta_n - \alpha_n}{L}.$$

By (4.31), $\psi(\mathcal{Z}_n) \to 0$ and $\alpha_n \to 0$; thus, $\eta_n \to 0$, and (3.6) follows. ■

Proof of Proposition 3.6 We shall use an idea from [4] in obtaining the separation estimates. For each $i = 0, 1, \dots, n$, consider the function

$$U_i(z) := \sum_{k \ne i} K(\|z - \tilde{z}_{k,n}\| \,;\, s), \quad z \in \Gamma.$$

The minimizing property of $\tilde{\mathcal{Z}}_n$ implies that, for any $z \in \Gamma$,

$$U_i(z) \ge U_i(\tilde{z}_{i,n}) \ge K(\|\tilde{z}_{i,n} - \tilde{z}_{i^*,n}\| \,;\, s) = \|\tilde{z}_{i,n} - \tilde{z}_{i^*,n}\|^{-s}, \tag{4.32}$$

where i^* is such that

$$\|\tilde{z}_{i,n} - \tilde{z}_{i^*,n}\| = \min_{k \ne i}\{\|\tilde{z}_{i,n} - \tilde{z}_{k,n}\|\}.$$

For $j \ne i$ and $\varepsilon > 0$ ($|\Gamma| > \varepsilon$), set

$$D_j := \Gamma \setminus B(\tilde{z}_{j,n}, \varepsilon), \quad D := \bigcap_{j \ne i} D_j,$$

where, as in (3.7), $B(z, r)$ denotes the ball centered at z and of the radius $r > 0$. Then the condition (3.7) implies that

$$|D| \ge |\Gamma| - M\varepsilon n. \tag{4.33}$$

Integrating (4.32) over D with respect to arclength $m_1(z)$ along Γ yields

$$\|\tilde{z}_{i,n} - \tilde{z}_{i^*,n}\|^{-s}|D| \le \int_D U_i(z)\, dm_1(z) = \sum_{k \ne i} \int_D \|z - \tilde{z}_{k,n}\|^{-s}\, dm_1(z)$$

$$\le \sum_{k \ne i} \int_{D_k} \|z - \tilde{z}_{k,n}\|^{-s}\, dm_1(z) =: \sum_{k \ne i} I_k. \tag{4.34}$$

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Each of the I_k's can be easily estimated by using (3.7) and the Theorem 1.13 of [6]. We have

$$I_k = \int_0^{\varepsilon^{-s}} m_1(\{z : \|z - \tilde{z}_{k,n}\|^{-s} > t\})\, dt = \int_0^{\varepsilon^{-s}} m_1\big(B(\tilde{z}_{k,n}, t^{-1/s})\big)\, dt$$
$$\le M \int_{|\Gamma|^{-s}}^{\varepsilon^{-s}} t^{-1/s}\, dt + |\Gamma| \int_0^{|\Gamma|^{-s}} dt \le \delta(\varepsilon, s) := \begin{cases} C_1 \varepsilon^{-s+1}, & \text{if } s > 1, \\ C_1 \ln(C_2/\varepsilon), & \text{if } s = 1, \end{cases}$$

with $C_1 = C_1(\Gamma, s)$, $C_2 = C_2(\Gamma, s)$. Thus, substituting this estimate into (4.34) and using (4.33) we conclude that

$$\|\tilde{z}_{i,n} - \tilde{z}_{i^*,n}\|^s \ge \frac{|\Gamma| - M\varepsilon n}{n\delta(\varepsilon, s)}.$$

Choosing $\varepsilon = c/n$ with $c > 0$ small enough (say, $c = |\Gamma|/(2M)$), we finally get

$$\|\tilde{z}_{i,n} - \tilde{z}_{i^*,n}\| \ge \frac{c_1}{\big(n\delta(c/n, s)\big)^{1/s}} = \begin{cases} c_2/n, & \text{if } s > 1, \\ c_2/\big(n \ln(C_3 n)\big), & \text{if } s = 1, \end{cases}$$

and (3.8) follows. ■

Proof of Theorem 3.7 We give the proof only for the case when Γ is an arc. An obvious modification of this proof for the case when Γ is a closed curve is left to the reader.

We need the following well-known elementary property of smooth arcs.

Lemma 4.7 *If Γ is a smooth arc, then*

$$\lim_{y \to x} \frac{\ell(x, y)}{\|x - y\|} = 1 \tag{4.35}$$

uniformly on $x \in \Gamma$.

