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# ISOMORPHIC GROUP RINGS OF FREE ABELIAN GROUPS

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**Introduction.** S. K. Sehgal ([**9**], Problem 26) proposed the following question: Let $A, B$ be rings and $X$ an infinite cyclic group. Does $AX \simeq BX$ imply $A \simeq B$? M. M. Parmenter and S. K. Sehgal (c.f. [**9**], Chapter 4) proved that, under some strong assumptions concerning rings $A$, $B$, the answer is affirmative. In this paper, we show that the assumptions concerning the ring $B$ may be omitted in the above mentioned results. Moreover, it is proven that if $(AX)X \simeq BX$ then $AX \simeq B$ for all rings $A, B$. If $A$ is commutative and noetherian then a partial answer to Problem 27, [**9**] follows from our results.

Recently, L. Grünenfelder and M. M. Parmenter constructed nonisomorphic rings $A, B$ for which the group rings $AX, BX$ are isomorphic, [**2**]. We give a new class of rings of this type. Our examples are noncommutative domains and algebras over finite fields. That also gives a negative answer to Problem 29, [**9**].

**1. Preliminaries.** In this paper rings with unity and unital homomorphisms will be considered. If $R$ is a ring then $P(R)$ will mean the prime radical [**3**], and $J(R)$ the Jadobson radical of $R$. The same notation as in [**9**] will be used for group rings.

In this section $K$ will be a commutative ring and $Y$ a torsion-free abelian group.

LEMMA 1.1. *Let $I \subset KY$ be an ideal. Then $I$ is a minimal prime ideal in $KY$ if and only if $I = I'Y$ where $I'$ is a minimal prime in $K$. Thus $P(KY) = P(K)Y$.*

*Proof.* Of course $(I \cap K)Y \subset I$. If $\bar{K}$ is a homomorphic image of the ring $K$ then $\bar{K}$ is a domain if and only if $\bar{K}Y$ is a domain since $Y$ is torsion free and abelian. Thus, the result follows easily.

It is well known [**3**] that if $e \in KY$ is an idempotent element then $e \in K$. Let $E = \{e_1, \ldots, e_n\}$ be a decomposition of the unity element in $K$, i.e., a decomposition into a sum of orthogonal idempotents. If $u \in U(KY)$ then we shall say that $E$ *splits* $u$ if there exist $v \in U(K)$, $y_1, \ldots, y_n \in Y$ (not necessarily different) and $p \in P(KY)$ such that

$$u = v \sum e_i y_i + p.$$

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LEMMA 1.2. *Let $u \in U(KY)$. Then there exists a decomposition of the unity $E = \{e_1, \ldots, e_n\}$ in $K$ which splits $u$.*

*Proof.* It follows from [**4**] (Lemma 5) that there exists a decomposition of unity $E = \{e_1, \ldots, e_n\}$ such that

$$u = \sum v_i y_i + p$$

where $v_i \in U(e_i K)$, $y_i \in Y$, $p \in P(KY)$. Then, for $v = \sum v_i$ we have $v \in U(K)$ and

$$v \sum e_i y_i + p = \sum v e_i y_i + p = \sum v_i y_i + p = u,$$

i.e., $E$ splits $u$.

LEMMA 1.3. *Let $E = \{e_1, \ldots, e_n\}$ be a decomposition of the unity in $K$. Then, the set of elements $u \in U(KY)$ which are split by $E$ is a subgroup in $U(KY)$.*

*Proof.* Let $G$ be the set of elements $u \in U(KY)$ which are split by $E$. Of course $1 \in G$ and $G$ is a semigroup. If

$$g = v \sum e_i y_i + p \in G$$

then it may be easily checked that

$$g^{-1} = v^{-1} \sum e_i y_i + q$$

for an element $q \in P(KY)$ and hence $g^{-1} \in G$.

It may be shown that if $E$, $F$ are decompositions of the unity in $K$ then there exists a decomposition of the unity in $K$ which refines $E$, $F$ [**4**]. Moreover, if $E$ splits $u \in KY$ and $F$ refines $E$ then $F$ also splits $u$. Thus, the following result is a consequence of Lemma 1.2. and Lemma 1.3.

