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THE RIGHT REGULAR REPRESENTATION OF A COMPACT RIGHT TOPOLOGICAL GROUP

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ABSTRACT. We show that for certain compact right topological groups, $\overline{r(G)}$, the strong operator topology closure of the image of the right regular representation of G in $L(H)$, where $H = L^2(G)$, is a compact topological group and introduce a class of representations, R, which effectively transfers the representation theory of $\overline{r(G)}$ over to G. Amongst the groups for which this holds is the class of equicontinuous groups which have been studied by Ruppert in [10]. We use familiar examples to illustrate these features of the theory and to provide a counter-example. Finally we remark that every equicontinuous group which is at the same time a Borel group is in fact a topological group.

1. **Introduction.** Let (G, τ) be a group with compact Hausdorff topology. We say that (G, τ) is a *right topological group* if for every $g \in G$ the right translation map,

$$\rho_g: G \longrightarrow G: h \longmapsto hg,$$

is continuous. If the topology, τ, is understood we shall simply write G instead of (G, τ). The set of continuous left translation maps,

$$\Lambda(G) = \{g \in G \mid \lambda_g: G \longrightarrow G: h \longmapsto gh \text{ is continuous}\},$$

is called the *topological centre* of (G, τ). Compact right topological groups with dense topological centres arise in the study of minimal distal flows (see, for example, [9]).

If $G = \Lambda(G)$, *i.e.*, if multiplication in G is separately continuous, then we say that G is *semitopological*. A famous result tells us that if G is a (locally) compact semitopological group then G is in fact a topological group (see [2]). The definition of a left topological group and the corresponding definition for the topological centre is arrived at by interchanging the roles of the maps ρ_g and λ_g above.

A compact right topological group, G, is said to be *equicontinuous* if the set of all right translation maps is equicontinuous. These groups were studied in [10] where examples, including a more general form of Example 4 in this paper, are discussed in detail. The following result is taken from [10] and characterizes equicontinuous groups.

THEOREM 1. *Let G be a compact right topological group. Then the following statements are equivalent:*

(i) G is equicontinuous;

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(ii) The closure in G^G of the set of all right translation maps, which we denote by $\overline{\rho_G}$, is a compact topological group;

(iii) The map $(g,h) \longmapsto gh$ is jointly continuous at the identity;

Furthermore if any of these conditions is satisfied then the inversion map $g \longmapsto g^{-1}$ is continuous at the identity.

Though it may appear curious at first glance that a compact right topological group may have any points at all at which joint continuity can occur the reader may be reassured, as we shall see later, that this is not entirely an uncommon phenomenon. When G is equicontinuous the closure, $\overline{\rho_G}$, is sometimes referred to as the *Ellis group* of G. However, it is a well-known fact that an equicontinuous group cannot have dense topological centre unless it is topological (see [10]).

We say that a probability measure μ on G is *left invariant* if it is invariant under all continuous left translates, *i.e.*, $\mu(gB) = \mu(B)$ for every Borel set B in G and $g \in \Lambda(G)$. Furthermore we say that μ is a *Haar measure* on G if μ is right invariant, *i.e.*, $\mu(Bg) = \mu(B)$ for every Borel set B in G and $g \in G$. It is a well-known result that the existence and uniqueness of Haar measure for G, which is also left invariant, follows from the existence of a *strong normal system* for G, *i.e.*, a family of subgroups, $\{L_\xi : \xi \leq \xi_0\}$, indexed by the set of ordinals less than or equal to an ordinal ξ_0 such that,

(i) $L_0 = G$, $L_{\xi_0} = \{e\}$ and for each $\xi < \xi_0$, $L_{\xi+1} \subsetneq L_\xi$, L_ξ is a normal subgroup of G and the quotient group G/L_ξ is a Hausdorff space in the quotient topology;

(ii) for each $\xi < \xi_0$, the function

$$G/L_{\xi+1} \times L_\xi/L_{\xi+1} \longrightarrow G/L_{\xi+1} : (sL_{\xi+1}, tL_{\xi+1}) \longmapsto stL_{\xi+1}$$

is continuous in the quotient topologies.

(iii) for each limit ordinal $\xi \leq \xi_0$, $L_\xi = \bigcap_{\eta<\xi} L_\eta$.