By Lemma 4.2, it is sufficient to show that

$$\limsup_{n\to\infty} \frac{E(\mathcal{Z}_n^* \,;\, s)}{r_n(s)} \le C(\Gamma \,;\, s).$$

Let Γ_j, $j = \overline{1, m}$, denote smooth closed subarcs of Γ in successive order (disjoint except for endpoints) such that $\bigcup_{j=1}^m \Gamma_j = \Gamma$. We set

$$\mathcal{Z}_{n,1}^* := \mathcal{Z}_n^* \cap \Gamma_1,\ \mathcal{Z}_{n,j}^* := (\mathcal{Z}_n^* \cap \Gamma_j) \setminus \mathcal{Z}_{n,j-1}^*, \quad j = \overline{2, m},$$

and denote $p(j, n) := \operatorname{card} \mathcal{Z}_{n,j}^*$. Obviously,

$$\lim_{n\to\infty} p(j, n)/n = \lambda_\Gamma(\Gamma_j). \tag{4.36}$$

Let $\Gamma_{j,n}$ be the subarc of Γ_j joining the first and the last point of $\mathcal{Z}^*_{n,j}$. Clearly,

$$|\Gamma_{j,n}| > |\Gamma_j| - 2|\Gamma|n^{-1}, \tag{4.37}$$

and $\mathcal{Z}^*_{n,j}$ are equally spaced points on $\Gamma_{j,n}$.

Now we refer the reader to the last part of the proof of Theorem 3.1. Fixing $\delta > 0$ and setting $\sigma = \sigma_n := 2|\Gamma|n^{-1}$, by Lemma 4.7 and (4.37), we can treat $\Gamma_{j,n}$ as $\Gamma_{\delta,\varepsilon}$ (satisfying (4.12) with $\varepsilon > 0$ independent of σ, that is, of n). Then the set $\mathcal{Z}^*_{n,j}$ will serve as $\tilde{\mathcal{Z}}_n$ in that proof, and so (4.18) becomes

$$E_{p(j,n)}(\mathcal{Z}^*_{n,j}\,;s) \le 2(1+\delta)^s(|\Gamma_j| - \sigma_n)^{-s}\tilde{\zeta}(s)r_{p(j,n)}(s) + O(n^2)\varepsilon^{-s}.$$

Thus, since ε does not depend on n and $\delta > 0$ is arbitrary, we conclude that

$$\limsup_{n\to\infty} \frac{E_{p(j,n)}(\mathcal{Z}^*_{n,j}\,;s)}{r_{p(j,n)}(s)} \le C(\Gamma_j\,;s),$$

i.e., $\{\mathcal{Z}^*_{n,j}\} \in \mathrm{AEM}(\Gamma_j\,;s)$. Therefore, using (4.36), we get

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{j=1}^m E_{p(j,n)}(\mathcal{Z}^*_{n,j}\,;s)}{r_n(s)} &= \sum_{j=1}^m \lim_{n\to\infty}\left(\frac{E_{p(j,n)}(\mathcal{Z}^*_{n,j}\,;s)}{r_{p(j,n)}(s)}\,\frac{r_{p(j,n)}(s)}{r_n(s)}\right)\\ &= \sum_{j=1}^m C(\Gamma_j\,;s)\lambda_\Gamma(\Gamma_j)^{1+s}\\ &= \sum_{j=1}^m C(\Gamma\,;s)\lambda_\Gamma(\Gamma_j) = C(\Gamma\,;s).\end{aligned} \tag{4.38}$$

To complete the proof, all that remains to show is that, for all $i < j$,

$$E^{(i,j)}(n\,;s) := \sum_{\substack{x\in\mathcal{Z}^*_{n,i}\\ y\in\mathcal{Z}^*_{n,j}}} K(\|x-y\|\,;s) = o\big(r_n(s)\big) \quad \text{as } n\to\infty. \tag{4.39}$$

Clearly, if $j \ne i+1$, then $E^{(i,j)}(n\,;s) = O(n^2)$ and (4.39) is satisfied. In the case when $j = i+1$, thanks to (3.9), we have

$$\begin{aligned} E^{(i,j)}(n\,;s) &\le C^s \sum_{\substack{x\in\mathcal{Z}^*_{n,i}\\ y\in\mathcal{Z}^*_{n,j}}} K\big(\ell(x,y)\,;s\big) = C_1 \sum_{k=1}^{p(i,n)}\sum_{q=1}^{p(j,n)} \big((k+q-1)|\Gamma|/n\big)^{-s}\\ &\le C_2 n^s \sum_{k=1}^{n}\sum_{q=k}^{n} q^{-s} =: C_2 n^s \mathbf{S}. \end{aligned} \tag{4.40}$$

