THE ITERATED EQUATION OF GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY, III

CONJUGATE GENERAL SOLUTIONS

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1. Introduction

The iterated equation of generalized axially symmetric potential theory (GASPT), in the notation of the first paper of this series [1] which will be designated I, is the equation

$$L_k^n(f) = 0, \tag{1}$$

where the operator L_k is defined by

$$L_k(f) \equiv \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + ky^{-1}\partial f/\partial y.$$

It was seen in I that generalized axially symmetric potential theory arises from the discussion of the generalized Stokes-Beltrami equations

$$\partial\phi/\partial x - y^{-p}\,\partial\psi/\partial y = 0, \quad \partial\psi/\partial x + y^p\,\partial\phi/\partial y = 0, \tag{2}$$

from which it is immediately deduced that

$$L_p(\phi) = 0, \quad L_{-p}(\psi) = 0. \tag{3}$$

In the application of these equations to hydrodynamics, ϕ is a velocity potential and ψ a stream function which each describe the pattern of a particular irrotational fluid motion which, for $p = 0$, is a two-dimensional flow in the $x-y$ plane and, for $p = 1$, is an axially symmetric flow with OX as the axis of symmetry and the $x-y$ plane as a meridian plane.

When $p = 0$, the Stokes-Beltrami equations (2) become the Cauchy-Riemann equations and ϕ and ψ are conjugate harmonic functions, the real and imaginary parts of an analytic function w of the variable $z = x+iy$.

When $p \neq 0$, a pair of function ϕ and ψ which satisfy the Stokes-Beltrami equations (2) and hence their respective GASPT equations (3) can also be described as *conjugate* functions. ϕ and ψ can again be combined to form a function, this time of both z and $\bar{z}$, by the relation (defined in the region $y > 0$)

$$w(z, \bar{z}) = \phi(x, y) + iy^{-p}\psi(x, y)$$

and w will be called a *p-analytic* function. (As can be seen in [3] and the references given there, this p-analytic function is a special case of the *pseudo-analytic* functions introduced by Beltrami [2] and the *generalized analytic* functions of Vekua. The name used here has advantages in the present context as it is easily adapted when a similar but more general, function is introduced in section 5.) Now that this new function w has been introduced, it is convenient to reverse the statement and say that if $w = \phi + iy^{-p}\psi$ is a p-analytic function, then $L_p(\phi) = 0$, $L_{-p}(\psi) = 0$ and ϕ, ψ are conjugate functions, satisfying the Stokes-Beltrami equations (2). It can easily be proved that if ϕ or ψ is known, the other member of the conjugate pair of functions can be found. This means that if ϕ is given such that $L_p(\phi) = 0$, then a p-analytic function can be found whose real part is ϕ; if ψ is given such that $L_{-p}(\psi) = 0$, then a p-analytic function can be found whose imaginary part is $y^{-p}\psi$; and if the two p-analytic functions are the same, then ϕ and ψ are conjugate functions. It can therefore be said that conjugate general solutions of equations (3) are given by the real part and y^p times the imaginary part respectively of a p-analytic function. All of these properties are of course familiar in the special case $p = 0$.

In the first part of this paper, a definition of conjugate solutions is set up which applies to the pair of iterated equations .

$$L_p^n(\phi) = 0, \quad L_{-p}^n(\psi) = 0 \tag{4}$$

and expressions are found for conjugate general solutions of these equations. The results just described for the case $n = 1$ are special cases of course but they do not give much help in deciding how they should be generalized. A better lead comes from another familiar special case, that of the two-dimensional biharmonic equation $\nabla^4 f = 0$ which is obtained from equation (1) by putting $k = 0$ and $n = 2$. It is well-known that a general solution of this equation is given by either the real part ϕ or the imaginary part ψ of a function of the form

$$\phi + i\psi = w^{(0)} + \bar{z}w^{(1)} \tag{5}$$

where $w^{(0)}$, $w^{(1)}$ are arbitrary analytic functions of z. It is reasonable to define ϕ and ψ given by (5) as *conjugate* biharmonic functions. When the general results for equations (4) are obtained (section 4.5) it will be seen that they follow naturally from these special cases.

The second part of the paper makes use of the ideas developed in the first part to generalize the work of Parsons [4], who, for the special case $n = p = 1$, in the context of axially symmetric flow of a perfect fluid, proved four theorems concerning conjugate solutions of equation (3).

