COMMUTATIVITY FOR MATRICES OF QUATERNIONS

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1. Introduction. For any ring $\mathscr{R}$ we shall denote by $\mathscr{R}_n$ the ring of all $n \times n$ matrices with elements from $\mathscr{R}$ and by $\mathscr{R}[x]$ the set of all polynomials in x with coefficients from $\mathscr{R}$.

$\mathfrak{Q}$ will denote the non-commutative four-dimensional division algebra of real quaternions with $1, i_1, i_2, i_3$ as generators

$$(i_1{}^2 = i_2{}^2 = -1,\ i_1 i_2 = -i_2 i_1 = i_3).$$

We shall not distinguish between the real field $\mathfrak{R}$ and the subfield of $\mathfrak{Q}$ generated by 1. Also, we shall not distinguish between the complex field $\mathfrak{C}$ and the subfield of $\mathfrak{Q}$ generated by 1 and i_1.

If $\mathfrak{D}$ is any subset of $\mathscr{R}_n$ we shall denote the centralizer of $\mathfrak{D}$ in $\mathscr{R}_n$ by $C(\mathfrak{D}) = \{A \in \mathscr{R}_n | AD = DA, \text{ for all } D \in \mathfrak{D}\}$. In particular, if $A \in \mathscr{R}_n$, then $C(A)$ is the set of all matrices in $\mathscr{R}_n$ which commute with A and $C^2(A) = C(C(A))$ is the set of all matrices in $\mathscr{R}_n$ which commute with every matrix which commutes with A. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are subsets of $\mathscr{R}_n$ such that $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then we observe that $C(\mathfrak{D}_1) \supseteq C(\mathfrak{D}_2)$.

Our main purpose in this paper is to prove that $C^2(A) = \mathfrak{R}[A]$ for any $A \in \mathfrak{Q}_n$. This is a generalization of the following well-known result for matrices over a field.

THEOREM 1. *Let $\mathfrak{F}$ be a field and $A \in \mathfrak{F}_n$. Then $C^2(A) = \mathfrak{F}[A]$* (**2**, Theorem 5–19).

The development in this paper will depend heavily upon the following theorem about matrices of quaternions due to Wiegmann (**5**, Theorem 1).

THEOREM 2. *For any $A \in \mathfrak{Q}_n$ there exists a non-singular matrix $P \in \mathfrak{Q}_n$ such that $P^{-1}AP = J \in \mathfrak{C}_n$ is in Jordan canonical form with characteristic values (the diagonal elements of J) having non-negative imaginary parts.*

Since the diagonal blocks in the Jordan form may appear in any order, we have the following corollary to Theorem 2.

COROLLARY 2.1. *For any $A \in \mathfrak{Q}_n$ there exists a non-singular matrix $P \in \mathfrak{Q}_n$ such that the matrix of Theorem 2 is*

$$P^{-1}AP = J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix},$$

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where J_{11} has real entries and the characteristic values of J_{22} have positive imaginary parts.

2. Commutativity in $\mathfrak{Q}_n$. We begin by investigating the structure of the subalgebra $C^2(J)$, where J is a Jordan canonical matrix of the type described in Corollary 2.1. We shall then be led directly to the more general result about $C^2(A)$ for any $A \in \mathfrak{Q}_n$.

THEOREM 3. *Let*

$$J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \in \mathfrak{C}_n \subset \mathfrak{Q}_n$$

be a Jordan canonical matrix, where J_{11} is real and J_{22} has characteristic values with positive imaginary parts. If $A \in C^2(J)$, then

(1) $A \in \mathfrak{C}[J]$,

(2) $A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$, *where A_{11} has only real entries.*

Proof. $C(J) \supseteq C(J) \cap \mathfrak{C}_n$; hence $C^2(J) \subseteq C(C(J) \cap \mathfrak{C}_n)$ and, by Theorem 1, $C^2(J) \cap \mathfrak{C}_n \subseteq C(C(J) \cap \mathfrak{C}_n) \cap \mathfrak{C}_n = \mathfrak{C}[J]$.

It remains to show that $C^2(J) \cap \mathfrak{C}_n = C^2(J)$. If $A \in C^2(J)$, then A commutes with every matrix in $C(J)$; in particular, $A(i_1 I) = (i_1 I)A$, which, since the centralizer of i_1 in $\mathfrak{Q}$ is $\mathfrak{C}$, implies that $A \in \mathfrak{C}_n \subset \mathfrak{Q}_n$. Therefore, $C^2(J) \subseteq C^2(J) \cap \mathfrak{C}_n \subseteq \mathfrak{C}[J]$, which completes the proof of (1).

If $A \in C^2(J)$, then $A \in \mathfrak{C}[J]$ so that A has the form

$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

The matrix

$$B = \begin{bmatrix} i_2 I & 0 \\ 0 & I \end{bmatrix}$$

is in $C(J)$; hence $AB = BA$ and $A_{11}(i_2 I) = (i_2 I)A_{11}$, which, since the centralizer of i_2 in $\mathfrak{C}$ is $\mathfrak{R}$, implies that A_{11} has only real entries, and the proof is complete.

From Theorem 3 we have that

$$(2.1) \qquad C^2(J) \subseteq \{p(J) \mid p(x) \in \mathfrak{C}[x] \text{ and } p(J_{11}) \text{ is real}\} \subseteq \mathfrak{C}[J].$$

We shall show that if $p(x) \in \mathfrak{C}[x]$ such that $p(J_{11})$ is real, then we can find a real polynomial $q(x)$ such that $q(J) = p(J)$.

THEOREM 4. *Let*

$$J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}$$

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be a Jordan canonical matrix such that J_{11} is real and the characteristic values of J_{22} have positive imaginary parts. Let $p(x) \in \mathfrak{C}[x]$ such that $p(J_{11})$ is real. Then there exists a polynomial $q(x) \in \mathfrak{R}[x]$ such that $p(J) = q(J)$.

