Fifth Meeting, March 11*th*, 1887.

Dr George Thom, President, in the Chair.

Historical Notes on a Geometrical Theorem and its Developments.

[18th century.]

By J. S. Mackay, LL.D.

The theorem is—

The distance between the circumscribed and the inscribed centres of a triangle is a mean proportional between the circumscribed radius and its excess above the inscribed diameter.

Or, in other words,

The potency of the inscribed centre of a triangle with respect to the circumscribed circle is equal to twice the rectangle under the inscribed and the circumscribed radii.

The notes are arranged as far as possible in chronological order, under the names of the various geometers who have turned their attention to the question. The first paper is given nearly in full, partly because the author of it seems to be totally unknown outside of the United Kingdom, and partly because the periodical in which the paper appeared is so rare and so difficult to get that it seems a sort of mockery to refer any one to it. Some few changes have been made on the lettering of the figures, and on the notation for certain lines.

WILLIAM CHAPPLE (1746).

The *Miscellanea Curiosa Mathematica* was begun, under the editorship of Francis Holliday, in the year 1745. It was to be published quarterly, but the fact that the first volume contained only nine numbers, and that the dedication prefixed to it is dated March 25, 1749, seems to show that it cannot have appeared at the regular intervals intended. Probably it is not rash to suppose that the fourth number was published in 1746. That number opens with "An Essay on the properties of triangles inscribed in, and circumscribed about two given circles; by Mr William Chapple" (pp. 117-124). Before entering upon his subject Chapple remarks:—

The following enquiry into the properties of triangles inscribed in, and circumscribed about given circles, has led me to the discovery of some things relating to them, which I presume have not been hitherto taken notice of, having not met with them in any author; though an ingenious correspondent of mine, in the isle of Scilly, to whom I communicated some of the propositions hereinafter demonstrated, informs me that he had begun to consider it some years ago, but did not go through with it; however, I must acknowledge that a query of his to me, relating thereto, gave me the first hint, and induced me to pursue the subject with more attention than perhaps otherwise I might have done.

PROPOSITION 1.

The areas of all triangles, circumscribed about the same circle, are as their perimeters.

For the areas of all are equal to their perimeters multiplied into half the radius.

PROPOSITION 2.

The areas of all triangles, inscribed in the same circle, are as the products of their sides.

For the areas of all are equal to the product of their sides divided by twice the diameter of their circumscribing circle.

COROLLARY.

The areas of all triangles, inscribed and circumscribed in and about the same circles, are not only as their perimeters, but also as the products of their sides.

PROPOSITION 3.

To inscribe and circumscribe a triangle in and about two concentric circles, the radius of the greater circle must be double the radius of the lesser, and the triangle will always be equilateral.

[This is so easily established that I omit Chapple's demonstration.]

PROPOSITION 4.

To inscribe and circumscribe a triangle in and about two eccentric circles, the radius of the lesser circle must be less than half the radius of the greater circle.

The area of any triangle is equal to the product of the radius of its inscribed circle into half the perimeter; therefore, if this proposition be true, the area of any triangle doth not exceed the product of half the radius of its circumscribed circle into half the perimeter. Now a triangle whose area is equal to the product of half the radius of its circumscribed circle into half the perimeter is equilateral, by Proposition 3. So that, if this proposition be true, the greatest triangle that can be inscribed in any circle will be equilateral; which we come now to demonstrate.

[This also is so easily established that I omit Chapple's demonstration.]

Proposition 5.

An infinite number of triangles may be drawn, which shall inscribe and circumscribe the same two circles; provided their diameters, with respect to each other, be limited as in the two last propositions.

For, put x, y, and z equal to the sides of any triangle circumscribed about a circle whose radius is r and inscribed in a circle whose radius is R. Then $xyz/4R =$ the area of the triangle,

and $(x+y+z)\,r/2$ = the area.