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It is straightforward that the double sum $\mathbf{S}$ in (4.40) can be estimated, for n large enough, by

$$\mathbf{S} \le C_3 \cdot \begin{cases} n^{2-s}, & \text{if } 1 \le s < 2, \\ \ln n, & \text{if } s = 2, \\ 1, & \text{if } s > 2, \end{cases}$$

and (4.39) follows. ■

Proof of Theorem 3.8 Repeating arguments of the previous proof, we arrive at (4.38) (with $m = 2$) and, again, all we need to show is that, under assumptions (3.10) and (3.11), (4.39) holds true.

First, we need two elementary auxiliary inequalities. Since Γ_1 and Γ_2 are smooth Jordan arcs, (3.10) is equivalent, by Lemma 4.7, to

$$\|z - y\| \ge c_1 \ell(z,\tau)^\rho, \quad \text{for all } z \in \Gamma_j,\ y \in \Gamma_i,\ i \ne j. \tag{4.41}$$

If, for $z \in \Gamma_j$, a point $\tilde{z} \in \Gamma_{3-j}$ satisfies

$$\|z - \tilde{z}\| = \min_{\zeta \in \Gamma_{3-j}} \|z - \zeta\|,$$

then, considering separately cases $\|\zeta - \tilde{z}\| \le \|z - \tilde{z}\|/2$ and $\|\zeta - \tilde{z}\| > \|z - \tilde{z}\|/2$, after a little algebra we obtain that

$$\|z - \zeta\| \ge \frac{1}{4}(\|z - \tilde{z}\| + \|\zeta - \tilde{z}\|), \quad \zeta \in \Gamma_{3-j},$$

and so, applying (4.41) and, once again, Lemma 4.7, we get

$$\|z - \zeta\| \ge c_2 \big(\ell(z,\tau)^\rho + \ell(\zeta,\tilde{z})\big). \tag{4.42}$$

Let $z \in \Gamma_j \setminus \{\tau\}$. Then (4.42) implies that

$$\begin{aligned} I(z) &:= \sum_{\zeta \in \mathcal{Z}^*_{n,3-j}} K(\|z - \zeta\|\,; s) \le C_1 \sum_{\zeta \in \mathcal{Z}^*_{n,3-j}} \big(\ell(z,\tau)^\rho + \ell(\zeta,\tilde{z})\big)^s \\ &= C_1 \Big(\sum_{\zeta \in \Gamma_{3-j}(\tau,\tilde{z})} + \sum_{\zeta \notin \Gamma_{3-j}(\tau,\tilde{z})} \Big). \end{aligned}$$

In each sum above, one can easily recognize a Riemann sum of the function $f(t) := (a+t)^{-s}$, $a > 0$, which is continuous and decreasing on $[0, +\infty]$, and obtain

$$\begin{aligned} I(z) &< 2C_1 \Big(f(0) + \frac{n}{|\Gamma|} \int_0^{|\Gamma_{3-j}|} f(t)\,dt \Big) \\ &< C_2 \left(\ell(z,\tau)^{-\rho s} + \begin{cases} n\ell(z,\tau)^{\rho(1-s)}, & s > 1 \\ n \ln\big(C_3/\ell(z,\tau)\big), & s = 1 \end{cases} \right) = C_2 F_n(\ell(z,\tau)\,; \rho, s), \end{aligned} \tag{4.43}$$

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where $C_2 = C_2(\Gamma, s)$, $C_3 = C_3(\Gamma)$. Note that $F(\cdot\,;\rho, s)$ is continuous and decreasing on $(0, +\infty)$.

Now we are ready to estimate $E^{(1,2)}(n\,;s)$. First, choose $j = j(n) \in \{1, 2\}$. Let ζ_1, $\zeta_2 \in \mathcal{Z}_n^*$ be two consecutive points such that $\zeta_j \in \Gamma_j$, $j = 1, 2$. Since $\tau \in \Gamma(\zeta_1, \zeta_2)$,

$$\ell(\zeta_1, \tau) + \ell(\zeta_2, \tau) = \ell(\zeta_1, \zeta_2) = |\Gamma|/n.$$

