Can. J. Math., Vol. XXXI, No. 2, 1979, pp. 396–418

BIPLANAR SURFACES OF ORDER THREE

TIBOR BISZTRICZKY

0. Introduction. A surface of order three, F, in the real projective three-space P^3 is met by every line, not in F, in at most three points. F is biplanar if it contains exactly one non-differentiable point ν and the set of tangents of F at ν is the union of two distinct planes, say τ_1 and τ_2. In the present paper, we classify and describe those biplanar F which contain the line $\tau_1 \cap \tau_2$.

We describe a surface by determining the tangent plane sections of the surface at the differentiable points. This approach was introduced in [**1**] and it is based upon A. Marchaud's definition of "surfaces of order three" in [**4**].

We denote the planes, lines and points of P^3 by the letters $\alpha, \beta, \ldots;\ L, M, \ldots;$ and $p, q, \ldots$ respectively. For a collection of flats $\alpha, L, p, \ldots;$ $\langle \alpha, L, p, \ldots \rangle$ denotes the flat of P^3 spanned by them. For a set $\mathscr{M}$ in P^3, $\langle \mathscr{M} \rangle$ is the flat of P^3 spanned by the points of $\mathscr{M}$.

1. Surfaces of order three.

1.1. A *surface of order three* F in P^3, is a compact and connected set such that every intersection of F with a plane is a curve of order $\leqq 3$ and there is a plane β such that $\beta \cap F$ is a curve of order three which does not contain any lines.

Plane curves are defined by means of parameter curves. A *parameter curve* C is a continuous map from a line $M = \{m, m', \ldots\}$ into a plane α. A line T is the *tangent* of C at $m \in M$ if $T = \lim \langle C(m), C(m') \rangle$ as $m' \neq m$ tends to m. C is *differentiable* if the tangent T of C at m exists for every $m \in M$ and $|T \cap C(M)| < \infty$. C is *degenerate* if C is injective and $C(M)$ is a line. C is *totally degenerate* if $C(M)$ is a point (isolated).

Let C be differentiable, $C(M) \subset \alpha$. Then $p \in C(M)$ is *simple* if $p = C(m)$ has a unique solution $m \in M$. We introduce (cf. [**6**]) the characteristic $(a_0(m), a_1(m))$ of $C(m)$, $a_i(m) = 1$ or 2, and say that L meets C at m with *multiplicity* $a_0(m) + a_1(m)[a_0(m)]$ if $C(m) \in L \subset \alpha$ and L is (is not) the tangent of C at m. C is of *order* n if n is the supremum of the number of points of M, counting multiplicities, mapped into collinear points by C.

If C is of order two (three), we denote $C(M)$ by $S^1[F_*{}^1]$. Every point of an S^1 is simple and an $F_*{}^1$ contains at most one point q (double point) such that $q = C(m) = C(m')$, $m \neq m'$. A simple point of an $F_*{}^1$ is an ordinary, inflection or cusp point if it has the characteristic (1, 1), (1, 2) or (2, 1) respectively; cf. [**3**] and [**1**]. A degenerate C is considered to have order one and an isolated point is counted with multiplicity two.

Received December 21, 1977 and in revised form June 23, 1978.

A (*plane*) *curve* Γ is the union of a finite collection of sets $C_\lambda(M)$ where the C_λ's are parameter curves. A line T is a *tangent* of Γ at p if T is the tangent of some C_λ at m, $p = C_\lambda(m) \subset C_\lambda(M) \subset \Gamma$. The *order* of Γ is the supremum of the number of points of Γ, counting multiplicities on each C_λ, lying on any line not in Γ.

Let Γ be of order n, $n \leqq 3$. If $n = 1$, then Γ is a straight line. If $n = 2$, then Γ is an S^1 or an isolated point or a pair of distinct lines. If $n = 3$, then Γ is (*i*) an F_*^1 or (*ii*) the disjoint union of an F_*^1 and either an S^1 or an isolated point or (*iii*) the union of a line and a curve Γ' of order two.

We denote a Γ of order three satisfying (*i*) or (*ii*) by F^1. Then there is an $F_*^1 \subseteq F^1$.

1.2. Let F be a surface of order three. Let α be a plane through p, $p \in F$. Then p is *regular* in $F[\alpha \cap F]$ if there is a line N in $P^3[\alpha]$ such that $p \in N$ and $|N \cap F| = 3$. Otherwise, p is *irregular* in $F[\alpha \cap F]$. We note that there is at most one point ν irregular in $\alpha \cap F$ if $\alpha \cap F$ is an F^1 and such a ν is a cusp, double point or isolated point of $\alpha \cap F$. Finally,

$$l(p, \alpha) = |\{L \subset \alpha| \quad p \in L \subset F\}| \leqq l(\alpha) = |\{L \subset \alpha| \quad L \subset F\}| \leqq 3.$$

If F is non-ruled, that is, F is not generated by lines, then $l(F) = |\{L \subset P^3| \ L \subset F\}| < \infty$ and F contains at most four irregular points.

Let $p \in F$. A line T is a *tangent* of F at p if T is a tangent of $\alpha \cap F$ at p for some α through p. Let $\tau(p)$ be the set of tangents of F at p. Then p is *differentiable* if p is regular in $\alpha \cap F$ and $\tau(p)$ is a plane $\pi(p)$; otherwise, p is *singular*.

Henceforth, we assume that every regular p in F is differentiable and $\pi(p)$ depends continuously on p.

Let p be a regular in F. Then $p \in T \subset \pi(p)$ implies that $T \subset F$ or $|T \cap F| \leqq 2$. Thus, $l(p) = |\{L \subset P^3| \quad p \in L \subset F\}| = l(p, \pi(p))$ and p is irregular in $\pi(p) \cap F$. If $l(p) = 0$, then p is an isolated point, cusp or double point of $\pi(p) \cap F$ and we call p *elliptic*, *parabolic* or *hyperbolic* respectively.

Let ν be irregular in F. If $l(F) < \infty$, then $\nu \in T \subset \tau(\nu)$ if and only if either $\nu \in T \subset F$ or $T \cap F = \{\nu\}$. Moreover, $\tau(\nu)$ is a plane or the union of two distinct planes or a cone of order two with the vertex ν; *cf.* [**5**].

Let $\mathscr{F}$ be a closed connected subset of S^1 or F_*^1. If the end points of F are distinct (equal), then F is a *subarc* (*subcurve*). We note that a subarc of F_*^1, containing only ordinary points in its interior, is of order two.

Let $p \in F$ be regular. Let $\mathscr{F}(p)$ be the set of all subarcs $\mathscr{F}$ of order two in F such that $p \in \mathscr{F} \not\subset \pi(p)$. Let $\{\mathscr{F}, \mathscr{F}'\} \subset \mathscr{F}(p)$. Then $\mathscr{F}$ and $\mathscr{F}'$ are *p-compatible* if there is a $\beta \subset P^3 \backslash \{p\}$ and an open neighbourhood $u(p)$ of p in P^3 such that $u(p) \cap (\mathscr{F} \cup \mathscr{F}')$ is contained in a closed half-space of P^3 bounded by $\pi(p)$ and β. Otherwise $\mathscr{F}$ and $\mathscr{F}'$ are *p-incompatible.*

A pair of subarcs $\mathscr{F}_1$ and F_2 are *compatible* [*incompatible*] if there is a $p \in \mathscr{F}_1 \cap \mathscr{F}_2$ such that $\{\mathscr{F}_1, \mathscr{F}_2\} \subset \mathscr{F}(p)$ and $\mathscr{F}_1$ and $\mathscr{F}_2$ are p-compatible [p-incompatible].

We consider a subcurve as an element of $\mathscr{F}(p)$ if it contains a subarc $\mathscr{F}$ such that $p \in \mathscr{F} \subset \mathscr{F}(p)$. In this sense, we say that a subcurve is compatible or incompatible with an element of $\mathscr{F}(p)$.

1.3. For proofs of the following results, we refer to [**1**] and [**2**].

1. If p is regular in F and isolated in $\alpha \cap F$, then p is elliptic and $\alpha = \pi(p)$.

2. Let p be regular in F, $l(p) = 0$. Then (*i*) p is elliptic if and only if $\mathscr{F}$ and $\mathscr{F}'$ are compatible for $\{\mathscr{F}, \mathscr{F}'\} \subset \mathscr{F}(p)$ and (*ii*) p is hyperbolic if and only if there exist incompatible $\mathscr{F}$ and $\mathscr{F}'$ in $\mathscr{F}(p)$ such that $p \in \text{int}(F) \cap \text{int}(F')$.

3. Let $\mathscr{F}' \subset F$ such that $\mathscr{F}' \in \mathscr{F}(p)$ for each $p \in \mathscr{F}'$. Let L be a line such that $L \not\subset \langle \mathscr{F}' \rangle$ and for each $p \in \mathscr{F}'$, there is an $\mathscr{F}p \in \mathscr{F}(p)$ with $L \subset \langle \mathscr{F}p \rangle$. If $\mathscr{F}p$ depends continuously on p, then $\mathscr{F}'$ and $\mathscr{F}p$ are either compatible for all $p \in \mathscr{F}'$ or incompatible for all $p \in \mathscr{F}'$.

4. Let $p_\lambda[\alpha_\lambda]$ be a sequence of points (planes) converging to $p(\alpha)$; $p_\lambda \in \alpha_\lambda$ for each λ.

(*a*) If $\alpha \cap F$ is not of order two or $\alpha \cap F$ does not contain an isolated point, then $\lim(\alpha_\lambda \cap F) = \alpha \cap F$.

(*b*) If p_λ is a cusp (isolated point) of $\alpha_\lambda \cap F$ for each λ, then $l(p) = 0$ implies that p is a cusp (isolated point or cusp) of $\alpha \cap F$ and $\alpha \cap F = L \cup S^1$ implies that $L \cap S^1 = \{p\}$.

5. Let $\gamma \cap F = L \cup L'$ such that $\gamma = \pi(p)$ for $p \in L \backslash L'$; $L \neq L'$. Let $\alpha_\lambda[\beta_\lambda]$ be a sequence of planes through $L[L']$ converging to γ; $\gamma \neq \beta_\lambda$ for each λ. Then $\lim(\alpha_\lambda \cap F) = \gamma \cap F$ and there is a subsequence β_λ' of β_λ such that either $\lim(\beta_\lambda' \cap F) = L \cup L'$ or $\lim \beta_\lambda' \cap F = L'$. (We shall simply say that $\lim(\beta_\lambda \cap F)$ is either $L \cup L'$ or L'.)

6. Let $\gamma \cap F$ be of order two. Then $\gamma \cap F = L \cup L'$, $L \neq L'$, and either $L' \subset \pi(p)$ for every regular $p \in L$ (in short, $L' \subset \pi(L)$) or $L \subset \pi(q)$ for every regular $q \in L'$ $(L \subset \pi(L'))$.

2. Biplanar surfaces.

2.0. Let F be a surface of order three. A point $\nu \in F$ is a *binode* if ν is irregular in F and $\tau(\nu)$ is the union of two distinct planes, say τ_1 and τ_2. F is *biplanar* if F is non-ruled and contains a binode ν as its only irregular point.

We wish to examine those biplanar F which contain the line $\tau_1 \cap \tau_2$. Unless stated otherwise, we assume that F is biplanar with the binode ν where $\tau(\nu) = \tau_1 \cup \tau_2$ and $\tau_1 \cap \tau_2 \subset F$. Since $\nu \in T \subset \tau(\nu)$ if and only if $\nu \in T \subset F$ or $T \cap F = \{\nu\}$, $l(\nu) \leqq l(\nu, \tau_1) + l(\nu, \tau_2) \leqq 6$. Then $M_0 = \tau_1 \cap \tau_2 \subset F$ implies that $1 \leqq l(\nu) \leqq 5$.

LEMMA 2.1. *Let* $\nu \in \beta$ *such that* $\beta \cap \tau_i$ *is a line* N_i; $i = 1, 2$.

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

1. If $M_0 = N_1 = N_2$, then either $\beta \cap F$ consists of M_0 and an S^1 such that $M_0 \cap S^1 = \{\nu\}$ or $\beta \cap F = M_0 \cup L$ where $\nu \notin L$ and $L \subset \pi(M_0)$.

2. If $l(\nu, \beta) = 0$, then ν is the double point of $\beta \cap F$.

3. If $N_i \subset F$ and $N_j \cap F = \{\nu\}$, then $\beta \cap F$ consists of N_i and an S^1 such that $|N_i \cap S^1| = 2$ and $\nu \in N_i \cap S^1$; $\{i, j\} = \{1, 2\}$.

4. If $l(\nu, \beta) = 2$, then $\beta \cap F = N_1 \cup N_2 \cup L_{12}$ where $\nu \notin L_{12}$.

Proof. We note that $\nu \in \pi(p)$ if and only if $\langle \nu, p\rangle \subset F$ for $p \in F\backslash\{\nu\}$ and $\nu \in L \not\subset \tau_1 \cup \tau_2$ implies that $|L \cap F| = 2$. The lemma now follows by listing all possible $(\beta \cap F)$'s.