Corresponding theorems will be obtained which apply to the general equations (4) and are valid for all positive integers n and all values of p. These theorems appear as theorems 5.2, 6, 7.1, 7.2.

2. Solutions of $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$ in terms of a given p-analytic function

The p-analytic function $w = \phi + iy^{-p}\psi$ is used to define two new functions of z and $\bar{z}$ with their corresponding real and imaginary parts as follows: for any integer $n \geqq 0$, let

$$\phi_n + iy^{-p}\psi_n \equiv w_n = wz^n, \tag{6}$$

$$\Phi_n + iy^{-p}\Psi_n \equiv W_n = w\bar{z}^n. \tag{7}$$

Two preliminary results are required.

LEMMA 2.1 *For any integer $n \geqq 2$, $\phi_n, \psi_n, \Phi_n, \Psi_n$ all satisfy the recurrence relation $f_n = 2xf_{n-1} - r^2 f_{n-2}$.*

It is easily verified that

$$w_n = (z+\bar{z})w_{n-1} - z\bar{z}w_{n-2}, \quad W_n = (z+\bar{z})W_{n-1} - z\bar{z}W_{n-2}.$$

Since $z+\bar{z} = 2x$ and $z\bar{z} = r^2$ are real, the required result follows at once.

LEMMA 2.2. *For integers $m \geqq 1$, $n \geqq 2$, all the functions $\phi_n, \psi_n, \Phi_n, \Psi_n$ satisfy the equation*

$$\begin{aligned} L_k^m(f_n) = {} & 2xL_k^m(f_{n-1}) + 4m\, \partial/\partial x\, L_k^{m-1}(f_{n-1}) - r^2 L_k^m(f_{n-2}) \\ & - 4mr\, \partial/\partial r\, L_k^{m-1}(f_{n-2}) - 2m(2m+k)L_k^{m-1}(f_{n-2}). \end{aligned}$$

This result is obtained by using the recurrence relation of Lemma 2.1 and the expressions for $L_k^m(xf)$ and $L_k^m(r^2 f)$ given for any function f by equations (5) and (14) of I.

It can now be shown that the functions defined in (6) and (7) are solutions of equations of the type (4).

THEOREM 2.1

(i) *For* $n \geqq 1$, $L_p^n(\phi_n) = 0$, $L_{-p}^n(\psi_n) = 0$.

(*ii*) *For* $n \geqq 0$, $L_p^{n+1}(\Phi_n) = 0$, $L_{-p}^{n+1}(\Psi_n) = 0$.

These results are proved by induction, using Lemma 2.2. For (i), it is easily verified that $L_p(\phi_1) = L_{-p}(\psi_1) = 0$ and so $L_k(f_1) = 0$ where, for $n \geqq 0$, f_n represents ϕ_n or ψ_n according as $k = p$ or $-p$. Lemma 2.2, with $m = n$, is used to prove that if $L_k^{n-1}(f_{n-1}) = 0$, then $L_k^n(f_n) = 0$. For (ii), it is clear that $L_k(f_0) = 0$ where now f_n represents Φ_n or Ψ_n according as $k = p$ or $-p$. Lemma 2.2, with $m = n+1$ this time, is used to prove that if $L_k^n(f_{n-1}) = 0$, then $L_k^{n+1}(f_n) = 0$. In each case the proof by induction can be completed.

Another set of results which will be needed later can now be proved.

THEOREM 2.2

(i) *For* $n \geqq 1$, $L_p^n(\phi_{n+1}) = -\frac{1}{2}p4^n n!\,\partial^{n-1}\phi/\partial x^{n-1}$,
$L_{-p}^n(\psi_{n+1}) = \frac{1}{2}p4^n n!\,\partial^{n-1}\psi/\partial x^{n-1}$.

(ii) *For* $n \geqq 1$, $L_p^n(\Phi_n) = 4^n n!\,\partial^n\phi/\partial x^n$,
$L_{-p}^n(\Psi_n) = 4^n n!\,\partial^n\psi/\partial x^n$.

For (i), theorem 2.1 (i) and lemma 2.2 with $m = n-1$ show that if f_n represents ϕ_n or ψ_n according as $k = p$ or $-p$ and $n \geqq 2$ then

$$L_k^{n-1}(f_n) = 4(n-1)\partial/\partial x L_k^{n-2}(f_{n-1}).$$

It follows that, for $n \geqq 1$,

$$L_k^n(f_{n+1}) = 4^{n-1}n!\,\partial^{n-1}/\partial x^{n-1} L_k(f_2).$$

Finally, it can be shown that $L_k(f_2) = -2kf$ (where f stands for ϕ or ψ according as $k = p$ or $-p$) and so the required results are obtained.