For details see [1] (Appendix C), [6], [7] or [9]. It follows from the conditions above that $L_\xi/L_{\xi+1}$ is a compact, Hausdorff topological group for each ξ. We say that the length of the strong normal system is the ordinal ξ_0. We mention in passing that for *any* ordinal, ξ, it is possible to construct a group, G_ξ, for which every strong normal system of G_ξ is of length at least ξ (see [8]).

For simplicity we shall only be interested, throughout this paper, in groups which possess strong normal systems of finite length. There are, however, groups which do not possess strong normal systems though nonetheless have Haar measure. An example of such a group is given in Example 5. Of interest later is the existence of a normal subgroup, $N(G)$, of G with the property that for any closed normal subgroup, H, of G such that G/H is a compact Hausdorff topological group we have $N(G) \subset H$ (for details see [9], Propositions 2.1 and 2.2). We warn the reader that $N(G)$ need not be a topological group. Indeed it is possible that $G = N(G)$, however, this situation never arises if the topological centre of G is dense.

Haar measure on G may be realized as a positive, invariant, linear functional of norm 1 on $C(G)$. For some ξ, let μ_{L_ξ} denote Haar measure on the compact, Hausdorff topological group $L_\xi/L_{\xi+1}$ and let $C_{L_{\xi+1}}(G)$ denote the subspace of $C(G)$ consisting of functions constant on the cosets of $L_{\xi+1}$. Then any function $f \in C_{L_{\xi+1}}(G)$ gives rise to a function $\bar{f} \in C(G/L_{\xi+1})$, defined by $\bar{f}(\bar{s}) = f(s)$ where $\bar{s}$ denotes the coset of s in $G/L_{\xi+1}$. The continuity of the map,

$$G/L_{\xi+1} \times L_\xi/L_{\xi+1} \longrightarrow G/L_{\xi+1},$$

ensures that for $f \in C_{L_{\xi+1}}(G)$, the function $\phi_{L_\xi}(f)$ defined by,

$$\phi_{L_\xi}(f)(s) = \int_{L_\xi/L_{\xi+1}} \bar{f}(\bar{s}\bar{t})\, d\mu_{L_\xi}(\bar{t}),$$

for $s \in G$, is continuous. By the invariance of Haar measure the function $\phi_{L_\xi}(f)$ can be viewed as a function in $C(G/L_\xi)$ and therefore ϕ_{L_ξ} maps $C_{L_{\xi+1}}(G)$ onto $C_{L_\xi}(G)$. We obtain Haar measure on G by (transfinite) induction. We start with Haar measure on G/L_1 and define ψ_0 as the integral with respect to the Haar measure on G/L_1. Having determined the map

$$\psi_{\xi-1}\colon C_{L_{\xi-1}}(G) \longrightarrow C_{L_0}(G) = \mathbb{C}$$

we define $\psi_\xi = \phi_{L_\xi} \circ \psi_{\xi-1}$. It can be shown that ψ_{ξ_0} is Haar measure on G and for the details the reader is referred to [6].

Many compact right topological groups, including those in this paper, are constructed as extensions of compact topological groups. We shall therefore make use throughout of Schreier's (algebraic) analysis of group extensions and introduce appropriate notation which is consistent with that used in [5]. Suppose that we have two groups G_1 and G_2, a map

$$G_2 \longrightarrow \mathrm{Aut}(G_1)\colon t \longmapsto \big(s \longmapsto t(s)\big)$$

and a function,

$$G_2 \times G_2 \longrightarrow G_1\colon (t', t) \longmapsto [t', t],$$

satisfying the conditions

$$[t, e] = [e, t] = e,$$
$$[t, t']tt'(s) = t\big(t'(s)\big)[t, t']$$

and

$$[t, t'][tt', t''] = t([t', t''])[t', t't''],$$

for all $t, t', t'' \in G_2$ and $s \in G_1$. Then the set $G = G_1 \times G_2$ endowed with the multiplication,

$$(s', t')(s, t) = \big(s't'(s)[t', t], t't\big), \tag{1}$$

is a group with normal subgroup $G_1 \times \{e\} \cong G_1$ and $G/G_1 \cong G_2$. Notice that the action defined by the map $G_2 \longrightarrow \mathrm{Aut}(G_1)$ is on the left.

