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A PRECONDITIONED METHOD FOR THE SOLUTION OF THE ROBBINS PROBLEM FOR THE HELMHOLTZ EQUATION

JIANG LE[✉1,2], HUANG JIN[1], XIAO-GUANG LV[1,2] and QING-SONG CHENG[2]

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Abstract

A preconditioned iterative method for the two-dimensional Helmholtz equation with Robbins boundary conditions is discussed. Using a finite-difference method to discretize the Helmholtz equation leads to a sparse system of equations which is too large to solve directly. The approach taken in this paper is to precondition this linear system with a sine transform based preconditioner and then solve it using the generalized minimum residual method (GMRES). An analytical formula for the eigenvalues of the preconditioned matrix is derived and it is shown that the eigenvalues are clustered around 1 except for some outliers. Numerical results are reported to demonstrate the effectiveness of the proposed method.

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1. Introduction

The Helmholtz equation arises in a variety of applications in mathematics, scientific computing and engineering, for instance, acoustic phenomena in aeronautics, underwater acoustics, photolithography in electromagnetic applications and geophysics [1, 5, 16, 25]. Motivated by these applications, many mathematicians and engineers develop specific algorithms for solving the Helmholtz equation; see [5] for details.

In this paper, we are mainly interested in the numerical solution of the two-dimensional Helmholtz equation with Robbins boundary conditions. Such problems frequently arise in computational physics and many other applied areas [5, 10, 22].

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[1]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, PR China; e-mail: jiangle@hhit.edu.cn, huangjin12345@163.com.
[2]School of Science, HuaiHai Institute of Technology, Lianyungang, Jiangsu, PR China; e-mail: xiaoguanglv@126.com, qingsongcheng@hhit.edu.cn.

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To solve such problems, a finite-difference discretization is applied, leading to an n-by-n system of linear equations of the form $Au = b$. The matrix A is large but sparse. Some direct methods based on LU-factorization can yield the solution of $Au = b$. These methods are well known for their robustness for general problems. However, they are not favourable for sparse linear systems. During the elimination process, zero entries in the matrix may be filled by nonzero entries.

As an alternative, iterative methods may be more attractive for the solution of a large system of linear equations. The solution is iteratively obtained from a recursion consisting of one matrix-vector multiplication, starting with a given initial solution. In particular, Krylov subspace methods might be used: these methods apply techniques that involve orthogonal projections onto subspaces of the form $\mathcal{K}_l(A, b) \equiv \mathrm{span}\{b, Ab, A^2b, \ldots, A^{l-1}b\}$. Krylov subspace methods are among the most important iterative methods currently available. Common schemes that use this idea are the method of conjugate gradients (CG) for symmetric positive-definite matrices, the method of minimum residuals (MINRES) for symmetric and possibly indefinite matrices, and the generalized minimum residual method (GMRES) for nonsymmetric matrices, although many other methods are available; see, for example, Greenbaum [8]. One common feature of the aforementioned methods is that the solution to the n-by-n linear system $Au = b$ will be found within n iterations in exact arithmetic; see Joubert and Manteuffel [14]. In particular, if a method converges after a finite, small number of iterations, the method is very efficient. Iterative methods, however, are not always guaranteed to have fast convergence and it is also observed that there are many situations in which they diverge [15]. In such cases, iterative methods do not offer any advantage as compared to direct methods.

Therefore, the problem of slow convergence must be overcome. One way to improve the convergence rate of a method is to precondition the linear equations. Thus, instead of solving the original system $Au = b$, we solve the preconditioned system $P^{-1}Au = P^{-1}b$. The matrix P, called a preconditioner for the matrix A, is chosen with the following two criteria in mind.

(i) The equation $Pr = d$ is easy to solve for any vector d.
(ii) The spectrum of $P^{-1}A$ is clustered and/or $P^{-1}A$ is well conditioned compared with A.

(See Axelsson and Barker [2].) In fact, the main idea of preconditioning is to attempt to improve on the spectral properties, that is, the clustering of the eigenvalues, such that the total number of iterations required to solve the system within some tolerance is substantially decreased.

In general, there are two classes of preconditioners for the Helmholtz equation. The first class comprises the matrix-based preconditioners. For this class, the preconditioners are based on approximations of the inverse of A, including incomplete LU and sparse approximate inverse [9, 19]. The second class comprises the operator-based preconditioners. In this class, the preconditioner is built based on an operator for which the spectrum of the preconditioned system $P^{-1}A$ is favourably clustered.

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This operator does not have to be a representation of the inverse of the Helmholtz operator. We refer the reader to [18] for a general discussion on this class of preconditioners. Included within this class are the Analytic *ILU* (or AILU) [7], the separation-of-variables [23] and the Laplace preconditioner [3, 6].