For (ii), theorem 2.1 (ii) and lemma 2.2 with $m = n$ show that if f_n represents Φ_n or Ψ_n according as $k = p$ or $-p$ and $n \geqq 2$ then

$$L_k^n(f_n) = 4n\,\partial/\partial x\, L_k^{n-1}(f_{n-1}).$$

It follows that, for $n \geqq 1$,

$$L_k^n(f_n) = 4^{n-1}n!\,\partial^{n-1}/\partial x^{n-1}\, L_k(f_1).$$

This time it can be shown that $L_k(f_1) = 4\,\partial f/\partial x$ (f again standing for ϕ or ψ according as $k = p$ or $-p$) and so the results are obtained.

3. General solutions of $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$

Theorem 2.1 gives a family of solutions of the equations $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$. In particular, it is evident that $L_p^n(\Phi_m) = 0$ and $L_{-p}^n(\Psi_m) = 0$ for any integer m in the range $0 \leqq m \leqq n-1$. These solutions can be used to build up general solutions of the equations.

THEOREM.

(i) *A general solution of the equation* $L_p^n(\phi) = 0$ *is given by*

$$\phi = \mathscr{R}\{w^{(0)} + \bar{z}w^{(1)} + \cdots + \bar{z}^{n-1}w^{(n-1)}\}, \tag{8}$$

where the functions $w^{(s)}$ *are* p*-analytic.*

(*ii*) *A general solution of the equation* $L_{-p}^n(\psi) = 0$ *is given by*

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$$y^{-p}\psi = \mathscr{I}\{w^{(0)}+\bar{z}w^{(1)}+\cdots+\bar{z}^{n-1}w^{(n-1)}\}, \tag{9}$$

where the functions $w^{(s)}$ are p-analytic.

Theorem 2.1 shows that any function given by (8) is a solution of the equation $L_p^n(\phi) = 0$ and any function given by (9) is a solution of $L_{-p}^n(\psi) = 0$. It remains to prove that any solutions of the equations $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$ can be expressed in the forms (8) or (9) respectively.

LEMMA. *Define the operator I_k so that*

$$I_k(f) = \int_a^x f(\xi, y)d\xi - \int_c^y \eta^{-k}d\eta \int_b^\eta \zeta^k \frac{\partial f}{\partial x}(a, \zeta)d\zeta,$$

where a, b, c are arbitrary constants. Then I_k has the properties

$$\partial/\partial x I_k(f) = f, \tag{10}$$

$$L_k I_k(f) = \int_a^x L_k(f)dx. \tag{11}$$

(The form of $I_k(f)$ is found by assuming that

$$I_k(f) = \int_a^x f(\xi, y)d\xi + F(y)$$

so that (10) is true, and then finding $F(y)$ so that (11) also holds.)

The two parts of the theorem have similar proofs and only part (i) is considered in detail. It has to be shown that any solution of the equation $L_p^n(\phi) = 0$ can be expressed in the form (8). The case $n = 1$ is easily proved and the general result is proved by induction.

If ϕ is a known function such that $L_p^n(\phi) = 0$, then $f = L_p^{n-1}(\phi)$ is a known function such that $L_p(f) = 0$. Repeated application of (11) shows that the function $\phi^{(n-1)}$ defined by

$$\phi^{(n-1)} = I_p^{n-1}(f)/4^{n-1}(n-1)! \tag{12}$$

satisfies the equation $L_p(\phi^{(n-1)}) = 0$. It has been pointed out (in section 1) that the function $\psi^{(n-1)}$ which is conjugate to $\phi^{(n-1)}$ can be constructed. Then $L_{-p}(\psi^{(n-1)}) = 0$ and the function $w^{(n-1)} = \phi^{(n-1)}+iy^{-p}\psi^{(n-1)}$ is p-analytic. New functions $\Phi_{n-2}(x, y)$, $\Psi_{n-2}(x, y)$ and $W_{n-2}(z, \bar{z})$ are now defined so that

$$\Phi_{n-2}+iy^{-p}\Psi_{n-2} = W_{n-2} = \bar{z}^{n-2}w^{(n-1)}.$$

Then theorem 2.2 gives

$$\begin{aligned} L_p^{n-2}(\Phi_{n-2}) &= 4^{n-2}(n-2)! \quad \partial^{n-2}\phi^{(n-1)}/\partial x^{n-2}, \\ L_{-p}^{n-2}(\Psi_{n-2}) &= 4^{n-2}(n-2)! \quad \partial^{n-2}\psi^{(n-1)}/\partial x^{n-2}. \end{aligned} \tag{13}$$