As we shall see later in the examples it is also necessary to introduce the dual notion for $G = G_1 \times G_2$, where this time G is an extension of G_2 by G_1 and so $\{e\} \times G_2$ is a

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normal subgroup of G and $G/G_2 \cong G_1$. This time we are dealing with automorphisms $t \longmapsto (t)s$ of G_2 and the multiplication is given by

$$(s', t')(s, t) = \big(s's, [s', s](t')st\big). \tag{2}$$

The difference here is that this time the action defined by the map $G_1 \longrightarrow \mathrm{Aut}(G_2)$ is on the right. Finally if the groups G_1 and G_2 have topologies τ_1 and τ_2 respectively, then we say that G is the Schreier product both algebraically and topologically of G_1 and G_2 if G is the isomorphic to either one of the Schreier products described above endowed with the product topology $\tau_1 \times \tau_2$.

It is worth pointing out at this stage that if $G = G_1 \times G_2$ is algebraically and topologically the Schreier product of a topological group G_1 and a *finite group* G_2 then G is equicontinuous. Indeed if $U = U_1 \times U_2$ is an open neighbourhood of the identity, then U contains $U_1 \times \{1\}$ which is open in G because G_2 is discrete. Since G_1 is topological we can find open neighbourhoods $V \times \{1\}$ and $V' \times \{1\}$ such that

$$(V \times \{1\})(V' \times \{1\}) \subset U_1 \times \{1\} \subset U.$$

From this observation we deduce continuity of multiplication in G at the identity and apply Theorem 1.

2**. Representation theory of compact right topological groups.** We turn our attention now to studying representations, π, of G by unitary operators on some Hilbert space $\mathcal{H}$. Let us assume that the group G has Haar measure, μ. Typically we shall be interested in the Hilbert space, $L^2(G)$, of square integrable functions with respect to Haar measure, μ, on G. The space $\mathcal{L}(\mathcal{H})$ shall be endowed with the strong operator topology. As has been pointed out already in [4], the continuity of π implies, as in the topological case, that π is the direct sum of irreducible finite dimensional representations. Moreover any continuous representation, π, factors through $G/N(G)$, where $N(G)$ is the normal subgroup of G mentioned in the introduction, and hence faithful representations cannot be continuous unless G is topological. Consequently there is an insufficient theory of continuous representations for a compact right topological group, G, to reproduce the sort of results we are accustomed to in the topological case. For example, unless G is topological we are unable to separate the points in G using only continuous representations. The situation is particularly acute if $G = N(G)$. In this case there are *no* non-trivial continuous representations. This is the situation with the group in Example 5 below (see also [6]).

It is therefore the aim of this section to introduce a class of representations, $\mathcal{R}$, for certain compact right topological groups which plays a role analogous to that played by continuous representations for topological groups. In order to do this it is necessary to study the *right regular representation*,

$$r \colon G \longrightarrow \mathcal{U}\big(L^2(G)\big) \colon g \longmapsto r_g,$$

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where $r_g f(h) = f(hg)$ for $f \in L^2(G)$. Since r is clearly faithful it is evident from the remarks above that r is not continuous unless G is topological.

For some of what follows it is instructive to study the continuity properties of the map,

$$G \longrightarrow \mathbb{C} \colon g \longmapsto \mu(B \triangle Bg^{-1}),$$

for all Borel sets, $B \subset G$. If G is a locally compact topological group then this map is continuous (see [3], Theorem 61.A). If, however, G is a compact right topological group then the continuity of the above map implies that G is topological. Indeed from the continuity of the map,

$$g \longmapsto \mu(B \triangle Bg^{-1}) = \|\chi_B - r_g \chi_B\|_2^2$$

and by approximating any $f \in L^2(G)$ by finite linear combinations of characteristic functions we easily deduce the continuity of the map $g \longmapsto \|f - r_g f\|_2$ and hence the continuity (in the strong operator topology) of the right regular representation, r.