LEMMA 2.2. *1. For $\{p, p'\} \subset M_0\backslash\{\nu\}$, $\pi(p) = \pi(p')$.*

2. If $l(\tau_i) = 2$ for $i = 1$ or 2, then $\tau_i = \pi(p)$ for $p \in M_0\backslash\{\nu\}$.

3. If $l(\nu) \geqq 3$, then $l(\tau_i) = 3$ for $i = 1$ or 2.

Proof. By 2.1.1, we may assume that $\pi(p)$ is τ_1 or τ_2 for $p \in M_0\backslash\{\nu\}$. As $\pi(p)$ depends continuously on p, *1* follows.

Let $l(\tau_i) = 2$ and put $\tau_i \cap F = M_0 \cup M_i$; $\{i, j\} = \{1, 2\}$. By 1.3.6, either $M_i \subset \pi(M_0)$ or $M_0 \subset \pi(M_i)$. Let $N_j \subset \tau_j$ such that $N_j \cap F = \{\nu\}$. By 2.1.3, $\langle M_i, N_j\rangle \cap F = M_i \cup S^1$ where $M_i \cap S^1 = \{\nu, p_i\}$, $\nu \neq p_i$. Then $\pi(p_i) = \langle M_i, N_j\rangle \neq \tau_i$ and $M_i \subset \pi(M_0)$.

Clearly, *2* implies *3*.

THEOREM 2.3. *Let F be biplanar with the binode ν, $\tau_1 \cap \tau_2 \subset F$. Then F is one of the following types: (1) $l(F) = l(\nu) = 1$; (2) $l(F) = 2$ and $l(\nu) = 1$; (3) $l(F) = l(\nu) = 2$; (4) $l(F) = l(\nu) = 3$; (5) $l(F) = 4$ and $l(\nu) = 3$; (6) $l(F) = 6$ and $l(\nu) = 4$; (7) $l(F) = 10$ and $l(\nu) = 5$.*

Proof. Apply 2.1 and 2.2 with each $l(\nu)$, $1 \leqq l(\nu) \leqq 5$.

2.4. It is easy to check that if F is biplanar with the binode ν and one of the types listed in 2.3, then $\tau_1 \cap \tau_2 \subset F$.

Let $\nu \in \beta$ such that $l(\nu, \beta) = 0$. By 2.1.2, ν is the double point of $\beta \cap F$; that is, $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$ where $\mathscr{L} \cap (\mathscr{F}_1 \cup \mathscr{F}_2) = \{\nu\}$, $\mathscr{F}_1 \cap \mathscr{F}_2 = \{\nu, p_\beta\}$ (p_β is the inflection point of $\beta \cap F$) and $\mathscr{L}$ is the loop of $\beta \cap F$. We note that $\mathscr{L}$ is a subcurve of order two and $\{\mathscr{F}_1, \mathscr{F}_2\} \subset \mathscr{F}(p_\beta)$. We will always assume that $\lim\langle \nu, r\rangle \subset \tau_i$ as r tends to ν in $F_i\backslash\{\nu\}$; $i = 1, 2$.

In the following sections, we examine the surfaces listed in 2.3 by determining the existence and distribution of the elliptic, parabolic and hyperbolic points. By way of preparation, we have the following definitions and results.

2.5. Let $S^1 \subset F$, $\alpha = \langle S^1\rangle$. We denote by int S^1, the open disk of α bounded by S^1, and we put ext $S^1 = \alpha\backslash\mathrm{Cl}(\mathrm{int}\, S^1)$.

Let $L \subset F$ and $r \in F\backslash L$ such that $\langle L, r\rangle \cap F$ consists of L and S^1. We denote this S^1 by $S^1(L, r)$.

Let $I(E)$ be the set of parabolic (elliptic) points of F. From 1.3.4, E is open

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

and $\{r \in \text{bd}(E) | l(r) = 0\} \subseteq I$. In each of the surfaces we examine, it will be immediate that $E = \emptyset$ if and only if $I = \emptyset$.

THEOREM 2.6. *Let F be a surface of order three. Let G be an open region in F such that $\alpha_0 \cap \bar{G} = \emptyset$ for some α_0, $\text{bd}(F \backslash G) = \text{bd}(G)$, $\langle \text{bd}(G) \rangle$ is a plane and r is regular in F with $l(r) = 0$ for each $r \in G$. Then $G \cap E \neq \emptyset$.*

Proof. We note that any line in a plane $\langle F_*^1 \rangle$ meets F_*^1 and thus, any line in P^3 meets F.

Let $r \in G$ and put $L = \alpha_0 \cap \langle \text{bd}(G) \rangle$. Then $L \cap \bar{G} = \emptyset$ implies that $L \cap (F \backslash \bar{G}) \neq \emptyset$ and $\langle L, r \rangle \cap G$ is an S^1 or an isolated point of $\langle L, r \rangle \cap F$. Obviously, $\alpha_0 \cap \bar{G} = \emptyset$ implies that there is an $r_0 \in G$ such that $\langle L, r_0 \rangle \cap G = \{r_0\}$. Then $r_0 \in E$ with $\pi(r_0) = \langle L, r_0 \rangle$ by 1.3.1.

3. F with one line.

3.0. Let F be biplanar with the binode ν, $l(F) = 1$. Then $M_0 = \tau_1 \cap \tau_2 \subset F$ and $\tau_i \cap F = M_0$; $i = 1, 2$. By 2.1.1, $\langle M_0, r \rangle \cap F = M_0 \cup S^1(M_0, r)$ with $M_0 \cap S^1(M_0, r) = \{\nu\}$ for $r \in \mathscr{F} \backslash M_0$. We note that $S^1(M_0, r) \in F(r)$; *cf.* 1.2.

Let $\beta \cap M_0 = \{\nu\}$. Then ν is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. We fix a point $\bar{r} \in L \backslash \{\nu\}$ and let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the open half-spaces of P^3 determined by τ_1 and τ_2. Put $F_i = \mathscr{P}_i \cap F$ and assume that $\bar{r} \in F_1$. Then $\beta \cap \bar{F}_1 = \mathscr{L}$, $\beta \cap \bar{F}_2 = \mathscr{F}_1 \cup \mathscr{F}_2$ and

$$F_1 \cup F_2 = \{r \in F | l(r) = 0\}.$$

We fix a point $\bar{p} \in M_0 \backslash \{\nu\}$ and choose $T \subset \tau_1$ such that $\bar{p} \in T \neq M_0$. Then $\beta_t = \langle \nu, \bar{r}, t \rangle$ is a plane for $t \in T$ ($\beta = \beta_{t'}$, say),

$$\beta_{\bar{p}} \cap F = M_0 \cup S^1(M_0, \bar{r}) \subset \bar{F}_1 \quad \text{and} \quad \beta_t \cap F = \mathscr{L}^t \cup \mathscr{F}_1^t \cup \mathscr{F}_2^t, \quad t \neq \bar{p}.$$

LEMMA 3.1. *$\mathscr{L}^t \subset \bar{F}_1$ for all $t \in T \backslash \{\bar{p}\}$.*

Proof. Let $T(i) = \{t \in T | L^t \subset \bar{F}_i\}$, $i = 1, 2$. Let t tend to $\bar{t} \neq \bar{p}$ in $T(i)$. Then $\beta_t \cap \bar{F}_i = \mathscr{L}^t$ converges to $\beta_{\bar{t}} \cap \bar{F}_i$, which is $L^{\bar{t}}$ or $F_1^{\bar{t}} \cup F_2^{\bar{t}}$. Since $\lim \mathscr{L}^t$ cannot be a curve of order three, we obtain that

$$\mathscr{L}^{\bar{t}} = \lim L^t = \lim \beta_t \cap \bar{F}_i = \beta_{\bar{t}} \cap \bar{F}_i.$$

Thus $\bar{t} \in T(i)$ and $T(i)$ is closed. Then $T \backslash \{\bar{p}\} = T(1) \cup T(2)$ and $t' \in T(1)$ imply that $T(2) = \emptyset$.

COROLLARY. *As $t \neq \bar{p}$ tends to $\bar{p}$, $\lim \mathscr{L}^t = S^1(M_0, \bar{r})$ and $\lim F_1^t \cup F_2^t = M_0$. In particular, $\bar{F}_2 = F_2 \cup M_0$.*

THEOREM 3.2. *$\bar{F}_1 = F_1 \cup \{\nu\}$ and every point of F_1 is elliptic.*

Proof. Let $p_\lambda \in F_1$ tend to $p \in M_0$ such that $\alpha_\lambda = \langle \nu, \bar{r}, p_\lambda \rangle$ is a plane for each p_λ. Let α be a limit plane of α_λ. Then $\{\nu, \bar{r}, p\} \subset \bar{F}_1$ and by 3.1, $\alpha \cap \bar{F}_1$ is either $S^1(M_0, \bar{r})$ or a loop $\mathscr{L}^t$. Thus $\alpha \cap \bar{F}_1 = (\alpha \cap F_1) \cup \{\nu\}$ and $p = \nu$.

Let $r \in F_1$. Since $\bar{F}_1 \cap \bar{F}_2 = \{\nu\}$ and $\pi(r) \cap M_0 \neq \{\nu\}$, $\pi(r) \cap F$ is not connected and r must be elliptic.

COROLLARY. *1. Let $\beta \cap M_0 = \{\nu\}$. Then $\beta \cap \bar{F}_1$ is the loop of $\beta \cap F$.*
2. Let $r \in F_2$. Then $\pi(r) \cap \bar{F}_1 = \emptyset$.

Proof. Clearly, *1* implies *2* and $\beta \cap \bar{F}_1$ or $\beta \cap \bar{F}_2$ is the loop of $\beta \cap F$. Since $F_1 \subseteq E$, $\beta \cap F_1$ does not contain any inflection points.

3.3. Let $\beta \cap M_0 = \{\nu\}$. Then $\beta \cap \bar{F}_2 = \mathscr{F}_1 \cup \mathscr{F}_2$ where $\mathscr{F}_1 \cap \mathscr{F}_2 = \{\nu, p_\beta\}$. Since p_β is the inflection point of $\beta \cap F$, $\mathscr{F}_1$ and $\mathscr{F}_2$ are incompatible. We may assume that $S^1(M_0, p_\beta)$ and $\mathscr{F}_1[\mathscr{F}_2]$ are compatible [incompatible]. Then 1.3.3 (with $L = M_0$) yields that $S^1(M_0, r)$ and $\mathscr{F}_1[\mathscr{F}_2]$ are compatible [incompatible] for all $r \in F_1[F_2]$; $r \neq \nu$.

Let $p \in M_0 \backslash \{\nu\}$. Let $u(p)$ be an open neighbourhood of p in $\bar{F}_2$ such that

$$(1) \quad u(p) = u_1(p) \cup (u(p) \cap M_0) \cup u_2(p)$$

where $u_1(p)$ and $u_2(p)$ are open disjoint regions not meeting M_0.

Let $p' \in u_i(p)$ be arbitrarily close to p; $\{i, j\} = \{1, 2\}$. Then p' is arbitrarily close to p in some $\alpha \cap \bar{F}_2$ where $\langle p, p' \rangle \subset \alpha$; $M_0 \not\subset \alpha$. Since $\pi(p)$ is τ_1 or τ_2, $(\pi(p) \cap \alpha) \cap F = \{p\}$ yields that p is an inflection point of $\alpha \cap F$. Thus $|\langle p, p' \rangle \cap u(p)| = 3$ and $\langle p, p' \rangle \cap u_j(p) \neq \emptyset$.

LEMMA 3.4. *Under the hypotheses of* 3.3, *let r_λ be a sequence in $\mathscr{F}_1 \backslash \{\nu\}$ $[\mathscr{F}_2 \backslash \{\nu\}]$ converging to ν. Then $S^1(M_0, r_\lambda)$ converges to $\nu[M_0]$.*

Proof. Since $\langle M_0, r_\lambda \rangle$ tends to τ_1 or τ_2, $\{\nu\} \subseteq \lim S^1(M_0, r_\lambda) \subseteq M_0$.

Let r_λ tend to ν in $\mathscr{F}_2$ and let $\mathscr{F}_{2,\lambda}$ be the subarc of $\mathscr{F}_2$ with the end points ν and r_λ. Then $\mathscr{F}_{2,\lambda}$ converges to ν and, from 3.3, $\mathscr{F}_{2,\lambda}$ and $S^1(M_0, r_\lambda)$ are incompatible for each r_λ. From 1.2, $\mathscr{F}_{2,\lambda}$ and $S^1(M_0, r_\lambda)$ are contained in different closed half-spaces bounded by τ_2 and $\pi(r_\lambda)$. Then r_λ close to ν and $\mathscr{F}_{2,\lambda}$ arbitrarily small imply that $S^1(M_0, r_\lambda)$ is arbitrarily large. Clearly, $S^1(M_0, r_\lambda)$ converges to M_0.

Let $p \in M_0 \backslash \{\nu\}$ and let $u(p)$ satisfy 3.3 (1). Then $p \in \lim S^1(M_0, r_\lambda)$ implies that $p \in \lim(u(p) \cap S^1(M_0, r_\lambda))$. In fact, 3.3. yields that

$$p \in \lim \ (u_i(p) \cap S^1(M_0, r_\lambda)); \quad i = 1, 2.$$

Obviously, there is a $u'(p) \subseteq u(p)$ satisfying 3.3 (1) such that for $p' \in u'(p) \backslash M_0$, $p' \in S^1(M_0, r_\lambda)$ for some $r_\lambda \in \mathscr{F}_2$. Then $u'(p) \cap S^1(M_0, r_\lambda') = \emptyset$ for $r_\lambda' \in \text{int}(\mathscr{F}_1)$ and the lemma follows.