Since $\phi^{(n-1)}$, $\psi^{(n-1)}$ are conjugate functions, it is easily proved that $\partial^{n-2}\phi^{(n-1)}/\partial x^{n-2}$, $\partial^{n-2}\psi^{(n-1)}/\partial x^{n-2}$ and hence $L_p^{n-2}(\Phi_{n-2})$, $L_{-p}^{n-2}(\Psi_{n-2})$ are also pairs of conjugate functions. Substituting in (13) the expression for $\phi^{(n-1)}$ given in its definition (12) and using (10) gives

$$L_p^{n-2}(\Phi_{n-2}) = I_p(f)/4(n-1)$$

so that a further application of (10) gives

$$4(n-1)\,\partial/\partial x\, L_p^{n-2}(\Phi_{n-2}) = f. \tag{14}$$

The function Φ_{n-1} is now defined so that

$$\Phi_{n-1} = \phi - x\Phi_{n-2} - y^{1-p}\Psi_{n-2} \tag{15}$$

and equations (5) and (35) of I are used to show that

$$\begin{aligned} L_p^{n-1}(\Phi_{n-1}) = L_p^{n-1}(\phi) - xL_p^{n-1}(\Phi_{n-2}) - 2(n-1)\,\partial/\partial x\, L_p^{n-2}(\Phi_{n-2}) \\ -y^{1-p}L_{-p}^{n-1}(\Psi_{n-2}) - 2(n-1)y^{-p}\,\partial/\partial y\, L_{-p}^{n-2}(\Psi_{n-2}). \end{aligned}$$

Since $L_p^{n-2}(\Phi_{n-2})$ and $L_{-p}^{n-2}(\Psi_{n-2})$ are conjugate functions,

$$\begin{gathered} L_p^{n-1}(\Phi_{n-2}) = 0, \quad L_{-p}^{n-1}(\Psi_{n-2}) = 0, \\ \partial/\partial x\, L_p^{n-2}(\Phi_{n-2}) = y^{-p}\,\partial/\partial y\, L_{-p}^{n-2}(\Psi_{n-2}). \end{gathered}$$

Thus $L_p^{n-1}(\Phi_{n-1}) = L_p^{n-1}(\phi) - 4(n-1)\,\partial/\partial x\, L_p^{n-2}(\Phi_{n-2})$ and the definition of f and equation (14) combine to show that $L_p^{n-1}(\Phi_{n-1}) = 0$.

The theorem is now assumed to be true for the equation $L_p^{n-1}(\phi) = 0$ so that Φ_{n-1} can be expressed in the form

$$\Phi_{n-1} = \mathscr{R}\{w^{(0)} + w^{(1)}\bar{z} + \cdots + w^{(n-2)}\bar{z}^{n-2}\},$$

where the functions $w^{(s)}$ are p-analytic. Since

$$x\Phi_{n-2} + y^{1-p}\Psi_{n-2} = \mathscr{R}(\bar{z}W_{n-2}) = \mathscr{R}(w^{(n-1)}\bar{z}^{n-1})$$

and $w^{(n-1)}$ is p-analytic, it follows from (15) that

$$\phi = \mathscr{R}\{w^{(0)} + w^{(1)}\bar{z} + \cdots + w^{(n-2)}\bar{z}^{n-2} + w^{(n-1)}\bar{z}^{n-1}\},$$

where all the functions $w^{(s)}$ are p-analytic. This is the result required to set up the inductive proof of part (i) and part (ii) can be proved similarly.

It will be observed that the general solutions of the equations $L_p^n(\phi) = 0$ and $L_{-p}^n(\psi) = 0$ which are given by the theorem include all the special cases mentioned in the introduction.

4. n-Conjugate functions and p^n-analytic functions

A function ϕ such that $L_p^n(\phi) = 0$ is given in terms of p-analytic functions by an expression of the form (8). Similarly, if ψ is such that $L_{-p}^n(\psi) = 0$,

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then $y^{-p}\psi$ is given by an expression of the form (9). If the p-analytic functions in these expressions are the same so that

$$\phi+iy^{-p}\psi \equiv w = w^{(0)}+w^{(1)}\bar{z}+\cdots+w^{(n-1)}\bar{z}^{n-1}, \tag{16}$$

where the functions $w^{(s)}$ are p-analytic, then the pair of functions ϕ and ψ will be described as *n-conjugate* functions.