THEOREM 2. *Let G be an equicontinuous compact right topological group. Then $\overline{r(G)}$ is a compact topological group.*

PROOF. First we prove that $\overline{r(G)}$ is compact. Since $\overline{r(G)}$ is got by taking the closure of $r(G)$ in $\prod\{\overline{r_G f} \mid f \in L^2(G)\}$, it suffices to show that for every $f \in L^2(G)$, the set $\overline{r_G f}$ is compact. In fact we need only show that the set $r_G f$ is totally bounded in the $\|\cdot\|_2$-norm. Let $f \in C(G)$. Since G is compact it follows that $r_G f$ is pointwise bounded. Moreover since G is equicontinuous the family $\{f \circ \rho_g \mid g \in G\}$ is equicontinuous. We can therefore apply Ascoli's theorem to deduce that $r_G f$ is sup-norm totally bounded in $C(G)$ and hence $\|\cdot\|_2$-norm relatively compact. The result now follows from the fact that the space of all $f \in L^2(G)$ for which $\overline{r_G f}$ is compact is a closed subspace of $L^2(G)$ (see [1], Chapter 6) containing the dense subset, $C(G)$, of $L^2(G)$. That $\overline{r(G)}$ is a topological group is now a simple consequence of the compactness of $\overline{r(G)}$ (see, for example, [1] Chapter 6).

THEOREM 3. *Let G be a compact right topological group with Haar measure μ, and a commutative subgroup, H, such that $G = G/H \times H$ is algebraically the Schreier product with multiplication given by* (2)*. Suppose furthermore that,*

(i) for every (relative topology) neighbourhood, U, of the identity of $\{e\} \times H$ we can find finitely many points, $\{t_i\}_{i=1}^n \in H$, such that the sets, $U(e, t_i)$, cover $\{e\} \times H$,

(ii) G/H is finite and

(iii) On H the map $g \longmapsto \mu(B \triangle Bg^{-1})$ is continuous for all Borel sets, B, of G.

Then $\overline{r(G)}$ is a compact topological group.

PROOF. As in the previous theorem it suffices to show that for every f in a dense set of functions in $L^2(G)$ the set $r_G f$ is $\|\cdot\|_2$-norm totally bounded. This time to prove the result we show that it holds for every characteristic function, $f = \chi_B$, where B is a Borel subset of G.

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Let $\epsilon > 0$. We need to find a finite set of points $\{g_i\}_{i=1}^n$ such that for any $g \in G$, $\|r_{g_i}\chi_B - r_g\chi_B\| < \epsilon$ for some i. Now for any pair $(s,t),(s',t') \in G/H \times H = G$ we have

$$\|r_{(s,t)}\chi_B - r_{(s',t')}\chi_B\| = \|r_{(s,e)}r_{(e,t)}\chi_B - r_{(s',e)}r_{(e,t')}\chi_B\|.$$

Letting $s = s'$ yields $\|r_{(e,t)}\chi_B - r_{(e,t')}\chi_B\|$ in the above expression. However, owing to the continuity of $g \longmapsto \mu(B\triangle Bg^{-1})$ on the subgroup $\{e\} \times H$ we can find an open neighbourhood of the identity, U, in $\{e\}\times H$ such that for $(e,h) \in H$, $\mu\big(B\triangle B(e,h^{-1})\big) < \epsilon$. Therefore by condition (i) above we can find finitely many points $\{(e,t_i)\}_{i=1}^n$ in $\{e\}\times H$ such that for any $t \in H$, $(e,t) \in U(e,t_i)$ for some i. Hence by the commutativity of H, $(e,{t_i}^{-1}t) = (e,t{t_i}^{-1}) \in U$ and

$$\begin{aligned}\|r_{(e,t)}\chi_B - r_{(e,t_i)}\chi_B\|_2^2 &= \int_G |r_{(e,t)}\chi_B(x) - r_{(e,t_i)}\chi_B(x)|^2\, d\mu(x),\\ &= \int_G \chi_{B(e,t^{-1})\triangle B(e,{t_i}^{-1})}(x)\, d\mu(x),\\ &= \mu\big(B\triangle B(e,{t_i}^{-1}t)\big) < \epsilon,\end{aligned}$$

for some i. Thus $G/H \times \{t_i\}_{i=1}^n$ is the required finite set.