3.5. Let $r \in F_2$. In view of 3.4, $S^1(M_0, r)$ is the boundary of an open region $F_2(M_0, r) \subset F_2$ such that $M_0 \cap F_2(M_0, r) = \emptyset$. Then $\lim S^1(M_0, r) = \{\nu\}$ implies that $\lim \text{Cl}(F_2(M_0, r)) = \{\nu\}$. Clearly, $F_2(M_0, r)$ satisfies 2.6 and thus contains elliptic points. Hence, $F_2 \cap E \neq \emptyset$ with $\nu \in \text{Cl}(F_2 \cap E)$.

From 3.3, 1.3.2 and 1.3.3, F_2 also contains hyperbolic and parabolic points. We note that $\tau_2 = \pi(p)$ for $p \in M_0 \backslash \{\nu\}$ from 3.4.

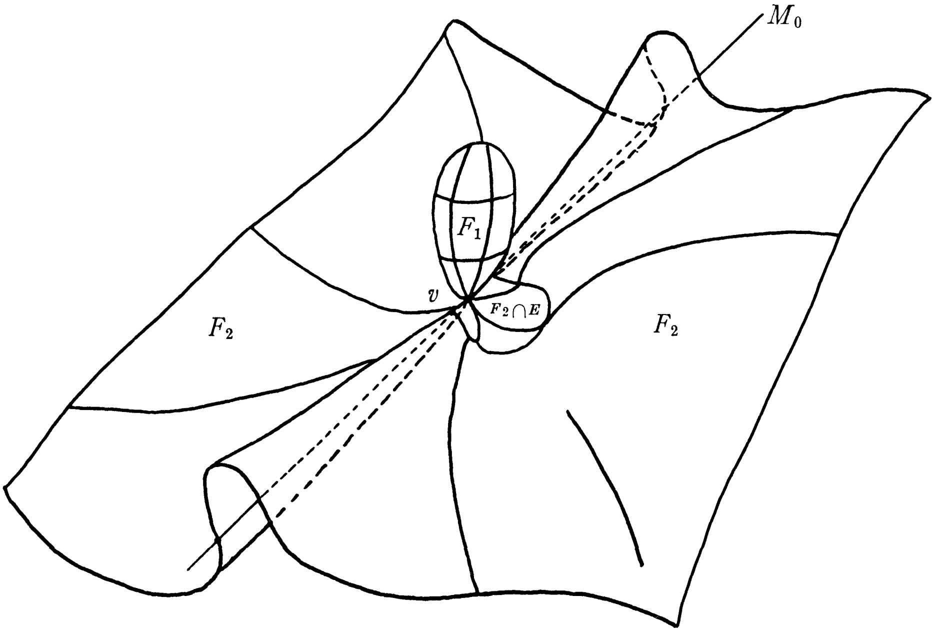

FIGURE 1

THEOREM 3.6. *Let F be biplanar with the binode ν, $l(F) = 1$. Then $F = \bar{F}_1 \cup \bar{F}_2$ where $\bar{F}_1 \cap \bar{F}_2 = \{\nu\}$, every point of F_1 is elliptic and F_2 is described in* 3.0 *and* 3.5.

We refer to Figure 1 for a representation of F. The surface in P^3 defined by $x_1^3 + x_2^3 + x_0^2x_2 + x_1x_2x_3 = 0$ satisfies 3.6 with $\nu \equiv (0, 0, 0, 1)$.

4. F with two lines; $l(\nu) = 1$.

4.0. Let F be biplanar with the binode ν; $l(F) = l(\nu) + 1 = 2$. Let L_0 and $M_0 = \tau_1 \cap \tau_2$ be the lines of F. Then $L_0 \cap M_0$ is a point $p_0 \neq \nu$, $\langle L_0, M_0\rangle \cap F = L_0 \cup M_0$ and $\tau_i \cap F = M_0$; $i = 1, 2$. By 2.1.1, $L_0 \subset \pi(M_0)$.

Let $r \in F$, $l(r) = 0$. Then $M_0 \cap S^1(M_0, r) = \{\nu\}$ and if $L_0 \not\subset \pi(r)$, $|L_0 \cap S^1(L_0, r)| \leqq 2$.

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the open half-spaces of P^3 determined by τ_1 and τ_2. We assume $L_0 \subset \bar{\mathscr{P}}_2$ and put $F_i = \mathscr{P}_i \cap F$. Then (*cf.* Figure 2) $L_0 \subset \bar{F}_2$.

LEMMA 4.1. *1. Let $q \in L_0$. Then $\pi(q) \cap F_1 = \emptyset$.*
2. Let $\beta \cap M_0 = \{\nu\}$. Then $\beta \cap \bar{F}_1$ is the loop of $\beta \cap F$.

Proof. Since $\pi(p_0) = \langle L_0, M_0\rangle \subset \bar{P}_2$, we take $q \neq p_0$. Then $L \subset \pi(q) \neq \pi(p_0)$ implies that $\pi(q) \cap F$ is connected and $(\pi(q) \cap \tau_i) \cap F = \{p_0\}$; $i = 1,2$. Thus $L_0 \subset \bar{F}_2$ yields that $\pi(q) \cap F \subset \bar{F}_2$.

Clearly, ν is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$ and $\mathscr{L}$ is either $\beta \cap \bar{F}_1$ or $\beta \cap \bar{F}_2$. Since $\pi(r) \cap (\mathscr{F}_1 \cap \mathscr{F}_2) \neq \emptyset$ for $r \in L\backslash\{\nu\}$, *1.* implies *2.*

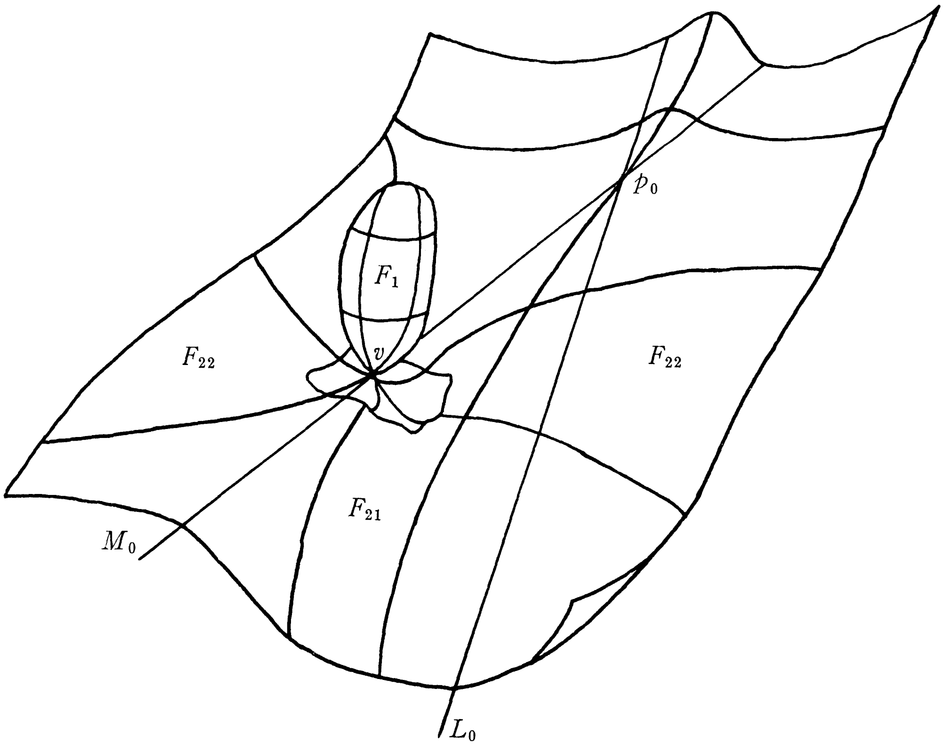

FIGURE 2

THEOREM 4.2. $\bar{F}_1 = F_1 \cup \{\nu\}$ *and every point of* F_1 *is elliptic.*

Proof. Fix $\bar{r} \in F_1$ and apply 4.1 as in the proof of 3.2.

LEMMA 4.3. *Let* r_λ *be a convergent sequence in* $F\backslash(M_0 \cup L_0)$. *If* $\lim \langle M_0, r\rangle = \langle M_0, L_0\rangle[\tau_i]$, *then* $\lim S^1(M_0, r_\lambda) = M_0 \cup L_0[\nu]$; $i = 1, 2$.

Proof. Since $L_0 \subset \pi(M_0)$, $\lim \langle M_0, r_\lambda\rangle = \langle M_0, L_0\rangle$ and 1.3.5 imply that $\lim (M_0 \cup S^1(M_0, r_\lambda)) = M_0 \cup L_0$. It is easy to check that in fact $\lim S^1(M_0, r_\lambda) = M_0 \cup L_0$.

Let $p \in M_0\backslash\{\nu, p_0\}$ and let $u(p) \subset \bar{F}_2$ satisfy 3.3 (1). Then (cf. the proof of 3.4) $u(p) \cap S^1(M_0, r_\lambda) = \emptyset$ for all $\langle M_0, r_\lambda\rangle$ sufficiently close to $\langle M_0, L_0\rangle$ by the preceding and the lemma follows.

4.4. From 4.1, there is an $r_1 \in F_1$ such that $\langle L_0, r_1\rangle \cap F = L_0 \cup \{r_1\}$. Thus there is an α_0 through L_0, sufficiently close to $\langle L_0, r_1\rangle$, such that $\alpha_0 \cap F = L_0$. Then α_0 and τ_1 or τ_2 decompose $\bar{F}_2$ into two open disjoint regions, say F_{21} and F_{22}, such that

$$\bar{F}_2 = \bar{F}_{21} \cup \bar{F}_{22}, \quad \bar{F}_{21} \cap \bar{F}_{22} = M_0 \cup L_0 \quad \text{and}$$
$$F_1 \cup F_{21} \cup F_{22} = \{r \in F | \, l(r) = 0\}.$$

Let $r \in F_{2i}$. Then $\alpha_0 \cap \bar{F}_2 = L_0$ implies that $S^1(M_0, r) \subset \bar{F}_{2i}$ and 4.3 implies that $S^1(M_0, r)$ is the boundary of an open region $F_{2i}(M_0, r) \subset F_{2i}$ such that $\mathrm{Cl}(F_{2i}(M_0, r))$ tends to ν as $S^1(M_0, r)$ tends to ν. Clearly, $F_{2i}(M_0, r)$ satisfies 2.6 and thus $F_{2i} \cap E \neq \emptyset$ with $\nu \in \mathrm{Cl}(F_{2i} \cap E)$; $i = 1, 2$.

The surface in P^3 defined by $x_1^3 + x_2^3 + x_0^2(x_0 + x_2) + x_1x_2x_3 = 0$ satisfies 4.0 with $\nu \equiv (0, 0, 0, 1)$, $M_0 \equiv x_1 = x_2 = 0$ and $L_0 \equiv x_1 + x_2 = x_3 = 0$.

THEOREM 4.5. *Let F be biplanar with the binode ν and the lines $M_0 = \tau_1 \cap \tau_2$ and L_0; $l(F) = l(\nu) + 1 = 2$. Then $F = \bar{F}_1 \cup \bar{F}_{21} \cup \bar{F}_{22}$ where $\bar{F}_1 = F_1 \cup \{\nu\}$, $\bar{F}_{2i} = F_{2i} \cup L_0 \cup M_0$, every point of F_1 is elliptic and F_{21} and F_{22} are described in* 4.4.

5. F with two lines; $l(\nu) = 2$.

5.0. Let F be biplanar with the binode ν; $l(F) = l(\nu) = 2$. Let $M_0 = \tau_1 \cap \tau_2$ and M_1 be the lines of F. Since $M_0 \cap M_1 = \{\nu\}$, we assume that $\tau_2 \cap F = M_0 \cup M_1$. Then $\tau_1 \cap F = M_0$ and $M_1 \subset \pi(M_0)$.

Let $r \in F$, $l(r) = 0$. Then $M_0 \cap S^1(M_0, r) = \{\nu\}$ and $|M_1 \cap S^1(M_1, r)| = 2$ by 2.1. If $M_1 \cap S^1(M_1, r) = \{\nu, q\}$, then $\pi(q) = \langle M_1, r\rangle$. Clearly, $\pi(q) \neq \pi(q')$ for $q \neq q'$ in $M_1\backslash\{\nu\}$.