In this paper, we consider the construction of the sine transform based preconditioners [4, 11, 12] for the Helmholtz equation with Robbins boundary conditions. We show that the construction cost of P is $O(1)$ and that the matrix-vector multiplication $P^{-1}v$ for any vector v can be done in $O(n \log n)$ operations. Furthermore, we conclude that we have finite termination in m iterations of the GMRES method, where m is the number of distinct eigenvalues of the preconditioned system. Thus P is an efficient preconditioner.

In Section 2, the discretization of the Helmholtz equation with Robbins boundary conditions is described. In Section 3, we consider the construction of the sine transform based preconditioners for Helmholtz problems, and also study the spectrum of the preconditioned matrix. In Section 4, numerical results are reported to illustrate the effectiveness of our method.

2. The discretized system

In this section, we derive the discrete linear system from the two-dimensional Helmholtz equation

$$-\Delta u - k^2 u = f \tag{2.1}$$

over the domain $0 \le x \le 1$, $0 \le y \le 1$ with the following Robbins boundary conditions:

$$\begin{aligned} u(0, y) &= g(y), \\ u(x, 0) &= \rho(x), \\ u_x(1, y) &= pu(1, y) + a(y), \\ u_y(x, 1) &= qu(x, 1) + c(x), \end{aligned}$$

where k, p and q are constants.

Given a natural number $m \ge 1$, consider the mesh of points

$$(x_i, y_j), \quad x_i = ih, \quad y_j = jh, \quad i, j = 1, 2, \ldots, m,$$

where $h = 1/m$ is the mesh size. Let $u_{i,j} = u(x_i, y_j)$, $f_{i,j} = f(x_i, y_j)$, $g_j = g(y_j)$, $\rho_i = \rho(x_i)$, $a_j = a(y_j)$ and $c_i = c(x_i)$. With three-point centred difference for the derivative boundary conditions, we obtain

$$\begin{aligned} \frac{1}{2h}(u_{m+1,j} - u_{m-1,j}) &\approx pu_{m,j} + a_j, \\ \frac{1}{2h}(u_{i,m+1} - u_{i,m-1}) &\approx pu_{i,m} + c_i. \end{aligned}$$

If $(x_i, y_j) \in [0, 1] \times [0, 1]$, then the Laplacian can be approximated by

$$(-\Delta u)(x_i, y_j) \approx \frac{1}{h^2}(4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}).$$

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The above formula is known as the classical five-points approximation to the Laplacian. When using the classical five-points difference scheme for the two-dimensional Helmholtz equation, we have

$$(4-k^2h^2)u_{i,j}-u_{i+1,j}-u_{i-1,j}-u_{i,j+1}-u_{i,j-1}=h^2f_{i,j}, \tag{2.2}$$

where $i,\ j=1,2,\ldots,m$; see [11, 12, 20] for more details.

With the imposed Robbins boundary condition, we have the matrix–vector equation

$$Au=b,$$

where the unknowns are

$$u=(u_{1,1},u_{1,2},\ldots,u_{1,m},u_{2,1},\ldots,u_{2,m},\ldots,u_{m,m})^T,$$

and $b=(b_1,\ldots,b_m)^T$ is a known n-dimensional vector, where $n=m^2$ and each b_i is an m-dimensional row vector. Let $\eta_i=h^2(f_{i,1},\ldots,f_{i,m})$ for $i=1,\ldots,m$ and $g=(g_1,\ldots,g_m)$, $a=(a_1,\ldots,a_m)$, $\beta_i=(\rho_i,0,\ldots,0,2hc_i)$. Then we obtain $b_1=g+\eta_1+\beta_1$, $b_m=2ha+\eta_m+\beta_m$ and $b_i=\eta_i+\beta_i$ for $i=2,\ldots,m-1$.

In addition, the nonsymmetric matrix A has a tensor product form

$$A=I_m\otimes B+C\otimes I_m, \tag{2.3}$$

where

$$B=\begin{pmatrix} 4-k^2h^2 & -1 & & & \\ -1 & 4-k^2h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4-k^2h^2 & -1 \\ & & & -2 & 4-k^2h^2-2hq \end{pmatrix}\in\mathbb{R}^{m\times m}$$

and

$$C=\begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & -1 \\ & & & -2 & -2hp \end{pmatrix}\in\mathbb{R}^{m\times m}.$$

We recall that the tensor product of $E=(e_{ij})\in\mathbb{R}^{s_1\times t_1}$ and $F=(f_{ij})\in\mathbb{R}^{s_2\times t_2}$ is defined as the matrix

$$E\otimes F=\begin{pmatrix} e_{11}F & \cdots & e_{1t_1}F \\ \vdots & \ddots & \vdots \\ e_{s_11}F & \cdots & e_{s_1t_1}F \end{pmatrix}\in\mathbb{R}^{s_1s_2\times t_1t_2}.$$

In fact, one may rewrite A as the tensor sum $A=B\oplus C$; see [17] for more details.