LEMMA 1.4. *Let $G \subset U(KY)$ be a finitely generated group. Then, there exists a decomposition of the unity $E$ in $K$ such that $E$ splits all elements of the group $G$.*

**2. Elementary homomorphisms.** In what follows $A, B$ will be rings and $C$, $D$ their centers. If $Y$, $Y'$ are abelian groups and $\varphi$: $AY \to BY'$ is a ring homomorphism then we shall say that $\varphi$ is *elementary* if $\varphi(CY) \subset DY'$ and for any $y \in Y$ the element $\varphi(y)$ is split by $\{1\}$, i.e., there exist $u \in U(D)$, $y' \in Y'$, $p \in P(DY')$ such that $\varphi(y) = uy' + p$.

LEMMA 2.1. *Let $Y$, $Y'$ be torsion free abelian groups and $\varphi : AY \to BY'$ be a ring isomorphism. If $Y$ is finitely generated then for an integer $n$ there exist ideals $A_i \subset A$, $B_i \subset B$, $1 \leqq i \leqq n$, such that*

$$A = A_1 \oplus \ldots \oplus A_n \qquad B = B_1 \oplus \ldots \oplus B_n$$

*and*

$$\varphi|_{A_iY} : A_iY \to B_iY'$$

*is an elementary isomorphism.*

*Proof.* Of course $\varphi(CY) = DY'$ and $\varphi(Y) \subset U(DY')$ is a finitely generated group. It follows from Lemma 1.4 that there exists a decomposition of the unity $F = \{f_1, \ldots, f_n\}$ in $D$ which splits all elements of $\varphi(Y)$.

Let

$$E = \varphi^{-1}(F) = \{\varphi^{-1}(f_1), \ldots, \varphi^{-1}(f_n)\}.$$

Let $B_i = Bf_i$ and $A_i = A\varphi^{-1}(f_i)$ for $1 \leqq i \leqq n$. Then

$$A = A_1 \oplus \ldots \oplus A_n \quad \text{and} \quad B = B_1 \oplus \ldots \oplus B_n.$$

Moreover, it follows from the choice of $F$ that $\varphi|_{A_iY}$ is an elementary isomorphism of $A_iY$ onto $B_iY$.

COROLLARY 2.2. *If, under the assumptions of Lemma* 2.1, *the ring $A$ has no nontrivial central idempotents then the isomorphism $\varphi$ is elementary.*

In the sequel $X = \langle x \rangle$ will be an infinite cyclic group. If $\varphi : AX \to BX$ is an elementary homomorphism and $\varphi(x) = ux^r + p$, $u \in U(D)$, $p \in P(DX)$ then the integer $|r|$ will be called the *degree* of $\varphi$ and it will be denoted by $|\varphi|$.

The following two results were, in fact, proven by M. M. Parmenter and S. K. Sehgal in [**7**] (the proof of Theorem 1).

THEOREM 2.3. *If $\varphi$ is an elementary $A$-endomorphism of $AX$ of degree* 1 *then $\varphi$ is an automorphism.*

COROLLARY 2.4. *If $\varphi : AX \to BX$ is an elementary isomorphism of degree* 1 *then the rings $A$, $B$ are isomorphic.*

As a consequence of Theorem 2.3 we also get

COROLLARY 2.5. *Let $\varphi : AX \to BX$ be an elementary isomorphism of degree $r$. Then there exist $u \in U(D)$, and an $A$-automorphism $\psi$ of $AX$ such that $\varphi\psi(x) = ux^r$.*

*Proof.* It follows from the assumption that there exist $u \in U(D)$, $p \in P(DX)$ such that $\varphi(x) = x^{\epsilon r} + p$ where $\epsilon = \pm 1$. Since $p \in P(DX)$ then $\varphi^{-1}(p) \in P(CX)$. Let $\psi_1$ be the $A$-endomorphism of $AX$ given by $\psi_1(x) = x - \varphi^{-1}(p)$. Then, it follows from Theorem 2.3 that $\psi_1$ is an automorphism and

$$\varphi\psi_1(x) = \varphi(x - \varphi^{-1}(p)) = ux^{\epsilon r} + p - p = ux^{\epsilon r}.$$

Now, let $\psi_2$ be the $A$-automorphism of $AX$ given by $\psi_2(x) = x^{\epsilon}$. Then it

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may be directly checked that $\psi = \psi_1\psi_2$ fulfils the conditions of the corollary.