LEMMA 5.1. *Let r_λ be a convergent sequence in $F\backslash(M_0 \cup M_1)$.*
1. If $\lim \langle M_0, r_\lambda\rangle = \tau_2[\tau_1]$, *then* $\lim S^1(M_0, r_\lambda) = M_0 \cup M_1[\nu]$.
2. If $\lim \langle M_1, r_\lambda\rangle = \tau_2$, *then* $\lim S^1(M_1, r_\lambda)$ *is either M_0 or ν.*

Proof. Since $M_1 \subset \pi(M_0)$, *1* follows as in the proof of 4.3. If $\lim \langle M_1, r_\lambda\rangle = \tau_2$, then $\lim (M_1 \cup S^1(M_1, r_\lambda)) = M_1 \cup \lim S^1(M_1, r_\lambda)$ is either $M_0 \cup M_1$ or M_1 by 1.3.5. Since $\tau_2 \neq \pi(q)$ for $q \in M_1\backslash\{\nu\}$, we obtain that

$$\lim(M_1 \cap S^1(M_1, r_\lambda)) = \{\nu\} \quad \text{and} \quad M_1 \not\subset \lim S^1(M_1, r_\lambda).$$

Thus $M_1 \cap \lim S^1(M_1, r_\lambda) = \{\nu\}$ and *2* follows.

5.2. Let $\beta \cap (M_0 \cup M_1) = \{\nu\}$. Then ν is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$ and (*cf*. 3.3) $S^1(M_0, r)$ is compatible (incompatible) with $\mathscr{F}_i[\mathscr{F}_j]$ for all $r \in \mathscr{F}_i[\mathscr{F}_j]$; $r \neq \nu$, $\{i, j\} = \{1, 2\}$. Similarly, $S^1(M_0, r)$ and $\mathscr{L}$ are either compatible for all $r \in L\backslash\{\nu\}$ or incompatible for all $r \in L\backslash\{\nu\}$.

Then (cf. the proof of 3.4) 5.1.1 implies that $S^1(M_0, r)$ and $\mathscr{F}_1[\mathscr{F}_2]$ are compatible [incompatible] for all $r \in F_1[F_2]$, $r \neq \nu$, and $S^1(M_0, r)$ and $\mathscr{L}$ are compatible for all $r \in \mathscr{L}\backslash\{\nu\}$.

5.3. Let $\mathscr{P}_0$ and $\mathscr{P}_1$ be the open half-spaces of P^3 determined by τ_1 and τ_2. Let $F_i = P_i \cap F$, $i = 1, 2$. Then $F_1 \cup F_2 = \{r \in F|\, l(r) = 0\}$.

Since $r \in F_i$ implies that $S^1(M_0, r) \subset \bar{F}_i$, $\mathrm{bd}(F_i) = M_0 \cup M_1$ by 5.1; $i = 1, 2$.

Let $\beta \cap (M_0 \cup M_1) = \{\nu\}$. Then $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$ and $\beta \cap \bar{F}_i$ is $\mathscr{L}$ or $\mathscr{F}_1 \cup \mathscr{F}_2$. In either case, there is $r_\lambda \in \beta \cap F_i$ tending to ν such that

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

$\lim \langle M_0, r_\lambda \rangle = \tau_1$ and $\lim S^1(M_0, r_\lambda) = \{\nu\}$. Thus $S^1(M_0, r_\lambda)$ is the boundary of an open region $F_i(M_0, r_\lambda) \subset F_i$ such that $\mathrm{Cl}(F_i(M_0, r_\lambda))$ tends to ν as $S^1(M_0, r_\lambda)$ tends to ν. Clearly, $F(M_0, r_\lambda)$ satisfies 2.6 for each r_λ and $\nu \in \mathrm{Cl}(F_i \cap E)$; $i = 1, 2$.

The surface in P^3 defined by $x_2^3 + x_0^2x_2 + x_0x_1^2 + x_1x_2x_3 = 0$ satisfies 5.0 with $M_0 \equiv x_1 = x_2 = 0$ and $M_1 \equiv x_0 = x_2 = 0$. We observe in Figure 3 that F has a 'fold' in the neighbourhood of M_1 due to $|M_1 \cap S^1(M_1, r)| = 2$ for $r \in F\backslash(M_0 \cup M_1)$ and the existence of loops (of $\beta \cap F$ where $l(\nu, \beta) = 0$) in $\bar{F}_1$ and $\bar{F}_2$. Clearly, both F_1 and F_2 contain hyperbolic and parabolic points.

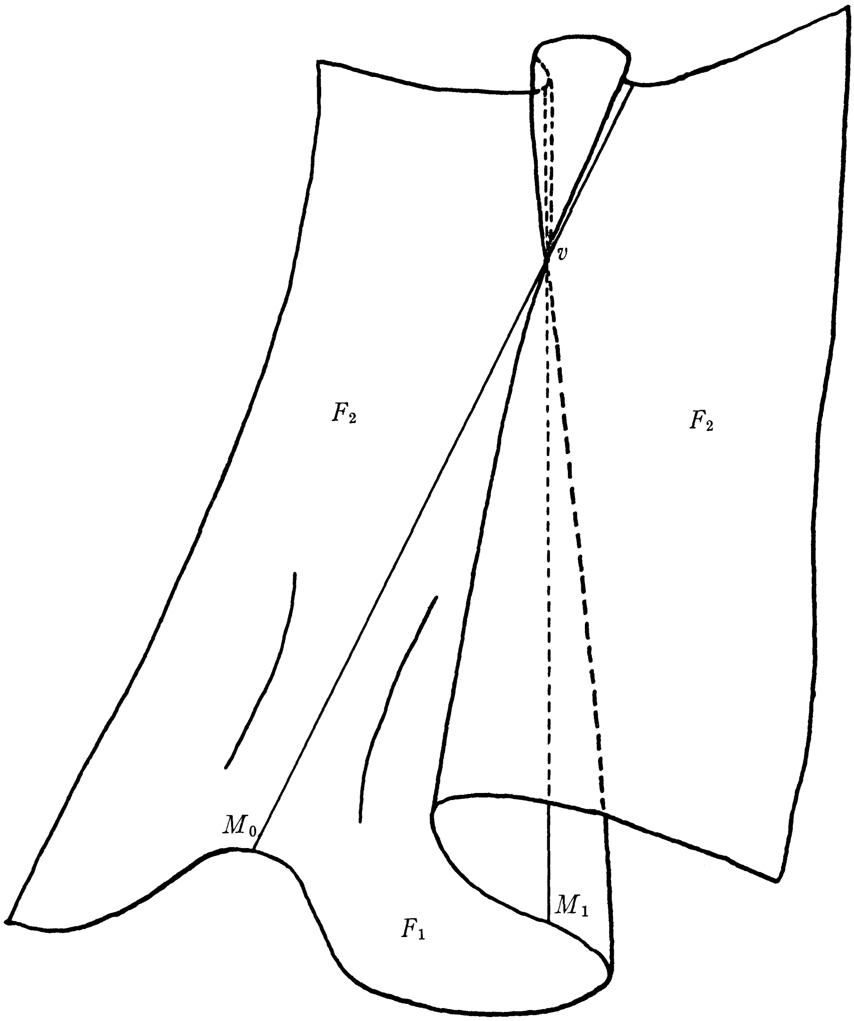

FIGURE 3

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

THEOREM 5.4. *Let F be biplanar with the binode ν and the lines M_1 and $M_0 = \tau_0 \cap \tau_2$; $l(F) = l(\nu) = 2$. Then $F = \bar{F}_1 \cup \bar{F}_2$ where $\bar{F}_1 \cap \bar{F}_2 = M_0 \cup M_1$ and $\nu \in \mathrm{Cl}(F_i \cap E)$; $i = 1, 2$.*

6. F with three lines.

6.0. Let F be biplanar with the binode ν; $l(F) = l(\nu) = 3$. Let $M_0 = \tau_1 \cap \tau_2$, M_1 and M_2 be the lines of F. 2.2.3, we may assume that $\tau_1 \cap F = M_0 \cup M_1 \cup M_2$ and $\tau_2 \cap F = M_0$. Then $\tau_2 = \pi(p)$ for all $p \in M_0\backslash\{\nu\}$ by 2.1.1. Let $r \in F$, $l(r) = 0$. Then $M_0 \cap S^1(M_0, r) = \{\nu\}$ and $|M_i \cap S^1(M_i, r)| = 2$; $i = 1, 2$.

LEMMA 6.1. *Let r_λ be a convergent sequence in $F\backslash\tau_1$.*
1. If $\lim \langle M_i, r_\lambda\rangle = \tau_1$, then $\lim S^1(M_i, r_\lambda) = M_j \cup M_k$; $\{i, j, k\} = \{1, 2, 3\}$.
2. If $\lim \langle M_0, r_\lambda\rangle = \tau_2$, then $\lim S^1(M_0, r_\lambda)$ is either M_0 or ν.

Proof. cf. 1.3.4 and the proof of 3.4.

6.2. Let ν be the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. As in the previous sections; $S^1(M_0, r)$ and $\mathscr{L}$ are either compatible for all $r \in \mathscr{L}\backslash\{\nu\}$ or incompatible for all $r \in L\backslash\{\nu\}$ and $S^1(M_0, r)$ and $\mathscr{F}_i[\mathscr{F}_j]$ are compatible [incompatible] for all $r \in \mathscr{F}_i[\mathscr{F}_j]$; $r \neq \nu$, $\{i, j\} = \{1, 2\}$.

Let $\mathscr{H}_0$ and $\mathscr{H}_1$ be the closed half-planes of τ_1 determined by M_1 and M_2. We assume that $M_0 \subset \mathscr{H}_0$. If r_λ is a sequence in $F\backslash\tau_1$ such that $\lim \langle M_0, r_\lambda\rangle = \tau_1$, then 6.1.1 and 2.1.1 imply that

$$\lim \mathrm{Cl}(\mathrm{int}\, S^1(M_0, r_\lambda)) = H_1.$$

LEMMA 6.3. *Let $\nu \in \beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$, $l(\nu, \beta) = 0$. If $\beta \cap \tau_1 \subset \mathscr{H}_0[\mathscr{H}_1]$, then $S^1(M_0, r)$ and $\mathscr{L}$ are compatible [incompatible] for all $r \in L\backslash\{\nu\}$, $S^1(M_0, r)$ and $\mathscr{F}_1$ are compatible [incompatible] for all $r \in F_1\backslash\{\nu\}$.*

Proof. There are $r_\lambda \neq \nu$ in $\mathscr{L}[\mathscr{F}_1]$ tending to ν such that $\beta \cap \tau_1 = \lim \langle \nu_1 r_\lambda\rangle$. Clearly,

$$\beta \cap \tau_1 \subset \mathscr{H}_1 = \lim \mathrm{Cl}(\mathrm{int}\, S^1(M_0, r_\lambda))$$

if and only if $S^1(M_0, r_\lambda)$ and $\mathscr{L}[\mathscr{F}_1]$ are incompatible for all r_λ close to ν. Now apply 6.2.

6.4. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the open half-spaces determined by τ_1 and τ_2. Let $F_i = \mathscr{P}_i \cap F$ and fix a β^* such that $\nu \in \beta^* \cap F = \mathscr{L}^* \cup \mathscr{F}_1^* \cup \mathscr{F}_2^*$, $l(\nu, \beta^*) = 0$ and $\beta^* \cap \tau_1 \subset \mathscr{H}_1$. We assume that $\beta^* \cap \bar{F}_1 = \mathscr{L}^*$ and $\beta^* \cap \bar{F}_2 = \mathscr{F}_1^* \cup \mathscr{F}_2^*$.

LEMMA 6.5. $\mathrm{bd}(F_1) = M_0 \cup M_1 \cup M_2$ *and* $\mathrm{bd}(F_2) = M_1 \cup M_2$.

Proof. Clearly, $M_1 \cup M_2 \subset \bar{F}_1 \cap \bar{F}_2$ by 6.1.1.

Let $r \in F_1$. Then $S^1(M_0, r)$ meets $L^*\backslash\{\nu\}$ at an r^*, $S^1(M_0, r) = S^1(M_0, r^*)$ and $S^1(M_0, r^*)$ and $\mathscr{L}^*$ are incompatible. If $r^* \in L^*\backslash\{\nu\}$ tends to ν such that

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

$\lim \langle M_0, r^* \rangle = \tau_2$, then (cf. the proof of 3.4) $\lim S^1(M_0, r^*) = M_0$ by 6.1.2 and 6.3. Thus $\mathrm{bd}(F_1) = M_0 \cup M_1 \cup M_2$.

By a similar argument, $r^* \in \mathrm{int}(F_2^*) \subset F_2$ tending to ν implies that $\lim S^1(M_0, r^*) = \{\nu\}$ and thus $\mathrm{bd}(F_2) = M_1 \cup M_2$.

THEOREM 6.6. *Every point of F_1 is hyperbolic.*

Proof. Let $r' \in F_1$. Then $S^1(M_0, r') = S^1(M_0, r^*)$ for some $r^* \in \mathscr{L}^*$.

Put $\beta^* \cap \mathscr{H}_1 = N^*$. Clearly, ν is the double point of $\langle N^*, r \rangle \cap F = L_r \cup F_{1,r} \cup F_{2,r}$ for each $r \in S^1(M_0, r')$, $r \neq \nu$. If $r \in S^1(M_0, r')$ tends to $\bar{r} \neq \nu$, then $\lim \mathscr{L}_{\bar{r}} = \mathscr{L}_r$ and $\lim \mathscr{F}_{1,r} \cup F_{2,\bar{r}} = F_{1,r} \cup F_{2,\bar{r}}$; cf. the proof of 3.1. Thus

$$\tilde{S} = \{r \in S^1(M_0, r') \mid r \neq \nu \quad \text{and} \quad r \in L_r\}$$

is open and closed in $S^1(M_0, r') \backslash \{\nu\}$, a connected set. Since $r^* \in \tilde{S}$, $\tilde{S} = S^1(M_0, r') \backslash \{\nu\}$ and $r' \in L_{r'} \subset \langle N^*, r' \rangle \cap F$ where $N^* \subset \mathscr{H}_1$. By 6.3 and 1.3.2, r' is hyperbolic.