Therefore $$xyz = 2\,Rr\,(x+y+z).$$

Hence it is plain that if the sides of the triangle were required to be found by the given diameters, the question would be capable of innumerable answers; for one of the sides at least is unlimited, and may be put equal to anything at pleasure, that doth not exceed the longest line that can be drawn within the great circle as a tangent to the lesser. And hence it appears, that if the distance of the given circles be so fixed, as that any one triangle may be inscribed and circumscribed, innumerable others may be inscribed and circumscribed in and about the same circles.

Proposition 6.

The nearest distance of the peripheries of two given circles, or, which amounts to the same, the distance of their centres, in order to render it possible to inscribe and circumscribe triangles, is fixed and will be always the same.

Let two circles be so situated as that a triangle ABC can be circumscribed and inscribed; then innumerable others may be inscribed and circumscribed; but it is evident by inspection that if the lesser circle be anyhow removed from its place, the sides of those triangles cannot be tangents to it; and therefore the situation of the

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two circles must be the same for them all. But if it be suspected that their situation may be altered, and yet other innumerable triangles may be circumscribed and inscribed (as the contrary is not yet made appear) let it be considered, that the lesser circle must, in its removal, move along on some one or other of the sides of the innumerable triangles that might be inscribed and circumscribed at its *first situation*, and then wherever it stops (unless it be at the same distance from the periphery of the great circle on the other side of the centre, which is in effect the same situation) the like inconveniency will follow. For it is well known that triangles circumscribed about equal circles and having one common base, will continually increase their altitude, the further the point of contact in the said base is removed from the middle thereof, till at length the two lines drawn from the extremities thereof become parallel to or diverging from each other, and the altitude of the triangle becomes infinite; consequently (when the two circles are eccentric) the vertices of *only two* of the circumscribing triangles that can be erected on the base AB can be at the periphery of the circumscribing circle; and in either of these the distance of the peripheries of the greater and lesser circles must be the same, and the triangles the same, having the same base and altitude. So that the proposition is abundantly proved; and hence we have the following

COROLLARY.

If one side of a triangle, inscribed in and circumscribed about two given circles, be given, the other two sides are thereby limited, and may from thence be found.

[For the method of finding them Chapple refers to the answer to a question he had proposed for solution in the *Miscellanea*. The question and answer will be given later on.]

PROPOSITION 7.

Of the innumerable triangles that may be inscribed and circumscribed in and about two given eccentric circles, two must of course be isosceles, the vertices of which will be in the common diameter of those circles, which will cut their bases at right angles; now the content of that isosceles triangle which hath the least base and greatest altitude will be the greatest, *and that of the other the* least *of all the triangles that can be inscribed and circumscribed in the given circles.*

See figure 71.

Let S and I be the centres of the greater and less circles, and let the two isosceles triangles ABC, DEF, whose vertices are at the ends of the common diameter AD, be inscribed in the one circle and circumscribed about the other; ABC is the greatest and DEF the least of the inscribed and circumscribed triangles.

Let AD meet BC at H; then H is the middle point of BC.
Find K the middle point of AB, and join CK, meeting AH in G.
Then $AG = \frac{2}{3}AH$, $CG = \frac{2}{3}CK$, $\triangle AGC = \frac{1}{3}\triangle ABC$.

Now innumerable lines may be drawn in the semicircle AED, all of which shall be tangents to the lesser circle, and sides of inscribed and circumscribed triangles, and such lines must always be less than AC, because farther removed from S, the centre of the great circle, and will continually decrease according as their extremities are farther removed from A, so that ED will be the least of all of them. Let LN be any one of those lines, and let the circumscribing and inscribed triangle LMN be drawn; also the lines LP and NQ bisecting MN and ML, and crossing each other in O, the centre of gravity of this triangle; so that $\triangle LON = \frac{1}{3}\triangle LMN$.

Because an angle of any triangle is greater or less according as it is nearer to or farther from the centre of the inscribed circle, therefore $\angle L$ is greater than $\angle A$;
therefore LP is less than AH;
therefore $\frac{2}{3}$ LP is less than $\frac{2}{3}$ AH, that is, LO is less than AG.
Similarly NO is less than CG.
Now LN, being also less than AC, the area of $\triangle LON$ is less than the area of $\triangle AGC$;
therefore $\triangle ABC$ is greater than $\triangle LMN$.