*Proof.** Let $r_1, r_2, \ldots, r_s$ be the distinct characteristic values of J_{11} (all real) and let $\lambda_1, \lambda_2, \ldots, \lambda_t$ be the distinct characteristic values of J_{22} (non-real with positive imaginary parts). Let the largest simple Jordan block associated with r_i be $k_i \times k_i$ and let the largest simple Jordan block associated with λ_j be $l_j \times l_j$.

Since $\bar{\lambda}_j \neq \lambda_j$, let $q(x) \in \mathfrak{C}[x]$ be the unique polynomial (Lagrange–Hermite interpolation polynomial) of degree less than $\sum k_i + 2\sum l_i$ which satisfies

$$(2.2) \qquad \begin{aligned} &q^{(n)}(r_i) = p^{(n)}(r_i), && i = 1, 2, \ldots, s;\ n = 0, 1, \ldots, k_i - 1, \\ &\left.\begin{aligned} q^{(m)}(\lambda_j) &= p^{m}(\lambda_j) \\ q^{(m)}(\bar{\lambda}_j) &= \overline{p^{m}(\lambda_j)} \end{aligned}\right\} && \begin{aligned} j &= 1, \ldots, t; \\ m &= 0, 1, \ldots, l_j - 1. \end{aligned} \end{aligned}$$

The first two of these inequalities imply that $p(J) = q(J)$ (**2**, p. 178). Now the polynomial $\bar{q}(x)$ is easily seen to fulfil the relations (2.2), e.g.

$$\bar{q}^{(m)}(\lambda_j) = \overline{q^{(m)}(\bar{\lambda}_j)} = p^{(m)}(\bar{\lambda}_j).$$

The uniqueness of $q(x)$ thus implies that $q(x) = \bar{q}(x)$ and hence that $q(x) \in \mathfrak{R}[x]$.

From equation (2.1) and Theorem 4 we have

COROLLARY 4.1. *If*

$$J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}$$

is a Jordan canonical matrix such that J_{11} is real and the characteristic values of J_{22} have positive imaginary parts, then $C^2(J) = \mathfrak{R}[J]$.

We conclude this section by showing that the conclusions of Corollary 4.1 hold for any $A \in \mathfrak{Q}_n$. This is a generalization of Theorem 1.

THEOREM 5. *For any $A \in \mathfrak{Q}_n$, $C^2(A) = \mathfrak{R}[A]$.*

Proof. Let $A \in \mathfrak{Q}_n$. Using Corollary 2.1 we can find a non-singular matrix $P \in \mathfrak{Q}_n$ such that

$$P^{-1}AP = J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}$$

is in Jordan canonical form, satisfying the hypotheses of Corollary 4.1. If $K \in C(J)$, then $KJ = JK$ and $PKP^{-1} = PJP^{-1}\,PKP^{-1}$; hence

$$(PKP^{-1})A = A(PKP^{-1})$$

so that $PKP^{-1} \in C(A)$.

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Choose $D \in C^2(A)$. Then

$$K(P^{-1}DP) = P^{-1}[(PKP^{-1})D]P = P^{-1}[D(PKP^{-1})]P = (P^{-1}DP)K$$

which shows that $P^{-1}DP \in C^2(J) = \mathfrak{R}[J]$.

Let $p(x) \in \mathfrak{R}[x]$ be such that $p(J) = P^{-1}DP$. Since $p(x)$ has real coefficients, it follows that $D = P(p(J))P^{-1} = p(PJP^{-1}) = p(A)$, showing that $C^2(A) \subseteq \mathfrak{R}[A]$. The other inclusion is apparent and hence

$$C^2(A) = \mathfrak{R}[A].$$

3. Applications to functions on $\mathfrak{Q}_n$. Theorem 6 has an interesting application to the theory of intrinsic functions on $\mathfrak{Q}_n$. A function F, with range and domain in $\mathfrak{Q}_n$, is said to be intrinsic if $F(\Omega A) = \Omega F(A)$ for every A in the domain of F and for every Ω in the group of automorphisms and anti-automorphisms of $\mathfrak{Q}_n$. Intrinsic functions on a general linear algebra were motivated and studied by Rinehart **(3, 4)**.

Let F be an intrinsic function on $\mathfrak{Q}_n$ with A in the domain of F and $B \in C(A)$. If B^{-1} exists, then $B^{-1}F(A)B = F(B^{-1}AB) = F(B^{-1}BA) = F(A)$, which implies that $F(A)B = BF(A)$. If B is singular, then we can find a real scalar r such that $(B + rI)^{-1}$ exists, and it follows, as above, that $F(A)B = BF(A)$. Thus $F(A) \in C^2(A) = \mathfrak{R}[A]$. We have proved the following theorem.

THEOREM 6. *If F is an intrinsic function on $\mathfrak{Q}_n$ and A is in the domain of F, then the functional value $F(A)$ is a real polynomial in A.*

Theorem 6 indicates the existence of an error in the characterization of intrinsic functions on $\mathfrak{Q}_n$ given by Cullen **(1)**. The example given in **(1)** to show that not every intrinsic function on $\mathfrak{Q}_n$ has the property described in Theorem 6 is incorrect. The function described there can also be interpreted as a function from $\mathfrak{C}_n$ to $\mathfrak{C}_n$, where $\mathfrak{C}_n$ is interpreted as a $2n^2$-dimensional algebra over $\mathfrak{R}$, and with this interpretation still provides an example of an intrinsic function, on an algebra $\mathscr{A}$, which is not a polyfunction (functional values are not polynomials in the argument with coefficients from the ground field).

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