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3. The sine transform based preconditioner

We begin this section by recalling some basic properties of the tensor product.

THEOREM 3.1 ([26]). *The tensor product has the following properties:*

(i) $(E \otimes F)^T = E^T \otimes F^T$;
(ii) $E \otimes (F \otimes G) = (E \otimes F) \otimes G$;
(iii) $E \otimes (F + G) = (E \otimes F) + (E \otimes G)$ *and* $(E + F) \otimes G = (E \otimes G) + (F \otimes G)$;
(iv) $(E \otimes F)(G \otimes H) = (EG) \otimes (FH)$;
(v) $(E \otimes F)^{-1} = E^{-1} \otimes F^{-1}$, *if* E *and* F *are nonsingular;*
(vi) *if* E *and* F *are orthogonal (banded, nonsingular), then* $E \otimes F$ *is orthogonal (banded, nonsingular).*

In this paper we consider the sine transform based preconditioner P defined by

$$P = I_m \otimes \hat{B} + C \otimes I_m, \tag{3.1}$$

where

$$\hat{B} = \begin{pmatrix} 4 - k^2h^2 & -1 & & & \\ -1 & 4 - k^2h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 - k^2h^2 & -1 \\ & & & -2 & 4 - k^2h^2 \end{pmatrix} \in \mathbb{R}^{m \times m}.$$

Obviously, the complexity of the construction of P is $O(1)$.

THEOREM 3.2. *The matrix* $\hat{B}$ *defined above can be diagonalized as follows:*

$$\hat{B} = Q_1 \Lambda_1 Q_1^{-1}, \tag{3.2}$$

where the i*th column of* Q_1 *is given by*

$$\left(\sin \frac{(2i-1)\pi}{2m}, \sin \frac{2(2i-1)\pi}{2m}, \ldots, \sin \frac{(m-1)(2i-1)\pi}{2m}, (-1)^{i+1} \right)^T$$

and Λ_1 *is a diagonal matrix whose diagonal entries are given by*

$$[\Lambda_1]_{ii} \doteq \lambda_i = 4 - k^2h^2 - 2\cos \frac{(2i-1)\pi}{2m} \quad \textit{for } i = 1, 2, \ldots, m.$$

The proof is given by Pickering and Harley [21], who also observe that Q_1 is not orthogonal but satisfies $Q_1^T \operatorname{diag}(2, \ldots, 2, 1) Q_1 = m I_m$. Thus the inverse of Q_1 is $Q_1^{-1} = (1/m) Q_1^T \operatorname{diag}(2, \ldots, 2, 1)$. Therefore, the matrix–vector multiplication involving Q_1 and Q_1^{-1} can be computed efficiently by using discrete sine transforms in $O(m \log m)$ operations.

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By Theorem 3.1, it is not difficult to see that P can be decomposed as

$$P = (I_m \otimes Q_1)T(I_m \otimes Q_1^{-1}), \tag{3.3}$$

where T is a block-tridiagonal matrix with diagonal blocks as follows:

$$T = I_m \otimes \Lambda_1 + C \otimes I_m. \tag{3.4}$$

A careful observation of (3.3) and (3.4) shows that the solution of the preconditioned system $Px = y$ can be obtained by using $2m$ fast discrete sine transforms of order m and solving m tridiagonal systems of order m after rearranging the unknowns. Consequently, the total complexity of this solution procedure is $O(n \log n)$.

We will analyse the convergence rate of the GMRES method and give a detailed discussion about the eigenvalues of the preconditioned matrix $P^{-1}A$. First, we have the following theorem.

THEOREM 3.3. *Let γ_i, $i = 1, \ldots, m$, denote the eigenvalues of C. Then the preconditioned matrix $P^{-1}A$ has:*

(i) *an eigenvalue at* 1 *with multiplicity $m(m-1)$;*
(ii) *m eigenvalues given by*

$$\lambda_i(A) = 1 - \frac{2hq}{m} \sum_{j=1}^{m} \frac{1}{4 - k^2h^2 - 2\cos((2j-1)/2m) + \gamma_i}, \quad i = 1, \ldots, m.$$

PROOF. Define the error matrix $A_e = A - P$. Observe that $A_e = I_m \otimes E_1$, where E_1 is a rank one matrix given by

$$E_1 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -2hq \end{pmatrix}. \tag{3.5}$$

We find

$$P^{-1}A = I_n + P^{-1}(I_m \otimes E_1).$$

It then follows that $\operatorname{rank}(P^{-1}A - I_n) = m$. Thus, it is not difficult to see that $P^{-1}A$ has an eigenvalue at 1 with multiplicity $m(m-1)$.