6.7. Let r^* tend to ν in $\mathrm{int}(\mathscr{F}_2^*)$. Then $\lim S^1(M_0, r^*) = \{\nu\}$ and $S^1(M_0, r^*)$ is the boundary of an open region $F_2(M_0, r^*) \subset F_2$ such that $\mathrm{Cl}(F_2(M_0, r^*))$ tends to ν. Clearly, $F_2(M_0, r^*)$ satisfies 2.6 for each r^* and $\nu \in \mathrm{Cl}(F_2 \cap E)$. By 6.3 and 1.3.2, F_2 also contains hyperbolic points.

The surface in P^3 defined by $x_1^3 - x_2^3 + x_0^2x_2 + x_1x_2x_3 = 0$ satisfies 6.0 with $M_0 \equiv x_1 = x_2 = 0$, $M_1 \equiv x_1 = x_0 + x_2 = 0$ and $M_2 \equiv x_1 = x_0 - x_2 = 0$. In Figure 4, we observe that the loops of $\beta \cap F(l(\nu, \beta) = 0$ and $B \cap \tau_1 \subset \mathscr{H}_1)$ form the boundary of a hole in F.

THEOREM 6.8. *Let F be biplanar with the binode ν and the lines M_1, M_2 and $M_0 = \tau_1 \cap \tau_2$; $l(F) = l(\nu) = 3$. Then $F = \bar{F}_1 \cup \bar{F}_2$ where* $\mathrm{bd}(F) = M_0 \cup M_1 \cup M_2$, $\mathrm{bd}(F_2) = M_1 \cup M_2$, *every point of F_1 is hyperbolic and* $\nu \in \mathrm{Cl}(F_2 \cap E)$.

7. *F* with four lines.

7.0. Let F be biplanar with the binode ν; $l(F) = l(\nu) + 1 = 4$. Let $M_0 = \tau_1 \cap \tau_2$, M_1 and M_2 be the lines of F through ν and let $L_0 \subset F$ with $\nu \notin L_0$. By 2.2.3, we may assume that $\tau_1 \cap F = M_0 \cup M_1 \cup M_2$ and $\tau_2 \cap F = M_0$. Then L_0 meets M_0 at $p_0 \neq \nu$, $L_0 \cap (M_1 \cup M_2) = \emptyset$, $\langle L_0, M_0 \rangle \cap F = L_0 \cup M_0$ and $L_0 \subset \pi(M_0)$.

Let $r \in F$, $l(r) = 0$. Then $M_0 \cap S^1(M_0, r) = \{\nu\}$, $|L_0 \cap S^1(L_0, r)| \leqq 2$ and $|M_i \cap S^1(M_i, r)| = 2$; $i = 1, 2$. Let r_λ be a convergent sequence in $F \backslash (\tau_1 \cup L_0)$.

1. If $\lim \langle M_i, r_\lambda \rangle = \tau_1$, then $\lim S^1(M_i, r_\lambda) = M_j \cup M_k$; $\{i, j, k\} = \{0, 1, 2\}$.
2. If $\lim \langle M_0, r_\lambda \rangle = \langle M_0, L_0 \rangle\, [\tau_2]$, then $\lim S^1(M_0, r_\lambda) = M_0 \cup L_0[\nu]$.
3. If $\lim \langle L_0, r_\lambda \rangle = \langle M_0, L_0 \rangle$, then $\lim S^1(L_0, r_\lambda)$ is either M_0 or ν.

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

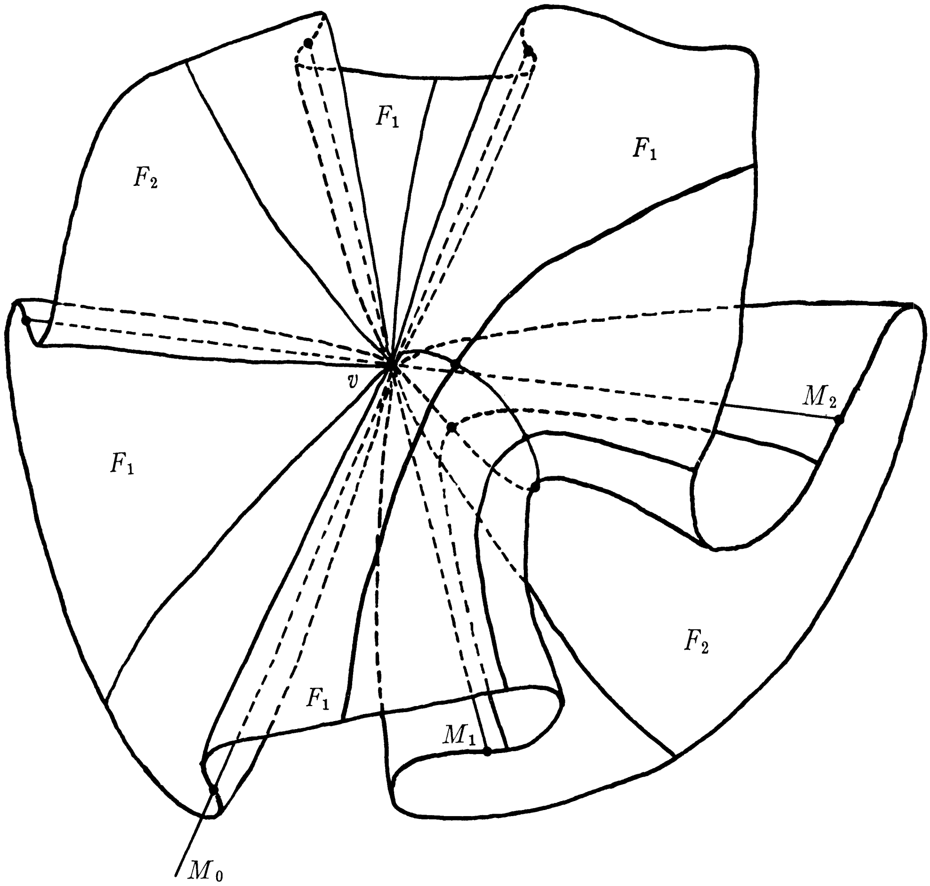

FIGURE 4

7.1. Let ν be the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. Let $r \in \mathscr{L}\backslash\{\nu\}$ tend to ν such that $\lim \langle M_0, r\rangle = \tau_2$. Then $\lim S^1(M_0, r) = \{\nu\}$ and (cf. the proof of 3.4) $S^1(M_0, r)$ and $\mathscr{L}$ are compatible for r sufficiently close to ν. Similarly, $S^1(M_0, r)$ and $\mathscr{F}_2$ are compatible for r sufficiently close to ν in $F_2\backslash\{\nu\}$.

By 1.3.3, $L_0 \cap \mathscr{L} = \emptyset$ implies that $S^1(M_0, r)$ and $\mathscr{L}$ are compatible for all $r \in L\backslash\{\nu\}$ and $L_0 \cap (\mathscr{F}_1 \cup \mathscr{F}_2) = \emptyset$ implies that $S^1(M_0, r)$ and $\mathscr{F}_2[\mathscr{F}_1]$ are compatible [incompatible] for all $r \in \mathscr{F}_2[\mathscr{F}_1]$, $r \neq \nu$.

Let $\mathscr{H}_0$ and $\mathscr{H}_1$ be the closed half-planes of τ_1 determined by M_1 and M_2, $M_0 \subset \mathscr{H}_0$.

LEMMA 7.2. *Let* $\nu \in \beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$, $l(\nu, \beta) = 0$. *Then* $L_0 \cap \mathscr{L} \neq \emptyset$ *if and only if* $\beta \cap \tau_1 \subset \mathscr{H}_1$.

Proof. If r_λ is a sequence in $F\backslash\tau_1$ such that $\lim \langle M_0, r_\lambda\rangle = \tau_1$, then $\lim \mathrm{Cl}(\mathrm{int}\, S^1(M_0, r_\lambda)) = \mathscr{H}_1$. Now apply 7.1 and compare the proof of 6.3.

7.3. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the open half-spaces determined by τ_1 and τ_2, $L_0 \subset \bar{\mathscr{P}}_1$. Then $\langle M_0, L_0\rangle$ is the common boundary of two open quarter-spaces of $\bar{P}_1$, say P_{11} and P_{12}. We assume that $\tau_1 \subset \bar{\mathscr{P}}_{1i}$; $i = 1, 2$.

Let $F_{1i} = \mathscr{P}_{1i} \cap F$ and $F_2 = \mathscr{P}_2 \cap F$. Then

$$F_{11} \cup F_{12} \cup F_2 = \{r \in F | \, l(r) = 0\}.$$

Fix a β^* such that $\nu \in \beta^* \cap F = \mathscr{L}^* \cup \mathscr{F}_1^* \cup \mathscr{F}_2^*$, $l(\nu, \beta^*) = 0$ and $\beta^* \cap \tau_1 \subset \mathscr{H}_1$. Then L_0 meets $\mathscr{L}^*$ at a point $l_0 \neq \nu$, $\beta^* \cap (\bar{F}_{11} \cup \bar{F}_{12}) = \mathscr{L}^*$ and $\beta \cap \bar{F}_2 = \mathscr{F}_1^* \cup \mathscr{F}_2^*$. Let $B \cap \bar{F}_{1i} = \mathscr{L}_i^*$. Then $\mathscr{L}_1^*$ and $\mathscr{L}_2^*$ are the subarcs of $\mathscr{L}^*$, with the end points ν and l_0 and $\mathscr{L}_1^* \cap \mathscr{L}_2^* = \{\nu, l_0\}$. If r^* tends to ν in $L_i^*\backslash\{\nu\}$, then $\lim \langle M_0, r^*\rangle = \tau_i$; $i = 1, 2$. From 7.1 and the proof of 6.3, $S^1(M_0, r^*)$ and $\mathscr{L}_1^*[\mathscr{L}_2^*]$ are incompatible [compatible] for all $r \in \operatorname{int}(\mathscr{L}_1^*)[\operatorname{int}(\mathscr{L}_2^*)]$.

LEMMA 7.4. $\operatorname{bd}(F_{11}) = L_0 \cup M_0 \cup M_1 \cup M_2$, $\operatorname{bd}(F_{12}) = L_0 \cup M_0$ *and* $\operatorname{bd}(F_2) = M_1 \cup M_2$.

Proof. If $r \in F_{11}$, then $S^1(M_0, r) = S^1(M_0, r^*)$ for some $r^* \in \operatorname{int}(\mathscr{L}_1^*)$. If $r^* \in \operatorname{int}(\mathscr{L}_1^*)$ tends to $l_0[\nu]$, then $\lim S^1(M_0, r^*) = M_0 \cup L_0[M_1 \cup M_2]$ from 7.0. Thus $\operatorname{bd}(F_{11}) = L_0 \cup M_0 \cup M_1 \cup M_2$.

By similar arguments, we obtain the other two boundaries.

THEOREM 7.5. *Let F be a biplanar surface satisfying* 7.0. *Then*

$$F = \bar{F}_{11} \cup \bar{F}_{12} \cup \bar{F}_2$$

where every point of F_{11} is hyperbolic, $\nu \in \operatorname{Cl}(F_{12} \cap E)$ and $\nu \in \operatorname{Cl}(F_2 \cap E)$.

Proof. cf. 6.6 and 6.7.

We observe in Figure 5 that the loops of $\beta \cap F(l(\nu, \beta) = 0$ and $\beta \cap \tau_1 \subset \mathscr{H}_1)$ again form the boundary of a hole. The surface in P^3 defined by $x_1^3 + x_2^3 + x_0^2(x_1 - x_2) + x_1x_2x_3 = 0$ satisfy 7.0 with $M_0 \equiv x_1 = x_2 = 0$, $M_1 \equiv x_1 = x_0 + x_2 = 0$, $M_2 \equiv x_1 = x_0 - x_2 = 0$ and $L_0 \equiv x_1 - x_2 = x_3 + 2x_1 = 0$.

8. F with six lines.

8.0. Let F be biplanar with the binode ν; $l(F) = l(\nu) + 2 = 6$. Let M_i, $0 \leqq i \leqq 3$, be the lines of F through ν, $M_0 = \tau_1 \cap \tau_2$. We assume that $\tau_1 \cap F = M_0 \cup M_1 \cup M_2$. Then $\tau_2 \cap F = M_0 \cup M_3$ and $M_3 \subset \pi(M_0)$.

By 2.1.4, $\langle M_3, M_j\rangle \cap F$ contains a third line L_j, $j = 1, 2$. Clearly, $L_1 \cap L_2 = \emptyset$ and $\nu \notin L_1 \cup L_2$. Let $L_j \cap M_j$ be the point q_{jj} and $L_j \cap M_3$ be the point q_{j3}.