Hence it plainly appears that the area of any circumscribing and inscribed triangle which hath one angle anywhere between A and E is less than the area of the isosceles triangle ABC, and greater than the isosceles triangle DEF. And that this takes in all the cases that can possibly happen will be evident to any one that considers that if the angle L be between A and E, the angle N will be somewhere between C and D, and the angle M somewhere between B and F; so that this takes in one-half of the periphery; and the arc AF being equal to AE, and BF equal to CE, it is but changing sides with the circle, and we have the same set of triangles again; for ...ver the triangle be, if it be not isosceles, we may have anether

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on the other side of the centre of the great circle, which shall be equal to it.

From the three foregoing propositions we may deduce the following

COROLLARY.

If the distance of the centres of the two given circles be not so fixed as that an isosceles triangle can be inscribed and circumscribed, no triangle whatsoever can be inscribed and circumscribed.

And this suggests an easy method of finding the distance of the centres of the said given circles, and consequently the nearest distance of their peripheries; for here we may always consider the triangle as isosceles, and then the distance is found as follows.

See figure 71.

Put $AH = x$, $CH = y$, $SA = R$, $IH = r$;

then $DH = 2R - x$, and $CH = y = \sqrt{2Rx - x^2}$;

also $AC = \sqrt{x^2 + y^2}$.

Now, sine of $\angle CAH = \frac{CH}{AC} = \frac{y}{\sqrt{x^2 + y^2}}$,

and sine of $\angle CAH = \frac{IT}{IA} = \frac{r}{x - r}$;

therefore $\frac{y}{\sqrt{x^2 + y^2}} = \frac{r}{x - r}$.

In this equation substitute for y the value $\sqrt{2Rx - x^2}$, and complete the solution; then

$$x - R - r = \sqrt{R^2 - 2Rr} = \text{the distance of the centres};$$

therefore $x = AH = R + r + \sqrt{R^2 - 2Rr}$

and $DH = R - r - \sqrt{R^2 - 2Rr}$, the nearest distance of the peripheries.

Note—That R must be equal to or greater than $2r$, or else the root is impossible, which is another proof of the fourth proposition.

The question proposed for solution by Chapple (Vol. I., p. 143) and answered by him (Vol. I., pp. 171-3) was—

Let the two diameters of the circumscribing and inscribed circles of a triangle be 94 *and* 40, *and one of its sides* 90; *quere the other two sides.*

Denoting the radius of the inscribed circle by r, that of the circumscribed by R, the given side by a, and the distance between the middle of the given side and the point of inscribed contact on that side by e, Chapple deduces the expression

$$\sqrt{2Rr+e^2+\frac{4R^2r^2}{a^2}}+\frac{2Rr}{a}\pm e$$

for the two unknown sides, the affirmative value of e being taken for the greater, and the negative for the less.

Another question proposed by Chapple (Vol. I., p. 185) and answered by him (Vol. I., p. 196) was—

Given the hypotenuse of a right-angled triangle equal to 100, *and the nearest distance of the right angle from the periphery of the inscribed circle equal to* 8 ; *required the legs and area of the triangle by a simple equation, with a general theorem for questions of this nature.*

The general theorem given by Chapple at the end of his solution for finding the legs of right-angled triangles is

$$R+r\pm e=\left.\begin{matrix}\text{greater}\\ \text{lesser}\end{matrix}\right\}\text{ leg.}$$

He adds a corollary: The sum of the two legs of any right-angled triangle is equal to the sum of the diameters of the inscribed and circumscribing circles.

In order to draw attention to his paper in the *Miscellanea Curiosa Mathematica*, Chapple proposed the following as the prize question in the *Ladies' Diary* for 1746 :—

A gentleman has a circular garden whose diameter is 310 *yards, in which is contained a circular pond whose diameter is* 100 *yards, so situated in respect of each other that their peripheries will inscribe and circumscribe an infinite number of triangles (*i.e., *whose sides shall be tangents to the pond, and angles in the fence of the garden). He being disposed to make enclosures for different uses, and farther ornaments on his scheme begun, in order thereto applies himself to the artists of Great Britain for the dimensions of the greatest and least triangles that can be inscribed and circumscribed as aforesaid ? and the nearest distance of the peripheries of the garden and pond ? and for a demonstration of the truth of the pond's situation ?*

Robert Heath (1747).