Otto [20] shows that there exists a nonsingular matrix Q_2 diagonalizing C, that is,

$$Q_2^{-1}CQ_2 = \operatorname{diag}(\gamma_1, \ldots, \gamma_m) \doteq \Lambda_2. \tag{3.6}$$

This implies

$$\begin{aligned}(Q_2^{-1} \otimes Q_1^{-1})P(Q_2 \otimes Q_1) &= (Q_2^{-1} \otimes Q_1^{-1})(I_m \otimes \hat{B} + C \otimes I_m)(Q_2 \otimes Q_1) \\ &= I_m \otimes Q_1^{-1}\hat{B}Q_1 + Q_2^{-1}CQ_2 \otimes I_m \\ &= I_m \otimes \Lambda_1 + \Lambda_2 \otimes I_m \doteq \Lambda.\end{aligned}$$

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Hence, the diagonal entries of the diagonal matrix Λ are the eigenvalues of the preconditioner P, namely $\lambda_i + \gamma_j$, where $i, j = 1, \ldots, m$. For simplicity, set

$$D_k = \operatorname{diag}(\lambda_k + \gamma_1, \ldots, \lambda_k + \gamma_m).$$

We have $\Lambda = \operatorname{diag}(D_1, \ldots, D_m)$. A similarity transformation of $P^{-1}A_e$ is denoted by M and given by

$$\begin{aligned} M &= (Q_2^{-1} \otimes I_m) P^{-1} A_e (Q_2 \otimes I_m) \\ &= (Q_2^{-1} \otimes I_m) P^{-1} (I_m \otimes E_1)(Q_2 \otimes I_m) \\ &= (I_m \otimes Q_1) \Lambda^{-1} (I_m \otimes Q_1^{-1} E_1) \\ &= \operatorname{diag}(Q_1 D_1^{-1} Q_1^{-1} E_1, \ldots, Q_1 D_m^{-1} Q_1^{-1} E_1) \\ &\doteq \operatorname{diag}(M_1, \ldots, M_m). \end{aligned}$$

Thus, the eigenvalues of $P^{-1}A_e$ equal the eigenvalues of M_i, $i = 1, \ldots, m$. Due to the special structure of E_1, it is easy to observe that M_i has only one nonzero column. We have

$$M_i = \begin{pmatrix} 0 & \cdots & 0 & M_i(1, m) \\ 0 & \cdots & 0 & M_i(2, m) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & M_i(m, m) \end{pmatrix}, \tag{3.7}$$

where

$$M_i(m, m) = -\frac{2hq}{m} \sum_{j=1}^{m} \frac{1}{4 - k^2h^2 - 2\cos((2j-1)/2m) + \gamma_i}.$$

The precise forms of $M_i(j, m)$, $i = 1, \ldots, m$ and $j \neq m$, are irrelevant for our argument. □

As is well known, clustering of the eigenvalues may be even more important than a condition number improvement. A careful computation shows that when $m \to \infty$, the other m eigenvalues always lie on a bounded curve. This is illustrated by our numerical examples in the next section. As the dimension of the nullspace of A_e is $m(m-1)$, it is clear that the eigenvalue 0 of $P^{-1}A_e$ has $m(m-1)$ linearly independent eigenvectors. By properties of the GMRES method [8, 24], we have the following theorem.

THEOREM 3.4. *When the GMRES method is applied for solving the preconditioned system $P^{-1}Au = P^{-1}b$, it converges within at most $m+1$ iterations in exact arithmetic.*

PROOF. In general, in the ith iteration, a residual vector $r^{(i)}$ generated by the GMRES method satisfies

$$\|r^{(i)}\|_2 = \min_{p_i \in \mathcal{P}_i,\, p_i(0)=1} \|p_i(P^{-1}A) r^{(0)}\|_2,$$