Let $r \in F$, $l(r) = 0$. Then $M_0 \cap S^1(M_0, r) = \{\nu\}$, $|M_i \cap S^1(M_i, r)| = 2$ for $i = 1, 2, 3$ and $|L_j \cap S^1(L_j, r)| \leqq 2$ for $j = 1, 2$. Let r_λ be a convergent sequence in F, $l(r_\lambda) = 0$ for each r_λ. Since $M_3 \subset \pi(M_0)$, $\lim \langle M_0, r_\lambda\rangle = \tau_2$ implies that $\lim S^1(M_0, r_\lambda) = M_0 \cup M_3$ and $\lim \langle M_3, r_\lambda\rangle = \tau_2$ implies that

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

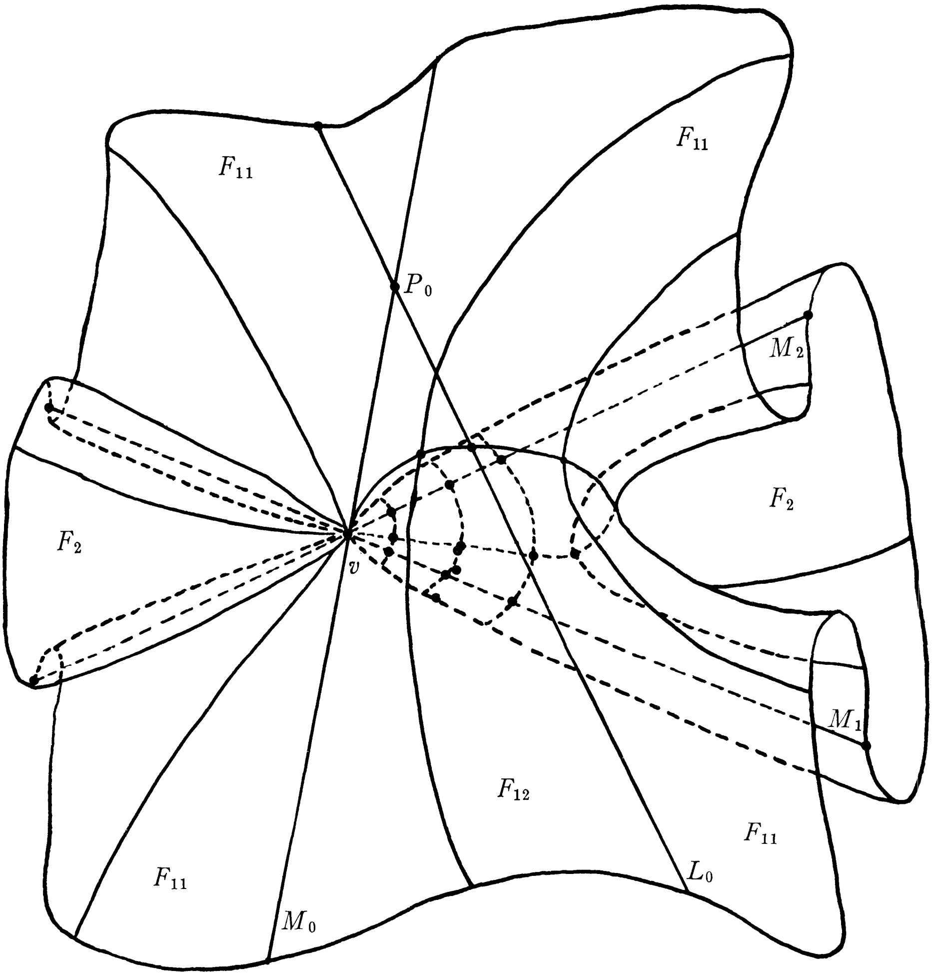

FIGURE 5

lim $S^1(M_3, r_\lambda)$ is either M_0 or ν. The limits of the plane sections through the other lines of F are immediate.

Let ν be the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. Since $M_0 \cap (L_1 \cup L_2) = \emptyset$, $S^1(M_0, r)$ exists for $r \in (\beta \cap F)\backslash\{\nu\}$. Let $\mathscr{H}_0$ and $\mathscr{H}_1$ be the closed half-planes of τ_1 determined by M_1 and M_2, $M_0 \subset \mathscr{H}_0$. We observe that 6.2 and 6.3 are true for this F. Arguing as in the proof of 6.3, we obtain

LEMMA 8.1. *Let* $\alpha = \langle M, \bar{r}\rangle$ *where* $\bar{r} \in F$, $l(\bar{r}) = 0$. *If* $\alpha \cap \tau_1 \subset \mathscr{H}_0[\mathscr{H}_1]$, *then* $S^1(M_0, r)$ *and* $S^1(M_3, \bar{r})$ *are compatible* [*incompatible*] *for all* $r \in S^1(M_3, \bar{r})\backslash M_3$.

8.2. Let $\mathscr{P}_0$ and $\mathscr{P}_1$ be the open half-spaces determined by $\langle M_2, M_1\rangle$ and $\langle M_3, M_2\rangle$, $M_0 \subset \overline{\mathscr{P}}_0$. Then $\overline{\mathscr{P}}_j \cap \tau_1 = \mathscr{H}_j$ $(j = 0, 1)$ and τ_2 is the common boundary of two open quarter-spaces of $\overline{\mathscr{P}}_0$, say $\mathscr{P}_{01}$ and $\mathscr{P}_{02}$. We assume that $\langle M_i, L_i\rangle \subset \bar{P}_{0i}$, $i = 1, 2$.

Let r_λ be a convergent sequence such that $l(r_\lambda) = 0$ for each r_λ and $\lim \langle M_3, r_\lambda\rangle = \tau_2$. We assume that $\lim S^1(M_3, r_\lambda) = M_0[\nu]$ as $\langle M_3, r_\lambda\rangle$ tends to τ_2 in $\overline{\mathscr{P}}_{02}[\overline{\mathscr{P}}_{01}]$.

Let $F_1 = \mathscr{P}_1 \cap F$, $F_{01} = \mathscr{P}_{01} \cap F$ and $F_{02} = \mathscr{P}_{02} \cap F$. Clearly;

$$F_1 \cup F_{01} \cup F_{02} = \{r \in F \mid l(r) = 0\},$$

$M_1 \cup M_2 \cup L_1 \cup L_2 \subset \mathrm{bd}(F_1)$, $M_1 \cup L_1 \subset \mathrm{bd}(F_{01})$ and $M_0 \cup M_2 \cup L_2 \subset \mathrm{bd}(F_{02})$.

LEMMA 8.3 *Let ν be the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$, $\beta \cap \tau_1 \subset \mathscr{H}_1$. Then $\mathscr{L} \cap \mathscr{P}_{01} = \emptyset$ and $(\mathscr{F}_1 \cup \mathscr{F}_2) \cap \mathscr{P}_{02} = \emptyset$.*

Proof. Since $\beta \cap \tau_1 \subset \mathscr{H}_1$, $\mathscr{L} \cap \mathscr{P}_1 \neq \emptyset$ and either $\mathscr{L} \cap \mathscr{P}_{01} = \emptyset$ or $\mathscr{L} \cap \mathscr{P}_{02} = \emptyset$. By 6.3, $S^1(M_0, r)$ and $\mathscr{L}$ are incompatible for all $r \in L\backslash\{\nu\}$.

Let $r_\lambda \in \mathscr{L}\backslash\overline{\mathscr{P}}_1$ tend to ν. For each r_λ, $\langle M_3, r_\lambda\rangle \subset \overline{\mathscr{P}}_0$, $\langle M_3, r_\lambda\rangle \cap \tau_1 \subset \mathscr{H}_0$ and thus $S^1(M_3, r_\lambda)$ and $\mathscr{L}$ are incompatible by 8.1 and the preceding. Then $S^1(M_3, r_\lambda)$ converges to M_0 (cf. 8.0 and 3.4) and $\mathscr{L} \subset \mathscr{P}_1 \cup \mathscr{P}_{02}$ from 8.2. Thus, $\mathscr{L} \cap \mathscr{P}_{01} = \emptyset$ and $(\mathscr{F}_1 \cup \mathscr{F}_2) \cap \mathscr{P}_{02} = \emptyset$.

THEOREM 8.4. *Every point of $F_1 \cup F_{02}$ is hyperbolic.*

Proof. Let $r \in F_1$. Then $\langle M_3, r\rangle \subset \overline{\mathscr{P}}_1$ and $\langle M_3, r\rangle \cap \tau_1 \subset \mathscr{H}_1$. By 8.1 and 1.3.2, r is hyperbolic.

Let $r \in F_{02}$ and choose $N \subset \mathscr{H}_1$ such that $N \cap F = \{\nu\}$. Then ν is the double point of $\langle N, r\rangle \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. By 8.3, 6.3 and 1.3.2, r is hyperbolic.

8.5. The points ν, q_{13} and q_{23} are mutually distinct. Let $M_3{}^*$ be the closed segment of M_3, with the end points ν and q_{13}, such that $q_{23} \notin M_3{}^*$. Let Δ_0 and Δ_1 be the triangles determined by M_1, L_1 and $M_3{}^*$. Then ν, q_{11} and q_{13} are the vertices of Δ_0 and Δ_1.

Let $r \in F_{01}$ and put $M_3 \cap S^1(M_3, r) = \{\nu, q_r\}$. As $S^1(M_3, r)$ tends to $M_1 \cup L_1[\nu]$, $q_r \neq \nu$ tends to $q_{13}[\nu]$ and thus $\bar{F}_{01} \cap M_3$ is either $M_3{}^*$ or $\mathrm{Cl}(M_3\backslash M_3{}^*)$. Since $\langle M_2, L_2\rangle \not\subset \bar{F}_{01}$, $q_{23} \notin \bar{F}_{01} \cap M_3$ and in particular

$$\mathrm{bd}(F_{01}) = L_1 \cup M_1 \cup M_3{}^* = \Delta_0 \cup \Delta_1.$$

As $|M_3 \cap S^1(M_3, r)| = 2$, this implies $S^1(M_3, r) \cap F_{01}$ is the union of two open disjoint sets. It is immediate that

$$F_{01} = G_0 \cup G_1$$

where G_0 and G_1 are open disjoint regions such that $S^1(M_3, r) \cap G_0$ and $S^1(M_3, r) \cap G_1$ are the maximal connected subsets of $S^1(M_3, r) \cap F_{01}$. Obviously, $\mathrm{bd}(G_0)$ is either Δ_0 or Δ_1. We assume that $\mathrm{bd}(G_0) = \Delta_0$, then $\mathrm{bd}(G_1) = \Delta_1$ and $\bar{G}_0 \cap \bar{G}_1 = M_3{}^* \cup \{q_{11}\}$.

From 8.2, there is a sequence r_λ in F_{01} such that $\lim S^1(M_3, r_\lambda) \cap \bar{G}_j = \{\nu\}$; $j = 0, 1$. Clearly, $S^1(M_3\ r_\lambda) \cap \bar{G}_j$ is the boundary of an open region G_j $(M_3,$

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

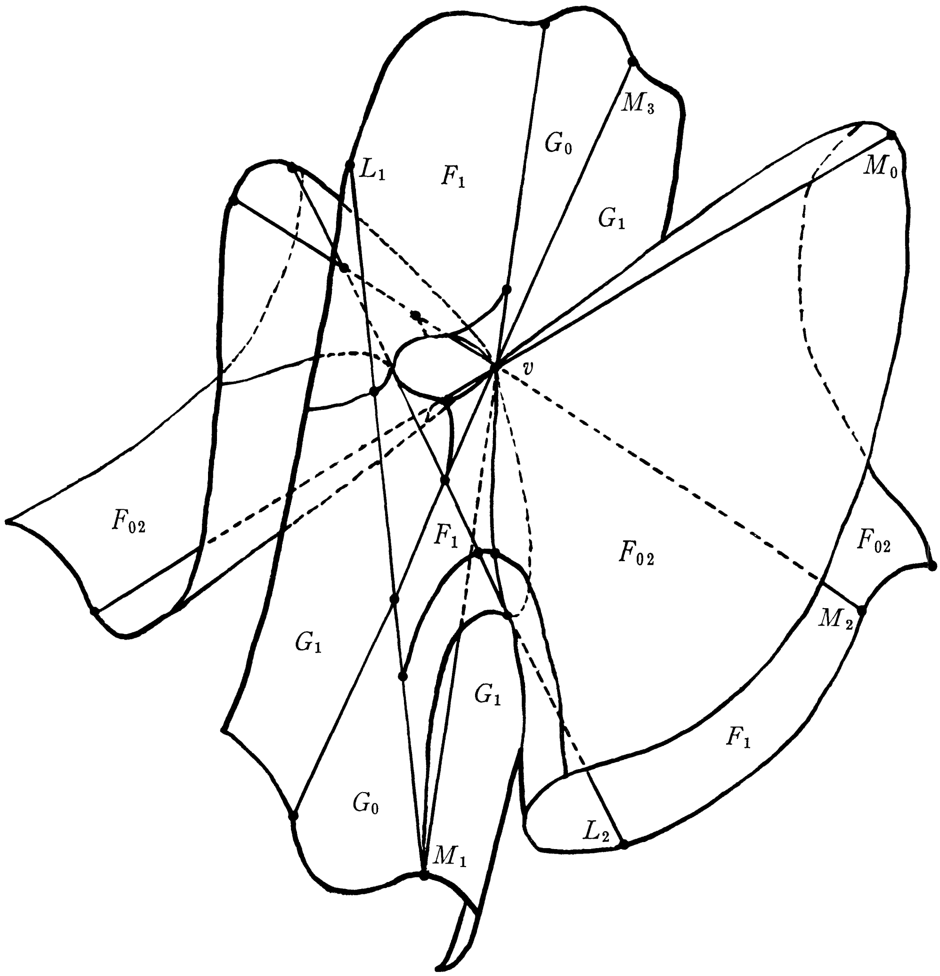

FIGURE 6

$r_\lambda) \subset G_j$ such that $\mathrm{Cl}(G_j(M_3, r_\lambda))$ tends to ν and $G_j(M_3, r_\lambda)$ satisfies 2.6 for each r_λ. Thus $\nu \in \mathrm{Cl}(G_j \cap E)$; $j = 0, 1$.