In the light of these results we define $\mathcal{R}$, whenever $\overline{r(G)}$ is a compact topological group, to be the class of representations of G of the form

$$\pi_{|r(G)} \circ r\colon G \longrightarrow \mathcal{L}(\mathcal{H})$$

where $\pi\colon \overline{r(G)} \longrightarrow \mathcal{L}(\mathcal{H})$ is a continuous representation. Thus in effect we are transferring the representation theory of $\overline{r(G)}$ over to G. For compact right topological groups for which $\overline{r(G)}$ is compact there are therefore enough representations in $\mathcal{R}$ to separate the points of G by virtue of the fact that there are sufficiently many continuous representations for $\overline{r(G)}$. Furthermore since we can always find an inner product on $\mathcal{H}$ with respect to which any continuous representation, $\pi\colon \overline{r(G)} \longrightarrow \mathcal{U}(\mathcal{H})$, is unitary, it follows that we can do the same for any representation of G in $\mathcal{R}$. Finally $\mathcal{H}$ decomposes into an orthogonal sum of finite dimensional subspaces which are invariant under the action of $\overline{r(G)}$ and hence G.

It is worth asking to what extent the class, $\mathcal{R}$, determines the structure of a compact right topological group, G. For example is it possible to construct a strong normal system using only the representations in $\mathcal{R}$? We have looked at this issue but have so far been unable to prove any results in this direction.

3**. Examples.** We present here some examples to illustrate the features outlined in the previous section. In the following example, where G is equicontinuous, we reproduce some results from [4] to provide a concrete description of the compact group $\overline{r(G)}$.

EXAMPLE 4. Let ϕ be a discontinuous automorphism of the circle group $\mathbb{T}$ satisfying $\phi^2 = 1$. For example if we regard $\mathbb{T}$ as the direct sum of the subgroup,

$$\{z \in \mathbb{T} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}\},$$

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and $\mathfrak{c}$ copies of $\mathbb{Q}$ then we may define ϕ as interchanging the coordinates of two fixed copies of $\mathbb{Q}$. Let $G = \mathbb{T} \times \{1, \phi\}$ be the semidirect product with multiplication,

$$(u, \epsilon)(v, \delta) = \big(u\epsilon(v), \epsilon\delta\big),$$

and the product topology. Then G is a compact right topological group with topological centre $\Lambda(G) = \mathbb{T} \times \{1\}$. Now let f_γ denote the function in $L^2(G)$ supported on $\mathbb{T} \times \{\gamma\}$ and determined by the function $f \in L^2(\mathbb{T})$. Then by direct calculation it is easy to show that the map $g \longmapsto r_g f_\gamma$ is given by

$$(v, 1) \longmapsto (r_{\gamma(v)} f)_\gamma$$

and

$$(v, \phi) \longmapsto (r_{\gamma(\phi(v))} f)_{\gamma\phi}.$$

Consider now the flow $\big(G, \rho(G)\big)$ defined by the action of G on itself by right translation, *i.e.*, $G \longrightarrow G^G: g \longmapsto \rho_g$. The family of maps, $\{\rho_g \mid g \in G\}$, is equicontinuous which implies that the Ellis group of the flow (G, ρ_G) is a compact topological group.

The Ellis group, $\overline{\rho_G}$, is given by the set,

$$\mathbb{T} \times \mathbb{T} \times \{1, \phi\},$$

with multiplication defined by,

$$(u_1, u_2, 1)(v_1, v_2, \delta) = (u_1 v_1, u_2 v_2, \delta)$$

and

$$(u_1, u_2, \phi)(v_1, v_2, \delta) = (u_1 v_2, u_2 v_1, \phi\delta).$$

The map $\theta: G \longrightarrow \overline{\rho_G}$ defined by,

$$(v, \delta) \longmapsto \big(v, \phi(v), \delta\big),$$

is a discontinuous isomorphism of G onto a dense subset of $\overline{\rho_G}$. Moreover the map $\pi: \overline{\rho_G} \longrightarrow U\big(L^2(G)\big)$ defined by

$$\pi(v_1, v_2, \delta) f_\gamma = \begin{cases} (r_{v_1} f)_1 & \text{if } \gamma = \delta \\ (r_{v_2} f)_\phi & \text{if } \gamma \neq \delta \end{cases}$$

is a continuous representation of $\overline{\rho_G}$ and $r = \pi \circ \theta: G \longrightarrow U\big(L^2(G)\big)$. Consequently, since $\theta(G)$ is dense in $\overline{\rho_G}$, it follows from the continuity of π that $r(G) = \pi \circ \theta(G)$ is dense in the compact group $\pi(\overline{\rho_G})$ and hence $\overline{r(G)} = \pi(\overline{\rho_G})$. The continuous representations of $\pi(\overline{\rho_G})$ therefore determine the representations of G which belong to the class $\mathcal{R}$. ■

Next we turn to an example, G, which satisfies the hypothesis of Theorem 3 and also shows that equicontinuity is not a necessary condition for the compactness of $\overline{r(G)}$.