LEMMA 2.6. *Let $\varphi : AX \to BX$ be an elementary isomorphism of degree* 0. *Then the rings $A, B$ are isomorphic.*

*Proof.* We shall exploit some ideas from [**5**]. In view of Corollary 2.5 we may assume that $\varphi(x) = u \in U(D)$. Let us suppose, for a moment, that $\varphi^{-1}$ is also elementary, i.e., $\varphi^{-1}(x) = vx^r + p$ where $v \in U(C)$, $p \in P(CX)$. Let $\psi$ be the $A$-automorphism of $AX$ given by $\psi(x) = vx$. Then

$$\begin{aligned}\varphi\psi(x) &= \varphi(vx) = \varphi((vx^r + p)x^{1-r} - px^{1-r}) \\ &= \varphi\varphi^{-1}(x) \cdot \varphi(x^{1-r}) - \varphi(px^{1-r}) = xu^{1-r} + \varphi(px^{1-r}).\end{aligned}$$

Since $u^{1-r} \in U(D)$ and $\varphi(px^{1-r}) \in P(DX)$ then $\varphi\psi$ is an elementary isomorphism of degree 1. Thus, in this case, the result follows from Corollary 2.4.

Now, if $\varphi^{-1}$ is not elementary then our considerations may be reduced to the above case with the use of Lemma 2.1.

We shall show that Corollary 2.4 and Lemma 2.6 can not be proved in the case of isomorphisms of higher degrees.

*Example* 2.7. Let $p$ be a prime number and $r \geqq 1$. Let $K$ be a field with $p^{r^2+1}$ elements and $\psi$ an automorphism of $K$ such that $\psi(k) = k^p$ for $k \in K$. If $Y = \langle y \rangle$ is an infinite cyclic group then let us consider the skew group rings [**9**], $A = K_\Theta(Y)$, $B = K_{\Theta'}(Y)$ where $\Theta(y) = \psi$ and $\Theta'(y) = \psi^r$. It is easily seen that $A$, $B$ are noncommutative domains and their groups of units are trivial. We shall show that $A$ and $B$ are not isomorphic. Let us suppose $\varphi : A \to B$ is an isomorphism. Since $K$ is a field then, similarly as in [**7**] Lemma 2, it may be checked that $\varphi(K) = K$ and so $\varphi|_K = \psi^s$ for some $s$, $1 \leqq s \leqq n$. Moreover $\varphi(y) = uy$ or $\varphi(y) = vy^{-1}$ for some $u, v \in K^*$.

Let us suppose $\varphi(y) = uy$. Then, for any $k \in K$ we get

$$\begin{aligned}\psi^{s+1}(k)uy &= \psi^s(\psi(k))uy = \varphi(\psi(k))\varphi(y) = \varphi(\psi(k)y) = \varphi(yk) \\ &= \varphi(y)\varphi(k) = uy\psi^s(k) = u\psi^r(\psi^s(k))y = \psi^{r+s}(k)uy\end{aligned}$$

and $\psi^{s+1}(k) = \psi^{r+s}(k)$. Thus $\psi = \psi^r$ which is impossible since the degree of $\psi$ equals $r^2 + 1$.

If $\varphi(y) = vy^{-1}$ then we get a contradiction by replacing $y$ by $y^{-1}$ and $r$ by $r^2 + 1 - r$ in the above reasoning. Now, we shall show that there exists an elementary isomorphism $\delta : AX \to BX$ of degree $r$. Let $\delta(k) = k$ for $k \in K$, $\delta(y) = y^r x^{-1}$ and $\delta(x) = y^{r^2+1}x^r$. Then, it is easy to check that $\delta$ is a well defined homomorphism of $AX$ into $BX$ and even an isomorphism. Of course, $\delta$ is elementary of degree $r$.

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The existence of an elementary isomorphism of degree $> 1$ of rings $AX$, $BX$ results in some connections between $A$ and $B$.