We observe in Figure 6 that there are two holes in this surface. Also, there is an $r' \in F_1$ such that $L_1 \cap S^1(L_1, r') = \emptyset$ and for any $r \in F$ with $l(r) = 0$, $|L_2 \cap S^1(L_2, r)| = 2$ with $q_{22} \in \mathrm{int}\, S^1(L_2, r)$.

The surface in P^3 defined by $x_1x_2x_3 + x_0^2x_2 + x_0x_1^2 - x_2^3 = 0$ satisfies 8.0 with $M_0 \equiv x_1 = x_2 = 0$, $M_1 \equiv x_1 = x_0 + x_2 = 0$, $M_2 \equiv x_1 = x_0 - x_2 = 0$, $M_3 \equiv x_0 = x_2 = 0$, $L_1 \equiv x_0 + x_2 = x_3 - x_1 = 0$ and $L_2 \equiv x_0 - x_2 = x_1 + x_3 = 0$.

THEOREM 8.6. *Let F be a biplanar surface satisfying* 8.0. *Then*

$$F = \bar{F}_1 \cup \bar{F}_{02} \cup \bar{G}_0 \cup \bar{G}_1$$

where every point of $F_1 \cup F_{02}$ is hyperbolic and $\nu \in \mathrm{Cl}(G_j \cap E)$; $j = 0, 1$.

9. F with ten lines.

9.0. Let F be biplanar with the binode ν; $l(F) = l(\nu) + 5 = 10$. Let M_i, $0 \leqq i \leqq 4$, be the lines of F with $M_0 = \tau_1 \cap \tau_2$, $\tau_1 \cap F = M_0 \cup M_1 \cup M_2$ and $\tau_2 \cap F = M_0 \cup M_3 \cup M_4$. By 2.1.1, there is an $L_0 \subset F$ such that $L_0 \cap M_0$ is a point $p_0 \neq \nu$, $\langle M_0, L_0 \rangle \cap F = M_0 \cup L_0$ and $L_0 \subset \pi(M_0)$.

By 2.1.4, $\langle M_i, M_j \rangle \cap F$ contains a third line L_{ij}; $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Clearly $M_0 \cap L_{ij} = \emptyset$, $L_0 \cap L_{ij}$ is a point l_{ij} and $L_{ij} \cap L_{kl} = \emptyset$ when $\{i, j\} \cap \{k, l\} \neq \emptyset$. Since $L_0 \subset \pi(M_0)$ and $L_{24} \cap M_2 \neq \emptyset$, 1.3.6 implies that $l(\langle L_0, L_{24} \rangle) = 3$ and $L_{24} \cap (L_{14} \cap L_{23}) = \emptyset$ implies that $L_{13} \subset \langle L_0, L_{24} \rangle$. Similarly, $L_{14} \subset \langle L_0, L_{23} \rangle$.

Let $r \in F$, $l(r) = 0$. Then $M_0 \cap S^1(M_0, r) = \{\nu\}$, $|M_k \cap S^1(M_k, r)| = 2$ for $k = 1, 2, 3, 4$, $|L_0 \cap S^1(L_0, r)| \leqq 2$ and $|L_{ij} \cap S^1(L_{ij}, r)| \leqq 2$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Let r_λ be a convergent sequence in F, $l(r_\lambda) = 0$ for each r_λ.

1. If $\lim \langle M_0, r_\lambda \rangle = \tau_1[\tau_2]$, then $\lim S^1(M_0, r_\lambda) = M_1 \cup M_2[M_3 \cup M_4]$.
2. If $\lim \langle M_0, r_\lambda \rangle = \langle M_0, L_0 \rangle$, then $\lim S^1(M_0, r_\lambda) = M_0 \cup L_0$.
3. If $\lim \langle L_0, r_\lambda \rangle = \langle M_0, L_0 \rangle$, then $\lim S^1(L_0, r_\lambda)$ is either M_0 or ν.

9.1. Let ν be the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. Let $\mathscr{H}_{10}$ and $\mathscr{H}_{11}[\mathscr{H}_{20}$ and $\mathscr{H}_{21}]$ be the closed half-planes of $\tau_1[\tau_2]$ determined by M_1, $M_2[M_3, M_4]$. We assume that $M_0 = \mathscr{H}_{10} \cap \mathscr{H}_{20}$. As in sections 6 and 7, we obtain the following:

1. If $\beta \cap \tau_1 \subset \mathscr{H}_{i1}[\mathscr{H}_{i0}]$ for $i = 1, 2$ then $L_0 \cap \mathscr{L} = \emptyset$, $S^1(M_0, r)$ and $\mathscr{L}$ are incompatible [compatible] for all $r \in L\backslash\{\nu\}$ and $S^1(M_0, r)$ and $F_k (k = 1, 2)$ are incompatible [compatible] for r sufficiently close to ν in $F_k\backslash\{\nu\}$.

2. Let $\{i, j\} = \{1, 2\}$. If $\beta \cap \tau_i \subset H_{i0}$ and $\beta \cap \tau_j \subset H_{ji}$, then L_0 meets $\mathscr{L}$ at a point l. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be the subarcs of $\mathscr{L}$ such that

$$\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2, \quad \mathscr{L}_1 \cap \mathscr{L}_2 = \{\nu, l\}$$

and $\lim \langle M_0, r \rangle = \tau_1[\tau_2]$ as $r \neq \nu$ tends to ν in $\mathscr{L}_1[\mathscr{L}_2]$. Then $S^1(M_0, r)$ and $\mathscr{L}_i[\mathscr{L}_j]$ are compatible [incompatible] for all $r \in \mathrm{int}(L_i)[\mathrm{int}(L_j)]$.

9.2. Since $L_0 \cap M_1 = \emptyset$, 9.0.3 clearly implies that

$$\tilde{M}_1 = \{m \in M_1 | \; m \in \pi(l) \text{ for some } l \in L_0\}$$

is a proper closed segment of M_1 with the end points ν and (say) m_0. Since $\pi(l)$ depends continuously on $l \in L_0$, it is easy to check that $m \in \pi(l)$ for exactly two $l \in L_0$ for each $m \in \mathrm{int}(\tilde{M}_1)$. *Let* $l_0 \in L_0$ such that $\pi(l_0) = \langle L_0, m_0 \rangle$.

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

As $\langle L_0, L_{14}\rangle \neq \langle L_0, L_{24}\rangle$, we may assume that $\pi(l_0) \neq \langle L_0, L_{14}, L_{23}\rangle$. Then $\langle L_{14}, L_{23}\rangle \cap \operatorname{int}(\tilde{M}_1) \neq \emptyset$, $l_{14} \neq l_{23}$ and $p_0 \in \hat{L}_0$, the closed segment of L_0 bounded by l_{14} and l_{23} which contains l_0.

Without loss of generality, we may assume that $\{l_{24}, l_{13}\} \subset \hat{L}_0$. Then l_0 is contained in the closed segment $L_0{}^*$ of $\hat{L}_0$ bounded by l_{24} and l_{13}. If $L_0{}^* = \{l_0\}$, then $\pi(l_0) = \langle L_{24}, L_{13}\rangle$ and L_0, L_{24} and L_{13} are concurrent. If $L_0{}^* \neq \{l_0\}$, then there is an $r_0 \in F$ such that $l(r_0) = 0$ and $L_0 \cap S^1(L_0, r_0) = \{l_0\}$.

9.3. Let $\mathscr{P}_0$ and $\mathscr{P}_1$ be the closed half-spaces of P^3 determined by $\langle M_1, M_3\rangle$ and $\langle M_2, M_4\rangle$, $M_0 \subset \mathscr{P}_0$. Then $p_0 \in \mathscr{P}_1$ and 9.2 imply that

$$\mathscr{P}_1 \cap L_0 = L_0{}^*, \quad L_{23} \cap L_{14} \subset \mathscr{P}_1 \quad \text{and} \quad \mathscr{P}_i \cap \tau_j = \mathscr{H}_{ji}, \quad i \in \{0, 1\} \quad \text{and} \quad j \in \{1, 2\}.$$

If $L_0{}^* \neq \{l_0\}$, let G_1 be the closed triangular region of $\mathscr{P}_1 \cap F$, bounded by $L_0{}^*$ and segments of L_{13} and L_{24}, which does not contain ν. If $L_0{}^* = \{l_0\}$, let $G_1 = \{l_0\}$. We put $F_1 = \operatorname{Cl}((P_1 \cap F)\backslash G_1)$. Then

$$\mathscr{P}_1 \cap F = F_1 \cup G_1.$$

THEOREM 9.4. *1. If $r \in F_1$ such that $l(r) = 0$, then r is hyperbolic.*
2. If $G_1 \neq \{l_0\}$, then $G_1 \cap E \neq \emptyset$.

Proof. Let $\beta = \langle \nu, l^*, r_1\rangle$ where $L_{13} \cap L_{24} = \{l^*\}$ and $r_1 \in F_1$, $l(r_1) = 0$. Then ν is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$ and $\beta \cap \tau_i \subset \mathscr{H}_{i1}$, $i = 1, 2$. By 9.1.1, $S^1(M_0, r)$ and $\mathscr{L}$ are compatible for all $r \in \mathscr{L}\backslash\{\nu\}$.

If $L_0{}^* = \{l_0\}$, then $l^* = l_0$ and $\pi(l_0) \cap F = L_0 \cup L_{13} \cup L_{24}$ imply that l^* is the inflection point of $\beta \cap F$. If $L_0{}^* \neq \{l_0\}$, put $\beta \cap L_0{}^* = \{l'\}$. Then $\beta \cap (F_1 \cap G_1) = \{l^*, l'\}$, $l' \in \pi(l^*)$ and $\nu \notin G_1$ imply that the inflection point of $\beta \cap F$ is contained in G_1. In either case, 9.1.1 clearly implies that $S^1(M_0, r)$ and $\mathscr{F}_k$ are incompatible for all $r \in \operatorname{int}(F_1 \cap \mathscr{F}_k)$; $k = 1, 2$. By 1.3.2, $r_1 \in F_1 \cap (\mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2)$ is hyperbolic.

From 9.3, it is immediate that a non-empty $\operatorname{int}(G_1)$ satisfies 2.6.

9.5. Let $\mathscr{P}_0{}'$ and $\mathscr{P}_1{}'$ be the closed half-spaces of P^3 determined by τ_1 and τ_2, $L_0 \subset \mathscr{P}_0{}'$. Then $L_{14} \cap L_{23} \subset \mathscr{P}_1{}'$ and $\{l^*\} = L_{13} \cap L_{24} \subset \mathscr{P}_0{}'$. We now examine $\operatorname{int}(P_0 \cap P_i{}') \cap F$.

Let $\beta = \langle \nu, l^*, r\rangle$, $r \in \operatorname{int}(P_0 \cap P_1{}') \cap F$. Then $\beta \subset P_0$, ν is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$, $\beta \cap \tau_i \subset \mathscr{H}_{i0}$ $(i = 1, 2)$ and $L_0 \cap \mathscr{L} = \emptyset$. Thus $L_0 \subset \mathscr{P}_0{}'$ implies that $\beta \cap \operatorname{int}(\mathscr{P}_1{}') \cap F = L\backslash\{\nu\}$. Finally, $L_{14} \cap L_{23} \subset \mathscr{P}_1 \cap \mathscr{P}_1{}'$ yields that

$$(L_{14} \cup L_{23}) \cap \operatorname{int}(\mathscr{P}_0 \cap \mathscr{P}_1{}') = \emptyset$$

and $l(r) = 0$.