Chapple's prize question was answered somewhat unsatisfactorily next year by Robert Heath, who was then editor of the *Ladies' Diary*. At the end of his solution he says :—This property of drawing triangles about circles I discovered some years ago, as may be seen in the *Monthly Oracle;* though the proposer has greatly deserved in a long account of it from Scilly, printed in a book called the *Quarterly Miscellanea Curiosa.*

Landen, who will be mentioned later on, seems also to have answered the question. He gave in an inconvenient form the expression

$$R + r \pm \sqrt{R^2 - 2Rr}$$

for the altitudes of the two isosceles triangles, and a construction for finding the centre of the pond.

JOHN TURNER (1748).

Another mathematical periodical, *The Mathematician* (London, 1751), edited, according to the title-page, "By a Society of Gentlemen," according to T. S. Davies by Turner, made its first appearance in 1745. Six numbers only were published, probably annually, and in the fourth of these Turner, who had been a frequent contributor to the *Miscellanea Curiosa Mathematica,* and who had answered in it the first of the questions here mentioned as proposed by Chapple, reproposed it (p. 256) in this form :—

One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically.

From his own solution given in the next number (pp. 311-313), I extract the construction, which is as simple as possible. His demonstration is not simple enough to be worth reproducing.

See figure 72.

Upon any point S, with an interval equal to the given radius of the circumscribing circle, let the circle BACV be described, in which apply the right line BC equal to the given side of the triangle. Bisect BC with the indefinite perpendicular UV; in which take HT equal to the radius of the inscribed circle. Through T draw a parallel to BC, and upon V the point where UV intersects the circle, and with the distance VB, describe an arch cutting the parallel to BC in I; then, if through I the line VIA be drawn meeting the circle in A, and BA CA be joined, ABC will be the triangle required.

JOHN LANDEN (1755).

In his *Mathematical Lucubrations* (London, 1755) Landen devotes Part I. pp. 1–24 to "an investigation of a remarkable property of triangles described in a certain manner about a circle or an ellipsis." The investigation is given in a proposition which is subdivided into three cases with six figures, and followed by eight corollaries and the solution of twenty examples. Two cases and two figures only are reproduced here, these being all that are really required.

PROPOSITION.

See figures 74, 75.

The two circles ABM, DEF, whose centres are O and I respectively, being given in magnitude and position, let any given chord AB in the circle ABM touch the circle DEF at F; and from the extremities of that chord let two other tangents, AC BC, be drawn to the circle DEF, touching it at E and D, and intersecting each other at C: it is proposed to find the radius AS of the circle ABU circumscribing the triangle ABC.

Bisecting AB by a right line HM at right angles thereto, that line will pass through O, the centre of the given circle ABM, and also through S, the centre of the circumscribing circle ABU. Draw ID, IE, IF to the points of contact D, E, F; join CI, and draw IT parallel to AB intersecting HM in T. Join AU.

Let $\quad AO = R,\ IF = r,\ IO = d,\ AH = b,\ AS = x;$

then $\quad OH = \pm\sqrt{R^2 - b^2},\ OT = \pm\sqrt{R^2 - b^2} - r,$

$$IT = FH = \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

$$AF = AE = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}.$$

$$BF = BD = b \sim \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

Case 1. When the circle DEF falls within the triangle ABC.

$$EF = \pm\sqrt{x^2 - b^2},\ GF = x \pm \sqrt{x^2 - b^2}.$$

From the similar triangles AHU, IEC an expression is found for CE or CD, which being added to the values of AE and BD found above, will give the values of AC and BC. Thence is obtained the value of AC + BC + AB, which being multiplied by $\frac{1}{2}r$ gives the area of triangle ABC.