LEMMA 2.8. *Let $\varphi : AX \to BX$ be an elementary isomorphism of degree $r > 1$. Then there exists a ring $B_1 \supset B$ such that $A$, $B_1$ are isomorphic and there exists an element $v$ in the center of $B_1$ such that $B_1 = B[v]$, $v^r \in U(D)$ and the elements $1, v, \ldots, v^{r-1}$ are independent over $B$.*

*Proof.* In view of Corollary 2.5. we may assume that $\varphi(x) = ux^r$, $u \in U(D)$. The isomorphism $\varphi$ induces an isomorphism of polynomial rings $AX[t]$, $BX[t]$ given by the formula

$$\psi(\sum a_i t^i) = \sum \varphi(a_i)t^i$$

where $a_i \in AX$. If $I$ is an ideal in $AX[t]$ generated by $t^r - x$ and $J$ is an ideal in $BX[t]$ generated by $t^r x^{-r} - u$ then $\varphi(I) = J$. Thus, $\psi$ induces an isomorphism $\bar{\psi}$ of rings $AX[t]_{/I}$, $BX[t]_{/J}$. Now, it is easy to see that $AX[t]_{/I}$ is the group ring of the infinite cyclic group generated by $t + I$ with the coefficient ring $A$ and $BX[t]_{/J}$ is the group ring of the infinite cyclic group generated by $x + J$ with the coefficient ring $B_1 = B[v]$ where $v = tx^{-1} + J$. Moreover

$$\bar{\psi}(t + I) = t + J = tx^{-1}x + J = v(x + J)$$

and hence $\bar{\psi} : AX \to B_1X$ is an isomorphism of degree 1. It follows from Corollary 2.4 that $A$ and $B_1$ are isomorphic. It is easy to check that

$$v^r = t^r x^{-r} + J = u + J \in U(D)$$

and the elements $1, v, \ldots, v^{r-1}$ are independent over $B$.

The following generalization of a result of Parmenter [**6**] follows from Lemmas 2.1, 2.6, 2.8 and Corollary 2.4.

THEOREM 2.9. *If $AX$ and $BX$ are isomorphic then the rings $A$, $B$ are subisomorphic. In fact, each of the rings $A$, $B$ is isomorphic to a finite, integral and central extension of another.*

**3. Uniqueness of coefficients.** We shall say that a ring $A$ is *$X$-invariant* (c.f. [**1**]) if for any ring $B$ we have $A \simeq B$ whenever $AX \simeq BX$.

THEOREM 3.1. *Let $A$ be a ring. Then the ring $AX$ is $X$-invariant.*

*Proof.* Since $X \otimes X$ is a free abelian group of rank 2 then the result follows from

LEMMA 3.2. *Let $Y$ be a free abelian group of rank 2. If the rings $AY$, $BX$ are isomorphic then the rings $AX$, $B$ are isomorphic.*

*Proof.* Let $\delta : AY \to BX$ be an isomorphism. In view of Lemma 2.1 we may assume that $\delta$ is elementary. Thus, for any $y \in Y$ we have

$\delta(y) = \alpha(y)\beta(y) + p(y)$ where $\alpha(y) \in U(D)$, $\beta(y) \in X$, $p(y) \in P(DX)$. It is easy to check that the transformation $\beta$ is a group homomorphism. Thus, we may choose a set of generators $\{y_1, y_2\}$ in $Y$ such that $\beta(y_2) = 1$. Then

$$AY = A\langle y_1, y_2\rangle = (A\langle y_1\rangle)\langle y_2\rangle.$$

Since $\beta(y_2) = 1$ then $\delta$ is an elementary isomorphism of rings $(A\langle y_1\rangle)\langle y_2\rangle$, $BX$ of degree 0. It follows from Lemma 2.6 that the rings $A\langle y_1\rangle$, $B$ are isomorphic. Thus $AX$, $B$ are isomorphic since $\langle y_1\rangle$ is infinite and cyclic.

Now we shall show that some properties of the center $C$ of $A$ cause the ring $A$ to be $X$-invariant.