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where $\mathcal{P}_i$ denotes the set of all polynomials of degree i and P is a preconditioner. In addition, if $P^{-1}A$ is diagonalizable then

$$\frac{\|r^{(i)}\|_2}{\|r^{(0)}\|_2} = \kappa(W_{P^{-1}A}) \min_{p_i \in \mathcal{P}_i,\, p_i(0)=1} \max_j |p_i(\delta_j)|,$$

where $W_{P^{-1}A}$ is the eigenvector matrix of $P^{-1}A$, $\kappa(\cdot)$ denotes the condition number and the δ_j are the eigenvalues of $P^{-1}A$; see [11, 24]. Therefore, if we can precondition our system such that $P^{-1}A$ has $m+1$ distinct eigenvalues, the GMRES method will converge to the true solution within $m+1$ iterations in exact arithmetic. □

In practical computations, we find that the number of preconditioned GMRES iterations required for convergence is far less than m. This cannot be explained by our theoretical results. However, all numerical tests in the following section show that the number of iterations required is apparently independent of m. This is a very useful property.

4. Numerical examples

We report on some numerical experiments to illustrate our results. All the experiments discussed in this section were performed in MATLAB 7.0, using the function "GMRES". In all tests, the zero vector is the initial guess and the stopping criterion is $\|r_l\|_2/\|r_0\|_2 < 10^{-6}$, where r_l is the residual in the lth iteration. We illustrate the efficiency of our preconditioner by solving the following problems arising from [6, 13, 27].

EXAMPLE 4.1. Consider the Helmholtz equation

$$-\Delta u - k^2 u = -(x^2 + y^2 + k^2)e^{xy}$$

over the domain $0 \le x \le 1$, $0 \le y \le 1$ with

$$g(y) = 1, \quad \rho(x) = 1, \quad p = 1,$$
$$a(y) = (y-1)e^y, \quad q = \tfrac{1}{2}, \quad c(x) = (x - \tfrac{1}{2})e^x.$$

EXAMPLE 4.2. Consider the Helmholtz equation

$$-\Delta u - k^2 u = \left(\frac{5\pi^2}{4} - k^2\right) \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)$$

over the domain $0 \le x \le 1$, $0 \le y \le 1$ with

$$g(y) = 0, \quad \rho(x) = 0, \quad p = -1,$$
$$a(y) = \sin(\pi y), \quad q = 1, \quad c(x) = -\pi \sin\left(\frac{\pi x}{2}\right).$$

EXAMPLE 4.3. Consider the Helmholtz equation

$$-\Delta u - k^2 u = -(4 + k^2x^2 + k^2y^2)$$

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TABLE 1. Iterations of different preconditioned GMRES methods for Example 4.1.

n	P_s				P				P_l				$ILU(0)$			
	1	5	10	20	1	5	10	20	1	5	10	20	1	5	10	20
2^6	7	7	8	7	5	5	5	5	8	8	10	10	13	15	Failed	Failed
2^8	10	10	8	11	5	5	5	4	10	10	10	13	23	35	75	Failed
2^{10}	13	13	11	16	5	5	5	6	13	13	13	18	51	118	119	$*$
2^{12}	18	18	15	19	5	5	5	6	18	18	16	22	634	559	694	$*$
2^{14}	26	26	20	29	6	5	5	6	38	31	22	37	$*$	$*$	$*$	$*$

over the domain $0 \le x \le 1$, $0 \le y \le 1$ with

$$g(y) = y^2, \quad \rho(x) = x^2, \quad p = 1, \quad a(y) = 1 - y^2, \quad q = -1, \quad c(x) = 3 + x^2.$$

In Figures 1, 2 and 3, subfigures (a)–(e) show the distributions of the eigenvalues for various preconditioned matrices for Examples 4.1, 4.2 and 4.3. In all cases, the mesh size is $1/16$, and $k = 5$. We denote by P_s, P, P_l and $ILU(0)$ the semi-Toeplitz preconditioner [11], our preconditioner (3.1), the shifted Laplace preconditioner [6] and the incomplete LU factorization preconditioner mentioned in [7, 9, 19]. It is easy to observe that the eigenvalues of the iteration matrix corresponding to our preconditioner are more clustered than those corresponding to the other three widely used preconditioners.