Another expression for the area of triangle ABC is got by considering it as equal to half the rectangle contained by AC, BC and the sine of angle ACB, which is b/x.

If these expressions be equated to each other, and the equation solved, the value of x or AS will be found to be

$$x = \frac{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}{4r} + \frac{b^2 r}{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}.$$

Landen remarks that if d^2 be equal to $R^2 - 2rR$, whatever b may be, x will be equal to R.

Case 2. When the circle DEF falls without the triangle ABC, the method of procedure is the same as before, the only changes being that GF is now $\pm\sqrt{x^2 - b^2} - x$, and that the area of triangle ABC is obtained by multiplying AC + BC − AB by $\frac{1}{2}r$. In this case the value of x or AS is found to be

$$x = \frac{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}{4r} + \frac{b^2 r}{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}.$$

Landen remarks that if in this case d^2 be equal to $R^2 + 2rR$, whatever b may be, x will be equal to R. He adds that the sign proper to $\sqrt{R^2 - b^2}$ or to $\sqrt{x^2 - b^2}$ is the upper or lower one, according as O and I, or according as S and I are on the same or contrary sides of AB.

Cor. 1. It follows from what has been said that, d being equal to $\sqrt{R^2 - 2rR}$ or $\sqrt{R^2 + 2rR}$, whatever b may be, S will fall in O, and the circle circumscribing the triangle always coincide with the given circle ABM; *a thing very remarkable!*

For this to happen and the circle DEF fall within the triangle, it is obvious R must not be less than $2r$. But, that circle falling without the triangle, the same thing may happen though R be less than $2r$, so that R be greater than $\frac{1}{4}r$.

Cor. 2. "In orthographic projections, circles having the same inclination to the plane of projection being projected into similar ellipses, and any tangent of a circle projected into an ellipsis being likewise, when projected therewith, a tangent to that ellipsis: it follows that, if, within or without any ellipsis whose transverse axis is T, a second concentric similar ellipsis be described with its transverse axis t in the same direction with T; and a third ellipsis be described, similar to the other two, with its centre anywhere in the periphery of the second ellipsis, and having its transverse axis equal to $\frac{T^2 \sim t^2}{2T}$, and parallel to the transverse axes of the other ellipses; any tangent being drawn to this third ellipsis and continued both ways till it intersects the periphery of the first ellipsis in two points, and two other tangents being drawn to the same third ellipsis from those points of intersection, the locus where these last tangents continued intersect each other will always be in the periphery of the first ellipsis.

"The drawing of tangents in that manner will be impossible unless t be less than $3T$."

Cor. 3. Here and in the examples, by *the* escribed circle is understood the circle touching the base and the other two sides produced; its radius is denoted by ρ, and the distance of its centre from the circumscribed centre by δ.

The corollary gives, in terms of the base, the circumscribed radius and the inscribed or escribed radius, twenty-six expressions for the other two sides of the triangle, the sum, difference, and product of those sides, the perimeter of the triangle, the area, the perpendicular from the vertex to the base, the distances of the circumscribed centre from the inscribed and escribed centres, and the distances of these latter centres from the vertex. The only expression important enough to quote is

$$\delta = \sqrt{R^2 + 2\rho R}.$$

Cor. 4. This consists of twenty-two expressions for the other two sides of the triangle, the sum, difference, and product of those sides, the perimeter of the triangle, the area, the perpendicular from the vertex to the base, and the distances of the circumscribed centre from the inscribed and escribed centres in terms of the base, the inscribed or escribed radius, and the sine of the vertical angle or the cotangent of half that angle.

Cor. 5. Here the triangle is supposed to be right-angled, the base becoming the hypotenuse. Seventeen expressions in terms of the circumscribed radius and the inscribed or escribed radius are given for the other two sides, their sum, difference, product, the perimeter, and the perpendicular from the vertex to the hypotenuse. The following are worth noting:

The product of the sides about the right angle

$$= 4rR + 2r^2 = 2r\rho = -4\rho R + 2\rho^2;$$

the perpendicular from the vertex to the hypotenuse

$$= 2r + \frac{r^2}{R} = \frac{r\rho}{R} = -2\rho + \frac{\rho^2}{R};$$

the perimeter $= 2\rho$.