THEOREM 3.3. *The ring $A$ is $X$-invariant in any of the following cases:*
(a) *$C$ has no nontrivial idempotents and $P(C) \neq J(C)$;*
(b) *$C$ is local;*
(c) *$C_{/P(C)}$ is regular;*
(d) *$U(C)$ is a divisible group.*

First we shall prove

LEMMA 3.4. *Let $K$, $L$ be commutative rings and $\varphi : KX \to LX$ an isomorphism. If $K$ is a field or $K$ is a domain such that $J(K) \neq 0$ then $|\varphi| = 1$.*

*Proof.* In both cases $KX \simeq LX$ is a domain and hence $L$ is a domain. Thus, it follows from Corollary 2.2 that $\varphi$ is elementary and the group $U(LX)$ is trivial. If $K$ is a field then, as in Lemma 2 [**7**], we get $\varphi(K) \subset L$. If $J(K) \neq 0$ then let $0 \neq j \in J(K)$. Then $1 + j + j^2 \in U(KX)$ and

$$1 + \varphi(j) + \varphi(j)^2 \in U(LX).$$

Since $L$ is a domain then $\varphi(j) \in L$. Now, if $0 \neq k \in K$ then $jk \in J(K)$ and so

$$\varphi(jk) = \varphi(j)\varphi(k) \in L.$$

Since $\varphi(j) \in L$ and $L$ is a domain then $\varphi(k) \in L$. Thus, in this case we also get $\varphi(K) \subset L$. Now, let $\varphi(x) = ux^r$ where $u \in U(L)$. Then

$$LX = \varphi(KX) \subset L\langle x^r\rangle \subset L\langle x\rangle$$

and $r = \pm 1$ which completes the proof of Lemma.

*Proof of Theorem* 3.3. Let $\delta : AX \to BX$ be an isomorphism. (a), (b) It follows from Corollary 2.2 that $\delta$ is elementary. Thus, in view of Corollary 2.4 it is enough to show that $|\delta| = 1$. Since $|\delta| = |\delta_{|CX}|$ we may assume that $A = C$. In both cases there exists a minimal prime ideal $Q \subset C$ such that $C_{/Q}$ is a field or $J(C_{/Q}) \neq 0$. Since $Q$ is a minimal prime ideal in $C$ it follows from Lemma 1.1 that there exists a minimal prime ideal $Q' \subset D$ such that $\varphi(QX) = Q'X$. Hence $\delta$ induces an isomorphism of rings $(C_{/Q})X$ and $(D_{/Q'})X$. It follows from Lemma 3.4 that $|\bar{\delta}| = 1$. However $|\bar{\delta}| = |\delta|$ which completes the proof in these cases.

(c) Since any regular commutative domain is a field then (c) may be proved by the same arguments as in (a), (b).

(d) In view of Lemma 2.1 we may assume that $\delta^{-1}$ is elementary and in view of Corollary 2.5 we may assume that $\delta^{-1}(x) = ux^r$ where $u \in U(C)$. If $r = 0$ then the result follows from Lemma 2.6. If $r > 0$ then there exists $v \in U(C)$ such that $v^r = u$ and hence $\delta^{-1}(x) = (vx)^r$. However, for $r > 1$, the element $x$ is not the $r$th power of any element in $DX$. Hence the isomorphism $\delta^{-1}$ is elementary of degree 1 and the result follows from Corollary 2.4.

In the case of integral group rings we get the following

THEOREM 3.5. *Let $G$ be a group. Then the ring $ZG$ is $X$-invariant in any of the following cases:*

(a) *$G \simeq H \otimes X$ for any group $H$;*

(b) *$G$ is abelian;*

(c) *$\phi(G)$ is torsion, where $\phi(G)$ is the set of elements of $G$ having only a finite number of conjugates in $G$.*

*Proof.* (a) follows directly from Theorem 3.1.

Let $\delta : (ZG)X \to BX$ be an isomorphism. It follows from the theorem of Kaplansky concerning traces of idempotents (c.f. [**8**], [**9**]) that $(ZG) \simeq Z(G \otimes X)$ has no nontrivial idempotents and hence $\delta, \delta^{-1}$ are elementary. Moreover, $ZG$ is semiprime ([**3**], [**9**]) and so it has no nontrivial central nilpotents. Hence $B$ has no nontrivial central nilpotents.

(b) Since $G$ is abelian then $B = D$ is commutative and the group $U(DX)$ is trivial. Hence, for any $g \in G$ we have $\delta(g) = \alpha(g)\beta(g)$ where $\alpha(g) \in U(D), \beta(g) \in X$. Of course $\beta : G \to X$ is a group homomorphism. If it is nontrivial and $H = \ker \beta$ then $G \simeq H \otimes X$ and the result follows from case (a). If $\beta$ is trivial then $\delta(G) \subset U(D)$ and so $\delta(ZG) \subset D$. Then, it is easy to check that $\delta : (ZG)X \to BX$ is an elementary isomorphism of degree 1 and $ZG$ is $X$-invariant by Corollary 2.4.