EXAMPLE 5. Let G be the algebraic Schreier product of $\{\pm 1\}$ and $\mathbb{T}$ with multiplication given by (2),

$$(\epsilon, u)(\delta, v) := (\epsilon\delta, u^\delta v).$$

Endow G with the topology, τ, for which a basic open and closed neighbourhood of the points $(1, e^{ia})$ or $(-1, e^{ib})$, where $a < b$, is of the form

$$U := \{(1, e^{ia}), (-1, e^{ib})\} \cup \{(\epsilon, e^{i\theta}) \mid \epsilon = \pm 1, a < \theta < b\}.$$

Then (G, τ) is a compact right topological group with trivial topological centre. However it is well-known (see for example [5]) that G is not equicontinuous. Unlike the other examples in this paper, G does not have a strong normal system (for details, see [6]). The group G does, however, have Haar measure given by $\mu(B) = \lambda\big(B \cap (\{1\} \times \mathbb{T})\big)$ for any Borel set $B \subset G$ where λ is the usual Lebesgue measure on $\mathbb{T} \cong \{1\} \times \mathbb{T}$. The map,

$$G \longrightarrow \mathbb{C} \colon g \longmapsto \mu(B \triangle Bg^{-1}),$$

when restricted to $\{1\} \times \mathbb{T}$ is continuous at the identity for all Borel sets, $B \subset G$. Indeed let $(1, e^{i\eta_n})$ be a sequence in G then

$$U(1, e^{-i\eta_n}) = \{(1, e^{i(a-\eta_n)}), (-1, e^{i(b-\eta_n)})\} \cup \{(\epsilon, e^{i(\theta-\eta_n)}) \mid \epsilon = \pm 1, a < \theta < b\}.$$

and hence

$$\mu\big(U \triangle U(1, e^{-i\eta_n})\big) = 2\mu\big(U \setminus U(1, e^{-i\eta_n})\big) = 2|\eta_n|.$$

Thus we deduce the continuity at the identity of the above (restricted) map for any open or closed subset, U, of G and the result for all Borel subsets, B, of G follows from the regularity of the measure, μ. ■

We end this section with a counterexample, G, where $\overline{r(G)}$ fails to be compact. In this example the topological centre is dense and thus G is not equicontinuous.

EXAMPLE 6. Let $G = \mathbb{T} \times \mathbb{T}^{\mathbb{T}}$ have the product topology and multiplication given by,

$$(t, f)(s, g) := (ts, f \circ \rho_s g).$$

Then G is a compact right topological group with dense topological centre $\Lambda(G) = \mathbb{T} \times C(\mathbb{T})$. The family $L_0 = G$, $L_1 = \{1\} \times \mathbb{T}^{\mathbb{T}}$ and $L_2 = \{(1, e)\}$ is a strong normal system for G and therefore Haar measure on G is given by the product of the measures on each of the groups $\mathbb{T}^{\mathbb{T}}$ and $\mathbb{T}$. Indeed G is the Schreier product of $G/L_1 \cong \mathbb{T}$ and $L_1 \cong \mathbb{T}^{\mathbb{T}}$ with multiplication given by (2). Now let $(s, g) \in G$. We denote a subbasic open neighbourhood of $\mathbb{T}^{\mathbb{T}}$ by

$$N(t; U) := \{f \in \mathbb{T}^{\mathbb{T}} \mid f(t) \in U\},$$

where U is open in $\mathbb{T}$. Choose any open set of the form,

$$W = V \times N(t; U) \subset \mathbb{T} \times \mathbb{T}^{\mathbb{T}},$$

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where the sets U and V are both open in $\mathbb{T}$. Then

$$W(s,g) = Vs \times N\big(s^{-1}t; g(t)U\big)$$

and hence for $s \neq e$,

$$\begin{aligned}
&\mu\big(W \triangle W(s,g)\big)\\
&\quad = 2\mu\big(W \setminus W(s,g)\big)\\
&\quad = 2\Big\{\mu\big((V \setminus Vs) \times N(t;U)\big) + \mu\Big((V \cap Vs) \times \Big(N(t;U) \cap N\Big(s^{-1}t; \big(g(t)U\big)^c\Big)\Big)\Big)\Big\}\\
&\quad = 2\lambda(U)\big\{\lambda(V \setminus Vs) + \big(1-\lambda(U)\big)\lambda(V \cap Vs)\big\},
\end{aligned}$$