The sixth, seventh, and eighth corollaries are of little interest.

The following is a statement of the examples with whose solution Landen's essay concludes:—

1. Let the radii of the circles circumscribed about, and inscribed in a triangle be 5 and 2·2 respectively, and one of its sides 8; to find the other two sides.

2. Given R, r, and the distance of the inscribed centre from the vertex, to find the base.

3. Given R, r, and the perpendicular from the vertex to the base, to find the base.

4. Given R, r, and ρ, to find the base.

5. Given R, r, and the perimeter, to find the corresponding triangle.

6. Given R, ρ, and the excess of the other two sides above the base, to find the corresponding triangle.

7. Given R and r, to find the triangle when its area is a maximum or a minimum.

8. Given the vertical angle, r, and ρ, to find the base.

9. Given the vertical angle, the sum of the sides containing it, and the perpendicular from it to the base, to find the base.

10. Given the vertical angle, the perpendicular from it to the base, and the perimeter, to find the base.

11. Given the vertical angle, the difference of the sides containing it, and the perimeter, to find the base.

12. Given the vertical angle, the product of the sides containing it, and the perimeter, to find the base.

13. Given the vertical angle, the sum, and the product of the three sides, to find the base.

14. Given the vertical angle, the distance of the inscribed centre from the vertex, and d, to find the base.

15. Given the vertical angle, d, and δ, to find the base.

16. Given the vertical angle and d, to find the triangle when the base is the greatest possible.

17. Given r, ρ, and the sum of the two sides, to find the base.

18. Given r, ρ, and the difference of the two sides, to find the base.

19. Given the three escribed radii, to find the triangle.

20. Given the inscribed radius and two escribed radii, to find the triangle.

LEONHARD EULER (1765).

In the *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tom. xi. (pp. 103–123), for the year 1765, occurs a paper* of Euler's, entitled, *Solutio facilis problematum quorumdam geometricorum difficillimorum.* He there determines the distance between the inscribed and circumscribed centres of a triangle in the following way.

See figure 73.

Let I and S be the inscribed and circumscribed centres of the triangle ABC. Draw IF, SL perpendicular to AB; join AS, and draw AM perpendicular to BC.

Denote BC, CA, AB by a, b, c, and let area $=$ A, $a+b+c=p$, $ab+ac+bc=q$, $abc=r$.

Then $$\mathrm{IF}=\frac{2\mathrm{A}}{a+b+c},\quad \mathrm{AF}=\frac{c+b-a}{2}.$$

From the similar triangles ACM, ASL,

$$\mathrm{AM} : \mathrm{CM} = \mathrm{AL} : \mathrm{SL};$$

that is, $$\frac{2\mathrm{A}}{a} : \frac{a^2+b^2-c^2}{2a} = \frac{c}{2} : \mathrm{SL};$$

therefore $$\mathrm{SL}=\frac{c(a^2+b^2-c^2)}{8\mathrm{A}}.$$

Hence $$\mathrm{AF}-\mathrm{AL}=\frac{c+b-a}{2}-\frac{c}{2}=\frac{b-a}{2};$$

and $$\mathrm{IF}-\mathrm{SL}=\frac{2\mathrm{A}}{a+b+c}-\frac{c(a^2+b^2-c^2)}{8\mathrm{A}},$$

$$=\frac{(a+b)c^3+(a^2+b^2)c^2-(a+b)(a^2+b^2)c-(a^2-b^2)^2}{8(a+b+c)\mathrm{A}}.$$

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*An abstract is given of it in the *Proceedings* for last year. See Vol. IV., pp. 51-55.