Any function $w(z, \bar{z})$ which can be expressed in terms of p-analytic functions as in (16) will be called *p^n-analytic* and it can now be said that a pair of n-conjugate general solutions of the equations $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$ is given by the real part and y^p times the imaginary part respectively of an arbitrary p^n-analytic function. This is the generalization of the classical result (obtained from the general result by taking $p = 0$, $n = 1$) that a pair of conjugate harmonic functions is given by the real and imaginary parts of any analytic function.

5. Solutions of $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$ in terms of a given p^m-analytic function

Theorem 2.1 which gives solutions of the equations $L_p^n(\phi) = 0$ and $L_{-p}^n(\psi) = 0$ in terms of a given p-analytic function can now be generalized. A given p^m-analytic function, $w = \phi+iy^{-p}\psi$, is used to define two new functions of z and $\bar{z}$ with their corresponding real and imaginary parts, exactly as in equations (6) and (7): for any integer $n \geqq 0$, let

$$\phi_n+iy^{-p}\psi_n \equiv w_n = wz^n, \quad \Phi_n+iy^{-p}\Psi_n \equiv W_n = W\bar{z}^n.$$

The following generalizations of theorem 2.1 can now be proved.

THEOREM 5.1

(i) *For $n \geqq 1$ and $1 \leqq m \leqq n$, $L_p^n(\phi_n) = 0$, $L_{-p}^n(\psi_n) = 0$;*
for $n \geqq 1$ and $n < m$, $L_p^m(\phi_n) = 0$, $L_{-p}^m(\psi_n) = 0$.

(ii) *For $n \geqq 0$ and $m \geqq 1$, $L_p^{m+n}(\Phi_n) = 0$, $L_{-p}^{m+n}(\Psi_n) = 0$.*

(i) Consider first the case $1 \leqq m \leqq n$. Since w is p^m-analytic, $w = \sum_{s=0}^{m-1} w^{(s)}\bar{z}^s$ where the functions $w^{(s)}$ are p-analytic. Hence

$$\phi_n+iy^{-p}\psi_n = wz^n = \sum_{s=0}^{m-1} w^{(s)}\bar{z}^s z^n = \sum_{s=0}^{m-1} w^{(s)} z^{n-s} r^{2s},$$

where $z\bar{z} = r^2$. Let $w^{(s)}z^{n-s} = \phi_s+iy^{-p}\psi_s$ so that

$$\phi_n = \sum_{s=0}^{m-1} r^{2s}\phi_s, \quad \psi_n = \sum_{s=0}^{m-1} r^{2s}\psi_s.$$

Since $w^{(s)}$ is p-analytic, theorem 2.1 shows that $L_p^{n-s}(\phi_s) = 0$, $L_{-p}^{n-s}(\psi_s) = 0$

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for $0 \leqq s \leqq n-1$. Theorem 3.5 of I shows that, for $0 \leqq s \leqq n-1$, $L_k^n(r^{2s}f) = 0$ provided $L_k^{n-s}(f) = 0$. Hence, for $0 \leqq s \leqq n-1$, $L_p^n(r^{2s}\phi_s) = 0$ and $L_{-p}^n(r^{2s}\psi_s) = 0$ and so, since $m \leqq n$, $L_p^n(\phi_n) = 0$ and $L_{-p}^n(\psi_n) = 0$ as required.

When $m > n$,

$$\begin{aligned}\phi_n + iy^{-p}\psi_n &= \sum_{s=0}^{m-1} w^{(s)}\bar{z}^s z^n \\ &= \sum_{s=0}^{n-1} w^{(s)} z^{n-s} r^{2s} + r^{2n} \sum_{t=0}^{m-n-1} w^{(n+t)}\bar{z}^t.\end{aligned}$$