In Tables 1, 2 and 3, we show the number of iterations for the restarted GMRES(20) method with different preconditioners for Examples 4.1, 4.2 and 4.3. (The "20" means that the GMRES method is restarted every 20 iterations.) Here "$*$" indicates that the number of iterations was more than one thousand, and "Failed" means that the iterative method failed. As the results show, the number of GMRES(20) iterations with the $ILU(0)$ preconditioner grows very quickly as the size of the coefficient matrix A increases from $n = 64$ to $n = 16384$ for $k = 1, 5, 10, 20$. The numbers of GMRES(20) iterations with the semi-Toeplitz and shifted Laplace preconditioners grow a little more slowly with increasing n, so these methods are often used. When the sine transform based preconditioner P is used, the convergence is generally faster. It is not difficult to see that the number of GMRES(20) iterations with preconditioner P is almost unchanged as the mesh size m and k increase. In order to more sufficiently illustrate that our method converges faster than the others, we plot the relative residual norm $\|r_l\|_2/\|b\|_2$ in Figures 1(f), 2(f) and 3(f) for $m = 16$ and $k = 5$. In Table 4, we give the relative error norm $\|u_{ij} - u(x_i, y_j)\|_2/\|b\|_2$ for Example 4.3, where $u(x_i, y_j)$ denotes the exact solution and u_{ij} is the solution calculated by our algorithm. It should be noted that the computations $P^{-1}v$ and Av are implemented by discrete sine transforms and fast Fourier transforms (FFTs), respectively. Therefore, the complexity of our method is less than those of the others.

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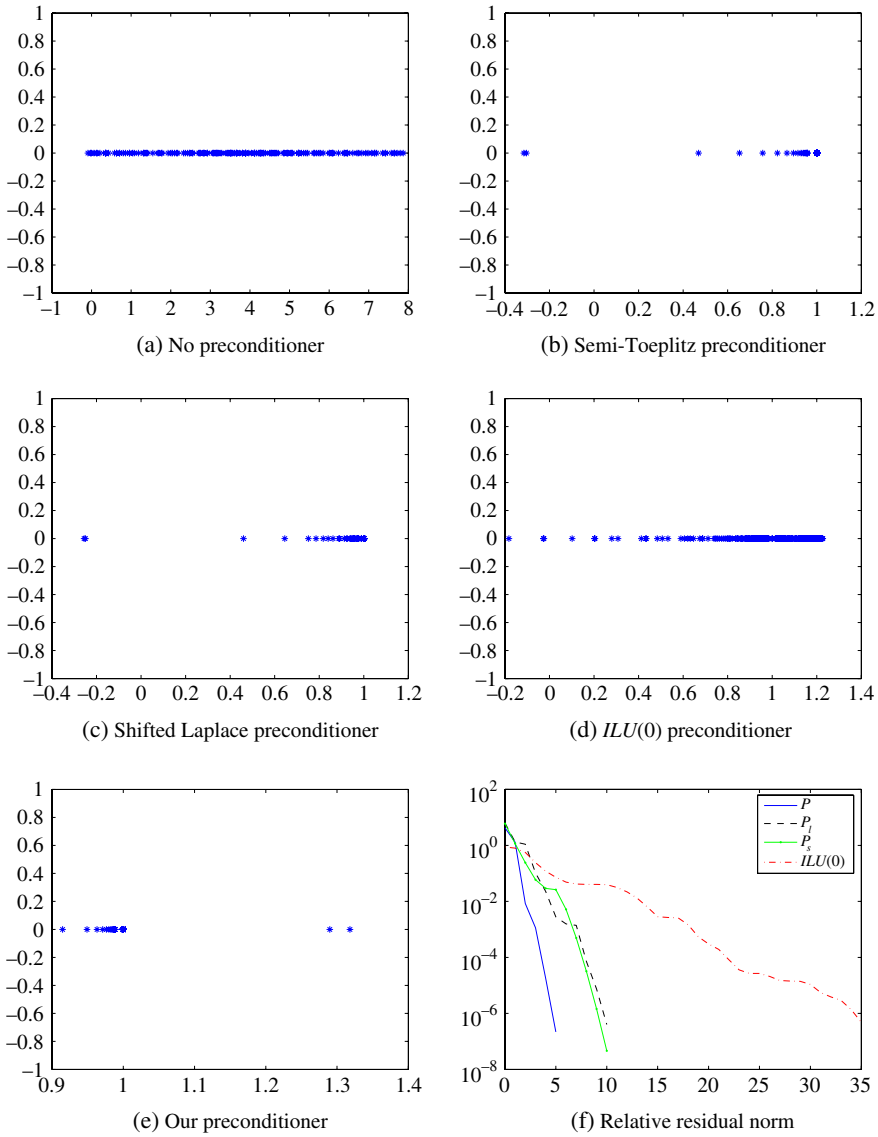

FIGURE 1. Distribution of the eigenvalues of the iteration matrices and relative residual norm for Example 4.1.