Now $IS = (AF - AL)^2 + (IF - SL)^2$,

$$= \frac{abc}{16(a+b+c)^2A^2}\left\{\begin{array}{l} + a^5 + a^4b + ab^4 + abc^3 - 2a^3b^2 - 2a^2b^3 \\ + b^5 + a^4c + ac^4 + ab^3c - 2a^3c^2 - 2a^2c^3 \\ + c^5 + b^4c + bc^4 + a^3bc - 2b^3c - 2^2b^2c^3 \end{array}\right\},$$

$$= \frac{abc}{16(a+b+c)A^2}\left\{\begin{array}{l} + a^4 + a^2bc + 2a^2b^2 \\ + b^4 + ab^2c - 2a^2c^2 \\ + c^4 + abc^2 - 2b^2c^2 \end{array}\right\},$$

$$= \frac{r}{16pA^2}(p^4 - 4p^2q + 9pr),$$

$$= \frac{r(p^3 - 4pq + 9r)}{16A^2},$$

$$= \frac{r^2}{16A^2} - \frac{r}{p}.$$

Nicolas Fuss (1794, 1798).

The inquiry is now extended to other figures than the triangle. In the tenth volume, pp. 103–125, of the *Nova Acta Academiae Scientiarum Imperialis Petropolitanae* (Petropoli, 1797), there is a paper by Fuss, entitled, *De Quadrilateris quibus circulum tam inscribere quam circumscribere licet*, which was read on the 14th August 1794. The following is a short abstract of it. The words "encyclic" and "pericyclic" are used for the phrases "that may be inscribed in a circle," and "that may be circumscribed about a circle."

The first problem and its corollaries show how, when the four sides of an encyclic quadrilateral are given, to express by means of the sides alone, the area, the radius of the circumscribed circle, the diagonals, and the angles of the figure. When, however, the quadrilateral is pericyclic, in addition to the four sides there must be given either a point of contact or an angle to determine the rest. This is shown in the second and third problems and their corollaries. In the fourth problem and its corollaries it is proved that, of all the quadrilaterals formed by four given sides, the one whose inscribed circle is the greatest is encyclic. The fifth problem is, About a given circle to circumscribe an encyclic quadrilateral; the sixth, In a given circle to inscribe a pericyclic quadrilateral. The seventh and eighth problems show how, when the four sides, or when two sides and the contained angle, are given to construct a quadrilateral which is both encyclic and pericyclic. The ninth and tenth problems show how, when the angles of an encyclic and pericyclic quadrilateral are given, to find the sides, the area, and the radii of the circumscribed and inscribed circles. The eleventh problem is, Given the inscribed and circumscribed radii of an encyclic and pericyclic quadrilateral, to find the distance of the centres. Two expressions are found for the square of this distance, namely,

$$R^2 + r^2 \pm r\sqrt{4^2 + rR^2},$$

where the upper sign holds for the circle which touches the sides produced of the quadrilateral, and the lower for the circle really inscribed in the quadrilateral. In the latter case Fuss remarks that $R^2 + r^2$ must be greater than $r\sqrt{4R^2 + r^2}$, that is, that R must be greater than $r\sqrt{2}$. He then adds the expression for the square of the distance between the inscribed and circumscribed centres of a triangle, namely, $R^2 - 2Rr$,

and gives the following demonstration.

See figure 72.

Let I be the centre of the circle inscribed in triangle ABC, S the circumscribed centre. From A draw AV through the centre I to meet the circumscribed circle in V. Through V draw the circumscribed diameter VU, and from I let fall IT perpendicular to VU.

Then $d^2 = \text{SI}^2 = \text{ST}^2 + \text{IT}^2$.

But $\text{ST} = \text{SV} - \text{TV} = R - \text{TV}$,

and $\text{IT}^2 = \text{IV}^2 - \text{TV}^2$;

therefore $d^2 = R^2 - 2R \cdot \text{TV} + \text{IV}^2$.

Draw the chords BU, BV, CV.

Then it is evident that $\angle\,\text{CHV} = \angle\,\text{UBV}$,

and $\angle\,\text{HCV} = \angle\,\text{BUV}$;

therefore triangle BUV is similar to triangle HCV;

therefore $\text{BV} : \text{UV} = \text{HV} : \text{CV}$;

therefore $\text{BV} \cdot \text{CV} = \text{UV} \cdot \text{HV}$,

$= 2R \cdot \text{HV}$.