Let $p \in M_0\backslash\{\nu\}$. Since $L_0 \subset \mathscr{P}_0{}' \cap F$ and $L_0 \subset \pi(p)$, there is an open neighbourhood $u(p)$ of p in F such that $u(p) \subset \mathscr{P}_0{}'$. Hence,

$$M_0 \cap \operatorname{Cl}(\operatorname{int}(\mathscr{P}_0 \cap \mathscr{P}_1{}') \cap F) = \{\nu\}.$$

Obviously, $\operatorname{int}(\mathscr{P}_0 \cap \mathscr{P}_0')$ is disconnected and $\operatorname{int}(\mathscr{P}_0 \cap \mathscr{P}_1') \cap F$ consists of two maximal open disjoint regions, say G_0 and G_0'. By the preceding, we may assume that

$$\operatorname{bd}(G_0) \subseteq M_1 \cup M_3 \cup L_{13} \quad \text{and} \quad \operatorname{bd}(G_0') \subseteq M_2 \cup M_4 \cup L_{24}.$$

Then

$$\operatorname{int}(\mathscr{P}_0 \cap \mathscr{P}_1') \cap F = G_0 \cup G_0' \quad \text{where} \quad \bar{G}_0 \cap \bar{G}_0' = \{\nu\}.$$

Let $\beta_\lambda \neq \langle M_1, M_3\rangle$ converge to $\langle M_1, M_3\rangle$ in $\mathscr{P}_0$. Then $\beta_\lambda \cap G_0' = \emptyset$ and $\beta_\lambda \cap \bar{G}_0$ is the loop of $\beta_\lambda \cap F$ for β_λ sufficiently close to $\langle M_1, M_3\rangle$. Thus $\lim(\beta_\lambda \cap \bar{G}_0)$ is a curve of order $\leqq 2$. It is easy to check that $\lim(\beta_\lambda \cap \bar{G}_0)$ is a triangle in $\langle M_1, M_3\rangle \cap \mathscr{P}_1' \cap F$ bounded by segments of M_1, M_3 and L_{13}.

Thus G_0 and (similarly) G_0' are bounded triangular regions in F. Clearly, each region satisfies 2.6 and thus contains elliptic points. From 9.0 and 1.3.4, each region also contains hyperbolic and parabolic points.

9.6. Let $\mathscr{P}_1^*$ and $\mathscr{P}_2^*$ be the closed quarter-spaces of $\mathscr{P}_0'$ determined by $\langle M_0, L_0\rangle$. We assume that $\tau_i \subset \mathscr{P}_i^*$ and put $F_{0i} = (\mathscr{P}_i^* \cap \mathscr{P}_0) \cap F$; $i = 1, 2$. Then

(1) $(\mathscr{P}_0 \cap \mathscr{P}_0') \cap F = F_{01} \cup F_{02}$,

(2) $\mathscr{P}_0 \cap F = F_{01} \cup F_{02} \cup G_0 \cup G_0'$ and

(3) $F = F_{01} \cup F_{02} \cup F_1 \cup G_0 \cup G_0' \cup G_1$.

THEOREM 9.7. *If $r \in F_{01} \cup F_{02}$ such that $l(r) = 0$, then r is hyperbolic.*

Proof. We recall that $\mathscr{P}_1$ is a closed half-space bounded by $\langle M_1, M_3\rangle$ and $\langle M_3, M_4\rangle$ such that $\mathscr{P}_1 \cap \langle M_1, M_2\rangle = \mathscr{H}_{11}$ and $P_1 \cap \langle M_3, M_4\rangle = \mathscr{H}_{21}$. It is easy to check that $p \in \mathscr{P}_1$ if and only if $\langle M_i, p\rangle \cap H_{21} \neq \{\nu\}$ for $i = 1, 2$ or $\langle M_j, p\rangle \cap \mathscr{H}_{11} \neq \{\nu\}$ for $j = 3, 4$.

Let $r \in F_{01}$ $l(r) = 0$. Since $r \notin \mathscr{P}_1$, we may assume that $\langle M_1, r\rangle \cap \mathscr{H}_{21} = \{\nu\}$. Then there is an $N_1 \subset \mathscr{H}_{11}$ arbitrarily close to M_1, such that $N_1 \cap F = \{\nu\}$ and $\langle N_1, r\rangle \cap \mathscr{H}_{21} = \{\nu\}$. Then $\langle N_1, r\rangle \cap \mathscr{H}_{20}$ is a line N_2; $N_2 \cap F = \{\nu\}$.

Let $\beta = \langle N_1, N_2\rangle$. By 2.1.2, ν is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$. By 9.1.2, $L_0 \cap \mathscr{L} \neq \emptyset$ and therefore $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2$, $(\beta \cap \mathscr{P}_1') \cap F = \mathscr{F}_1 \cup \mathscr{F}_2$ and

$$(\beta \cap \mathscr{P}_0') \cap F = [\beta \cap (\mathscr{P}_1^* \cup \mathscr{P}_2^*)] \cap F = \mathscr{L}_1 \cup \mathscr{L}_2.$$

Clearly, $(\beta \cap \mathscr{P}_1^*) \cap F = \mathscr{L}_1$ and $(\beta \cap \mathscr{P}_2^*) \cap F = \mathscr{L}_2$. As $r \in \beta \cap F_{01} \subset \beta \cap \mathscr{P}_1^* \cap F = \mathscr{L}_1$, $S^1(M_0, r)$ and $\mathscr{L}_1$ are incompatible by 9.1.2 and r is hyperbolic by 1.3.2.

By a similar argument, we prove the theorem for the points of F_{02}.

In Figure 7, we represent the lines of F with $L_0^* \neq \{l_0\}$ and in Figure 8, we represent F with $L_0^* = \{l_0\}$. We observe that there are two holes in this surface, $|L_{14} \cap S^1(L_{14}, r)| = |L_{23} \cap S^1(L_{23}, r)| = 2$ for any $r \in F$ with

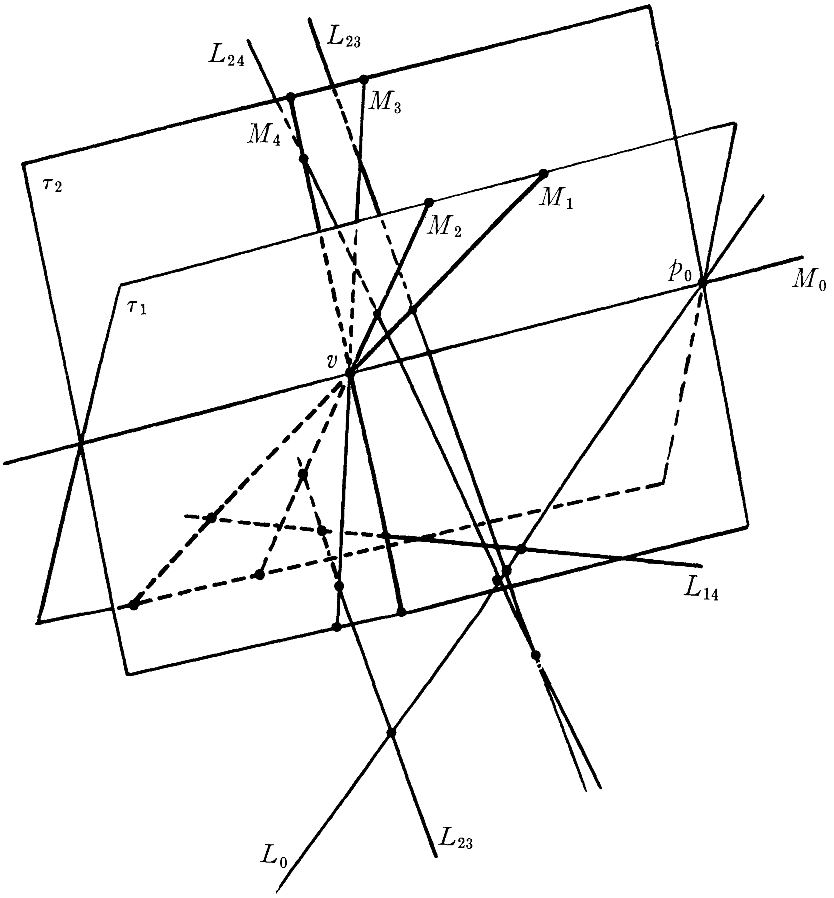

FIGURE 7

$l(r) = 0$ and there are points r' and r'' in F such that $L_{24} \cap S^1(L_{24}, r') = L_{13} \cap S^1(L_{13}, r'') = \emptyset$.

The surface in P^3 defined by $x_1x_2x_3 = (x_1 + x_2)(x_1^2 + x_2^2 - x_0^2)$ satisfies 9.0 with $M_0 \equiv x_1 = x_2 = 0$, $M_1 \equiv x_1 = x_0 - x_2 = 0$, $M_2 \equiv x_1 = x_0 + x_2 = 0$, $M_3 \equiv x_2 = x_0 - x_1 = 0$, $M_4 \equiv x_2 = x_0 + x_1 = 0$, $L_0 \equiv x_3 = x_1 + x_2 = 0$, $L_{14} \equiv x_3 - 2(x_1 + x_2) = x_0 + x_1 - x_2 = 0$, $L_{23} \equiv x_3 - 2(x_1 + x_2) = x_0 - x_1 + x_2 = 0$, $L_{13} \equiv x_3 + 2(x_1 + x_2) = x_0 - x_1 - x_2 = 0$, $L_{24} \equiv x_3 + 2(x_1 + x_2) = x_0 + x_1 + x_2 = 0$ and $L_0^* = \{l_0\} \equiv (0, 1, -1, 0)$.

THEOREM 9.8. *Let F be a biplanar surface satisfying* 9.0. *Then*

$$F = F_{01} \cup F_{02} \cup F_1 \cup G_0 \cup G_0' \cup G_1$$

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

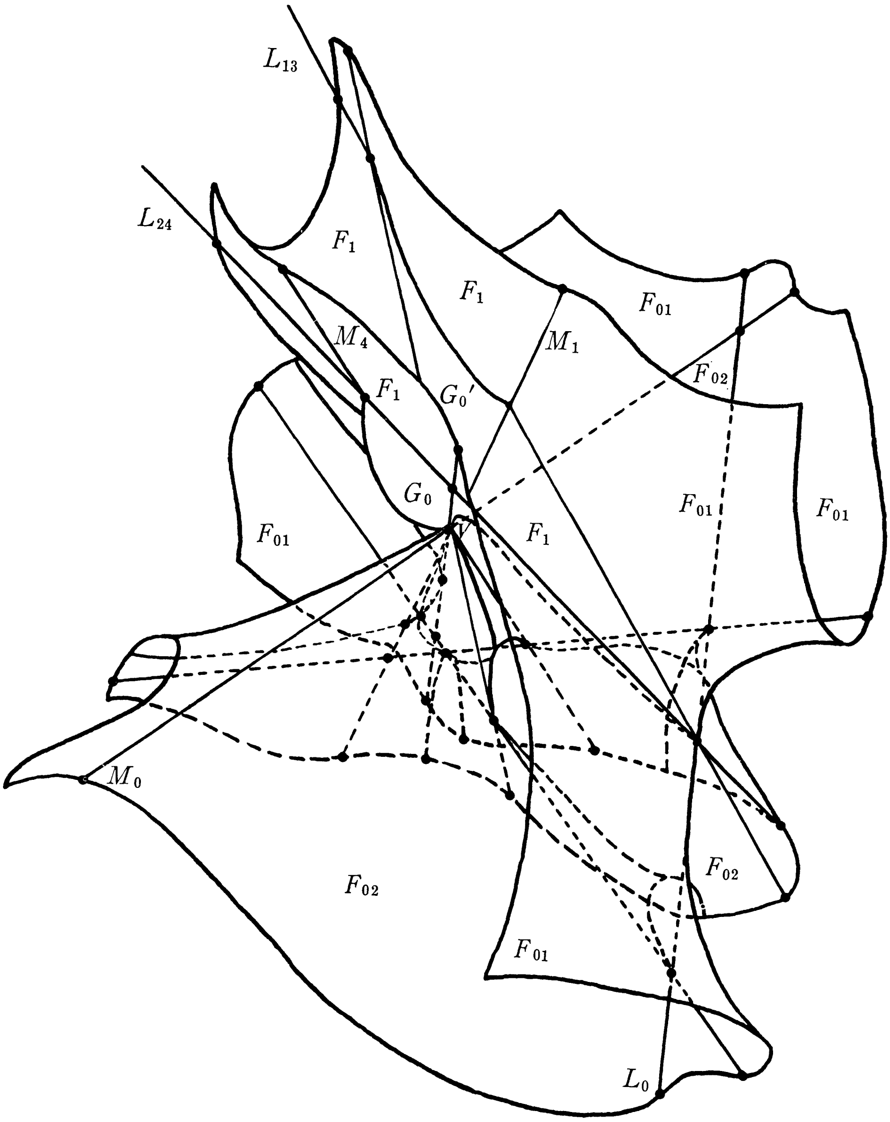

FIGURE 8

where every $r \in F_{01} \cup F_{02} \cup F_1$ *with* $l(r) = 0$ *is hyperbolic and* G_0, G_0' *and* G_1 *are described in* 9.3 *through* 9.5.

REFERENCES

1. T. Bisztriczky, *Surfaces of order three with a peak*, I. J. of Geometry, *11*/1 (1978), 55–83.
2. ——— *Uniplanar surfaces of order three*, to appear.
3. O. Haupt and H. Künneth, *Geometrische ordnungen* (Springer-Verlag, Berlin, 1967).

https://doi.org/10.4153/CJM-1979-044-3 Published online by Cambridge University Press

4. A. Marchaud, *Sur les surfaces du troisième ordre de la Géométrie finie*, J. Math. Pur. Appl. *18* (1939), 323–362.

5. ——— *Sur les propriétés différentielles du premier ordre des surfaces simples de Jordan et quelques applications*, Ann. Ec. Norm. Sup. *63* (1947), 81–108.

6. P. Scherk, *Über differenzierbare kurven und bögen I. zum begriff der charakteristik*, Časopis Pěst. Mat. *66* (1937), 165–171.

University of Calgary,
Calgary, Alberta