5. Concluding remarks

We have presented an iterative method involving the sine transform based preconditioner for the numerical solution of the Robbins problem for the Helmholtz equation. First, the nonsymmetric linear systems arising from discretization of

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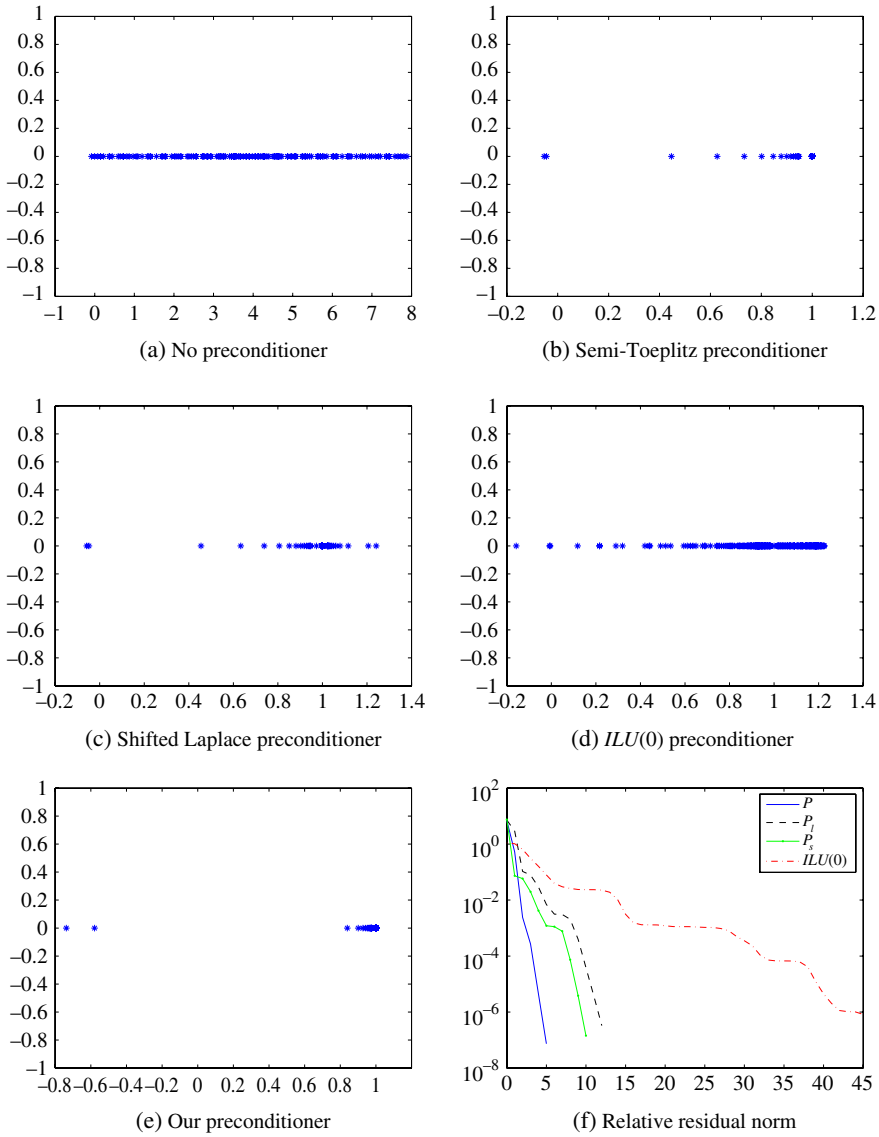

FIGURE 2. Distribution of the eigenvalues of the iteration matrices and relative residual norm for Example 4.2.

the two-dimensional Helmholtz partial differential equation are obtained. Then the generalized minimal residual method is applied for solving the linear system with a preconditioner based on the fast sine transform. We discuss the operation cost and convergence rate of our method and provide a spectral analysis of the