where λ is the usual Lebesgue measure on $\mathbb{T}$. In particular if we choose $V = \mathbb{T}$ then $\mu\big(W \triangle W(s,g)\big) = 2\lambda(U)\big(1-\lambda(U)\big)$. Thus by choosing $\epsilon < 2\lambda(U)\big(1-\lambda(U)\big)$ it follows from the above calculation and the identity,

$$\begin{aligned}
\|r_{(t,f)}\chi_W - r_{(t',f')}\chi_W\| &= \|\chi_W - r_{(t,f)^{-1}(t',f')}\chi_W\|_2\\
&= \|\chi_W - r_{(t^{-1}t', f^{-1}\circ\rho_{t't^{-1}}f')}\chi_W\|_2
\end{aligned}$$

that it is impossible to find finitely many elements in G to furnish an open cover of $r_G\chi_U$ by ϵ-balls since any such choice of elements must include a set of the form $\big\{(t,f_t) \mid t \in \mathbb{T}\big\}$ which is infinite! The set $\overline{r(G)}$ is therefore not compact. ∎

4. **A remark on Borel groups.** Let (G,τ) be a compact, right topological group and suppose moreover that (G,τ) has Haar measure μ. We say that G is a *Borel group* if the map,

$$G \times G \longrightarrow G\colon (g,h) \longmapsto g^{-1}h,$$

is Borel measurable. Note that if G is a Borel group then for $f \in L^1(G)$ and $g \in L^\infty(G)$ then,

$$f \diamond g(s) := \int_G f(st)g(t^{-1})\,d\mu(t),$$

is well-defined and is finite for almost all $s \in G$.

THEOREM 7. *Let G be an equicontinuous group which is also a Borel group. Then G is a topological group.*

PROOF. We begin by showing that for $f \in L^1(G)$ and $g \in L^\infty(G)$ the map,

$$s \longmapsto f \diamond g(s) = \int_G f(st)g(t^{-1})\,d\mu(t),$$

is continuous. Let $f \in C(G)$. Then since G is equicontinuous, inversion is continuous at the identity and therefore f is uniformly continuous with respect to the uniformity (see [10]),

$$\mathcal{V} := \{(s,t) \mid ts^{-1} \in V, V \text{ an open neighbourhood of } e \text{ in } G\}.$$

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This implies that for $\epsilon > 0$, there exists an open neighbourhood, V, of the identity such that for any $s, s' \in G$ such that $s's^{-1} \in V$,

$$\begin{aligned} |f \diamond g(s) - f \diamond g(s')| &\leq \int_G |f(st) - f(s't)||g(t^{-1})|\, d\mu(t) \\ &< \epsilon \|g\|_\infty. \end{aligned}$$

Hence $s \longmapsto f \diamond g(s)$ is continuous. Now let f be any element in $L^1(G)$. Then for any $\epsilon > 0$ there exists a continuous function, ϕ, such that $\|f - \phi\|_1 < \epsilon$. Since,

$$\begin{aligned} \|f \diamond g - \phi \diamond g\|_\infty &= \sup_{s\in G} |(f - \phi) \diamond g(s)| \\ &\leq \|f - \phi\|_1 \|g\|_\infty \\ &< \epsilon \|g\|_\infty, \end{aligned}$$

$f \diamond g$ is the uniform limit of continuous functions and is therefore itself continuous.

Finally we apply this result to the characteristic functions χ_{B_1} and χ_{B_2}, where B_1 and B_2 are Borel sets in G, to get that the map,

$$s \longmapsto \chi_{B_1} \diamond \chi_{B_2}(s) = \mu(s^{-1}B_1 \cap {B_2}^{-1}),$$

is continuous. In particular, for any Borel set, B, if we let $B_1 = B$ and $B_2 = B^{-1}$ we deduce the continuity of the map, $s \longmapsto \mu(s^{-1}B \triangle B) = \|l_s\chi_B - \chi_B\|_2$ where l is the faithful antirepresentation $l_s f(t) := f(st)$. This implies that G is topological.

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