Now if CI be joined, it is clear that

$\angle\,\text{CIV} = \angle\,\text{IAC} + \angle\,\text{ICA}$,

and $\angle\,\text{ICV} = \angle\,\text{ICH} + \angle\,\text{VCH}$.

But $\angle\,\text{ICH} = \angle\,\text{ICA}$, and $\angle\,\text{VCH} = \angle\,\text{IAC}$;

therefore $\angle\,\text{CIV} = \angle\,\text{ICV}$,

and $$IV = CV = BV.$$
Hence $$IV^2 = BV \cdot CV,$$
$$= 2R \cdot HV.$$
Substituting this value, we have
$$d^2 = R^2 - 2R \cdot (TV - HV),$$
$$= R^2 - 2R \cdot ID,$$
$$= R^2 - 2Rr.$$

In the thirteenth volume, pp. 166–189, of the *Nova Acta* (Petropoli, 1802), Fuss returns to the inquiry regarding the inscription and circumscription of circles to certain polygons. The title of his paper is *De Polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis;* it was read on the 19th April 1798. He says that he tried various ways of resolving the same problems, as he had done in the case of the quadrilateral, for polygons of more than four sides, but without success. The fundamental formulae became so perplexed that oil and toil ("oleum et opera") were fruitlessly spent in clearing them up. Laying aside, therefore, the general problem as beset with very great difficulties, he betook himself to polygons, which may be called *symetrically irregular*, that is, which have a diameter passing through the centres of the inscribed and circumscribed circles, and dividing the polygons into two equal and similar figures. Even with this limitation, he observes, the problem is a knotty one.

The following are the enunciations of the ten problems of which the paper consists:—

1. Given the angles of a pentagon, to find its sides so that a circle of given size can be circumscribed about it.

2. Given the angles of a pentagon, to find its sides so that a circle of given size can be inscribed in it.

3. Given the sides of a pentagon and any one point of contact, to find the radius of the inscribed circle.

4. If the angles of an encyclic and pericyclic pentagon are known, to find the relation between the radii R and r.

5. To find the distance of the centres of the circles inscribed and circumscribed to a symmetrically irregular pentagon.

6. In a given circle to inscribe a symmetrically irregular pentagon which shall be pericyclic.

7. To find the distance of the centres of the circles inscribed and circumscribed to a symmetrically irregular hexagon.

8. In a given circle to inscribe a symmetrically irregular hexagon which shall be pericyclic.

9. In a given circle to inscribe a symmetrically irregular heptagon which shall be pericyclic.

10. In a given circle to inscribe a symmetrically irregular octagon which shall be pericyclic.

Geometrical Notes.

By R. E. ALLARDICE, M.A.

1. Some points of difference between polygons with an even number and polygons with an odd number of sides.

A polygon with an odd number of sides is determined when its angles are given and it is such that a circle of given radius may be circumscribed about it; while a polygon with an even number of sides is not *determined by these conditions.*

To prove this it is sufficient to show that in general one polygon and only one with given angles can be inscribed in a given circle if the number of sides be odd; and that if the number of sides be even either it is impossible to inscribe any such polygon or it is possible to inscribe an infinite number.

Let ABCDE (fig. 76) be a polygon with an odd number of sides. In a circle take any point A′; make an arc A′C′ to contain an angle equal to B; an arc C′E′ to contain an angle equal to D; an arc E′B′ to contain an angle equal to A; and so on. Thus the problem is in general a possible one; but it is evident that, though no definite relation among the angles is required (except that connecting the angles of any n-gon), there are limits in which the angles must lie in order that a solution be possible. Thus when the above construction is made, the point E′ must fall between C′ and A′, the point B′, between A′ and C′, and so on.

If the angles are $A_1, A_2, A_3 \ldots\ldots A_{2n+1}$, the necessary and sufficient conditions are of the form

$$A_2 + A_4 + \ldots\ldots + A_{2n} > (n-1)\pi$$
$$A_2 + A \ + \ldots\ldots + A_{2n} + A_1 < n\pi.$$

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