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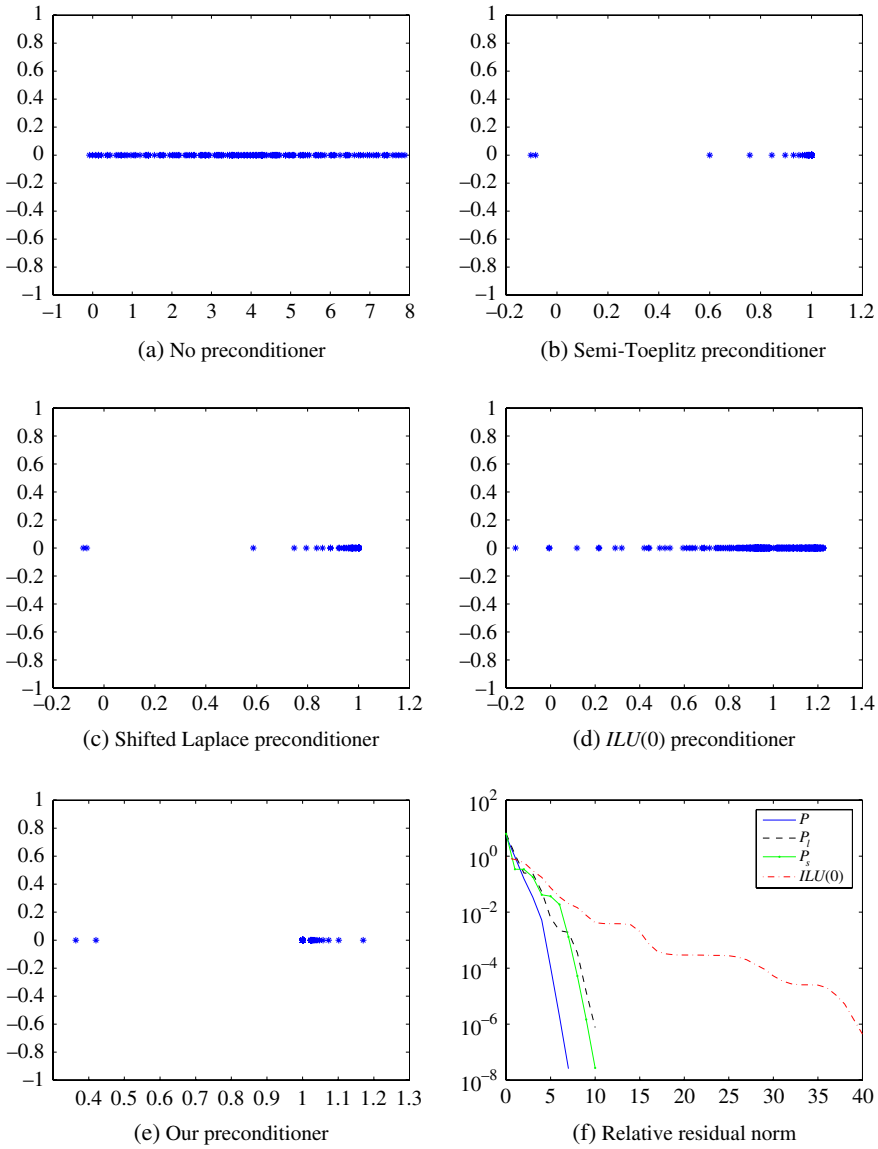

FIGURE 3. Distribution of the eigenvalues of the iteration matrices and relative residual norm for Example 4.3.

preconditioned matrix. In numerical experiments, we compare our preconditioner with other frequently used preconditioners. Numerical results show the efficiency of our preconditioned iterative method. It appears that the performance is much improved in terms of both convergence and operation cost compared to other iterative solvers.

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TABLE 2. Iterations of different preconditioned GMRES methods for Example 4.2.

n	P_s				P				P_l				$ILU(0)$			
	1	5	10	20	1	5	10	20	1	5	10	20	1	5	10	20
2^6	7	8	7	7	4	5	4	4	8	10	10	9	11	16	Failed	23
2^8	8	10	8	11	4	5	4	3	10	12	11	13	20	45	59	Failed
2^{10}	11	13	10	12	4	5	3	4	13	15	13	15	63	199	195	$*$
2^{12}	15	18	13	15	4	5	3	3	17	20	15	17	98	$*$	567	$*$
2^{14}	20	32	17	20	3	5	3	3	23	53	20	26	852	$*$	$*$	$*$

TABLE 3. Iterations of different preconditioned GMRES methods for Example 4.3.

n	P_s				P				P_l				$ILU(0)$			
	1	5	10	20	1	5	10	20	1	5	10	20	1	5	10	20
2^6	6	7	7	8	5	6	5	5	7	8	9	11	12	17	Failed	Failed
2^8	8	10	8	11	6	7	5	5	9	10	10	12	20	40	61	Failed
2^{10}	11	13	11	14	6	7	5	7	12	14	12	16	62	231	130	$*$
2^{12}	16	18	14	19	6	7	6	7	16	19	16	20	93	587	719	$*$
2^{14}	22	38	19	37	6	7	6	7	26	34	20	40	686	1000	$*$	$*$

TABLE 4. The relative error norm for Example 4.3.

m	k			
	1	5	10	20
2^3	6.6862e-007	3.0346e-007	2.5527e-006	2.8402e-006
2^4	3.7366e-008	3.9208e-008	1.9463e-007	1.2261e-006
2^5	8.5293e-008	1.2767e-007	6.6747e-007	1.8890e-007
2^6	1.4360e-007	2.7673e-007	9.9293e-009	1.8229e-008
2^7	1.6179e-007	3.3886e-007	1.0126e-008	2.8494e-008

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