(c) Let $\delta^{-1}(x) = ux^s$ where $u$ is central and invertible in $ZG$. Then there exists a finitely generated F.C.-group $H \subset \phi(G)$ such that $u \in ZH$. It follows from the assumption that $H$ is torsion and hence $H$ is finite ([**8**], [**9**]). Thus the element $u \in ZH$ is a root of a unitary polynomial $f(t) \in Z[t]$ and so $\delta(u) \in DX$ is a root of the same polynomial. Since $D$ has no nontrivial nilpotents nor idempotents then $\delta(u) = vx^k$ where $v \in U(D)$. Since $f(vk^k) = 0$ then $k = 0$ and $\delta(u) \in U(D)$. Now, if $\delta(x) = wx^r$ where $w \in U(D)$ then

$$x = \delta(\delta^{-1}(x)) = \delta(ux^s) = \delta(u)(wx^r)^s = \delta(u)w^s x^{rs}.$$

Since $\delta(u) \in U(D)$ then $rs = 1$ and $\delta$ is an elementary isomorphism of degree 1. Thus, the result follows from Corollary 2.4.

It is known that there exists a group $G$ such that the ring $ZG$ is not $X$-invariant [**2**].

The above results concerning uniqueness of coefficients may be extended to the case of group rings of abelian groups of finite rank.

LEMMA 3.6. *Let $Y_n$ be a free abelian group of rank $n$ for $n \geqq 1$. Then the following conditions are equivalent:*

(a) $AY_1 \simeq BY_1$;

(b) $AY_n \simeq BY_n$ *for any* $n \geqq 1$;

(c) $AY_n \simeq BY_n$ *for some* $n \geqq 1$.

*Proof*. Of course (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a). Let $m \geqq 1$ be the smallest integer such that $AY_m \simeq BY_m$. Let us suppose $m > 1$. Then

$$Y_m \simeq Y_{m-2} \otimes Y_2 \simeq Y_{m-1} \otimes Y_1.$$

Hence

$$(AY_{m-2})Y_2 \simeq (BY_{m-1})Y_1$$

and it follows from Lemma 3.2 that

$$AY_{m-1} \simeq (AY_{m-2})Y_1 \simeq BY_{m-1}$$

which contradicts the choice of $m$. Thus $m = 1$ and $AY_1 \simeq BY_1$.

**4. On some connected questions.** Problem 27, formulated by Sehgal in [**9**] may be stated as follows: Is a commutative noetherian ring $X$-invariant? The next theorem follows from results of Section 3.

THEOREM 4.1. *Let $C$ be a commutative noetherian ring. Then $C$ is $X$-invariant in any of the following cases:*

(a) *$C = DX$ where $D$ is commutative and noetherian;*

(b) *$C$ has no nontrivial idempotents and $P(C) \neq J(C)$;*

(c) *$C$ is artinian;*

(d) *$C = ZG$ where $G$ is finitely generated and abelian;*

(e) *$C$ is a finite direct sum of rings $C_i$ fulfilling one of the above conditions* (a), (b), (c), (d).

Sehgal [**9**] (Problem 29) proposed the following question: Are the polynomial rings $A[t]$, $B[t]$ isomorphic whenever $AX$, $BX$ are isomorphic? The next example gives a negative answer to this question.

*Example* 4.2. Let $A$, $B$ be rings as defined in Example 2.7. Then $AX$, $BX$ are isomorphic. Now, the ring $A$ has no non-trivial nilpotents and its additive group is generated by the set of invertible elements. Since $A$, $B$ are not isomorphic then it follows from Corollary 2 [**1**], that the rings $A[t]$, $B[t]$ are not isomorphic.

Let us notice that if $K$ is a commutative ring and if we replace rings by $K$-algebras in our considerations then some of our results may be considered in algebraic geometry.

*Added in proof.* Problem 27 [**9**] was also discussed by K. Yoshida in Osaka J. Math. *17* (1980), 769–782.

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