Therefore,

$$\max\{\ell(\zeta_1, \tau), \ell(\zeta_2, \tau)\} \geq |\Gamma|/(2n), \tag{4.44}$$

and we select j, for which the maximum in (4.44) is attained. Then, using (4.43), we conclude that

$$E^{(1,2)}(n\,;s) = \sum_{z \in \mathcal{Z}_{n,j}^*} I(z) \leq C_2 \sum_{z \in \mathcal{Z}_{n,j}^*} F_n\big(\ell(z, \tau)\,;\rho, s\big)\,. \tag{4.45}$$

Passing again from a Riemann sum to an integral and taking into account (4.44), we get

$$\begin{aligned} E^{(1,2)}(n\,;s) &< C_2\left(F_n\big(\ell(\zeta_j, \tau)\,;\rho, s\big) + \frac{n}{|\Gamma|}\int_{\ell(\zeta_j,\tau)}^{C_4} F_n(t\,;\rho, s)\,dt \right) \\ &\leq C_2\left(F_n\big(|\Gamma|/(2n)\,;\rho, s\big) + \frac{n}{|\Gamma|}\int_{|\Gamma|/n}^{C_4} F_n(t\,;\rho, s)\,dt \right). \end{aligned} \tag{4.46}$$

Estimates of the integral in (4.46) depend on values of parameters ρ and s. We omit these calculations and just state the final estimates.

$$\int_{|\Gamma|/n}^{C_4} F_n(t\,;\rho, s)\,dt \leq C_5 \cdot \begin{cases} n^{\max\{\rho s-1, 1\}}, & \rho(s-1) < 1, \\ n^{\rho s-1} + n\ln n, & \rho(s-1) = 1, \\ n^{\rho s-1}, & \rho(s-1) > 1. \end{cases}$$

Clearly,

$$F_n\left(\frac{|\Gamma|}{n}\,;\rho, s\right) \leq C_6 \cdot \begin{cases} n^{\rho} + n\ln n, & s = 1, \\ n^{\rho s}, & s > 1. \end{cases}$$

Combining these two estimates with (4.46) we finally obtain

$$E^{(1,2)}(n\,;s) \leq C_7 \cdot \begin{cases} n^{\max\{\rho s, 2\}}, & \rho(s-1) < 1, \\ n^{\rho s}\ln n, & \rho(s-1) = 1, \\ n^{\rho s}, & \rho(s-1) > 1. \end{cases} \tag{4.47}$$

Note that (3.11) implies that

$$1 + s > \rho s. \tag{4.48}$$

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If $s > 1$, then $1 + s > 2$, and taking into account (4.48) from (4.47) we conclude that (4.39) holds true. The case $s = 1$ falls into the first case in (4.47). For $\rho \in [1, 2]$, we have $\max\{\rho, 2\} = 2$ and, again, (4.39) is valid. ∎

Verification of Example 2 Using notations and arguments of the proof of Theorem 3.7, we obtain an analog of (4.38), and so

$$\liminf_{n\to\infty} \frac{E(\mathcal{Z}_n^* ; s)}{r_n(s)} = C(\Gamma^{(\rho)} ; s) + 2 \liminf_{n\to\infty} \frac{E^{(1,2)}(n ; s)}{r_n(s)}. \tag{4.49}$$

For n even, say, $n = 2k$,

$$E^{(1,2)}(2k ; s) > K(\|z_{k-1,2k} - z_{k+1,2k}\|) > \frac{2^{-s}}{|\Gamma^{(\rho)}|^{\rho s}} (2k)^{\rho s},$$

and (4.49) becomes

$$\liminf_{k\to\infty} \frac{E(\mathcal{Z}_{2k}^* ; s)}{r_{2k}(s)} \geq C(\Gamma^{(\rho)} ; s) + \frac{2^{1-s}}{|\Gamma^{(\rho)}|^{\rho s}} \liminf_{k\to\infty} \frac{(2k)^{\rho s}}{r_{2k}(s)}.$$

If $s > s^*(\rho)$, then $\rho s > 1 + s$, and the lower limit in the right-hand side of the above inequality is $+\infty$; if $s = s^*(\rho)$, then $\rho s = 1 + s$, and the mentioned limit is 1 for $s > 1$. In either case,

$$\liminf_{k\to\infty} \frac{E(\mathcal{Z}_{2k}^* ; s)}{r_{2k}(s)} > C(\Gamma^{(\rho)} ; s),$$

and so $E(\mathcal{Z}_n^* ; s) \notin \mathrm{AEM}(\Gamma^{(\rho)} ; \rho)$. ∎

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