Let

$$\sum_{s=0}^{n-1} w^{(s)} z^{n-s} r^{2s} = f_1 + iy^{-p} g_1, \quad \sum_{t=0}^{m-n-1} w^{(n+t)}\bar{z}^t = f_2 + iy^{-p} g_2$$

so that $\phi_n = f_1 + f_2$, $\psi_n = g_1 + g_2$. From the first part of the proof, $L_p^n(f_1) = 0$, $L_{-p}^n(g_1) = 0$ and so *a fortiori*, $L_p^m(f_1) = 0$, $L_{-p}^m(g_1) = 0$. The function $f_2 + iy^{-p}g_2$ is seen to be p^{m-n}-analytic so $L_p^{m-n}(f_2) = 0$, $L_{-p}^{m-n}(g_2) = 0$ and theorem 3.5 of I is used again to deduce that $L_p^m(r^{2n}f_2) = 0$, $L_{-p}^m(r^{2n}g_2) = 0$. Thus, $L_p^m(\phi_n) = 0$ and $L_{-p}^m(\psi_n) = 0$.

(ii) Since w is p^m-analytic and so is given by $w = \sum_{s=0}^{m-1} w^{(s)}\bar{z}^s$, $W_n = w\bar{z}^n$ can be expressed in the form $W_n = \sum_{t=n}^{n+m-1} w^{(t-n)}\bar{z}^t$ and so is seen to be p^{m+n}-analytic which implies that $L_p^{m+n}(\Phi_n) = 0$ and $L_{-p}^{m+n}(\Psi_n) = 0$ as required. Moreover, Φ_n and Ψ_n are $(m+n)$-conjugate functions.

A particular case of this theorem, obtained by taking $m = n$, enables new solutions of $L_p^n(\phi) = 0$ and $L_{-p}^n(\psi) = 0$ to be constructed in terms of a given pair of n-conjugate functions. Thus, the first of Parsons' theorems [4] is generalized:

THEOREM 5.2 *If $\phi(x, y)$ and $\psi(x, y)$ are n-conjugate functions so that $w(z, \bar{z}) = \phi(x, y) + iy^{-p}\psi(x, y)$ is p^n-analytic, and functions $\phi_n(x, y)$, $\psi_n(x, y)$ and $w_n(z, \bar{z})$ are defined so that $\phi_n + iy^{-p}\psi_n \equiv w_n = wz^n$, then $L_p^n(\phi_n) = 0$, $L_{-p}^n(\psi_n) = 0$.*

6. Construction of new solutions of $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$ by changing variables

New variables ξ, η are introduced as in section 5 of I so that $\zeta = \xi + i\eta$ is related to $z = x + iy$ by the equation $\zeta = a^2/\bar{z}$. Thus if $z = re^{i\theta}$, $\zeta = \rho e^{i\theta}$ where $\rho r = a^2$. The operator in the ξ, η plane which corresponds to L_k in the $x-y$ plane is defined as $\Lambda_k \equiv \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + k\eta^{-1}\,\partial/\partial\eta$. In section 5.4 of I it is proved that for any function $f(r, \theta)$ and the corresponding function $\boldsymbol{f}(\rho, \theta) \equiv f(a^2/\rho, \theta)$,

(17) $$\Lambda_k^n\{(\rho/a)^{-k-2+2n}f\} = (r/a)^{k+2+2n}L_k^n(f).$$

Parsons' second theorem [4] can now be generalized:

THEOREM. *If* $\phi(x, y)$ *and* $\psi(x, y)$ *are n-conjugate functions so that* $w(z, \bar{z}) = \phi(x, y)+iy^{-p}\psi(x, y)$ *is* p^n*-analytic, and functions* $\hat{\phi}(\xi, \eta)$, $\hat{\psi}(\xi, \eta)$ *and* $\hat{w}(\zeta, \bar{\zeta})$ *are defined so that* $\hat{\phi}+i\eta^{-p}\hat{\psi} \equiv \hat{w} = (r/a)^{p+2-2n}\bar{w}$, *then* $\Lambda_p^n(\hat{\phi}) = 0$, $\Lambda_{-p}^n(\hat{\psi}) = 0$.

Since $r/a = (\rho/a)^{-1}$ and $\eta = a^2y/r^2$, it is easily verified that if $\Phi(\rho, \theta) \equiv \phi(r, \theta)$ and $\Psi(\rho, \theta) \equiv \psi(r, \theta)$, then

$$\hat{\phi}(\rho, \theta) = (\rho/a)^{-p-2+2n}\Phi(\rho, \theta), \quad \hat{\psi}(\rho, \theta) = -(\rho/a)^{p-2+2n}\Psi(\rho, \theta).$$

Hence, since $L_p^n(\phi) = 0$, $L_{-p}^n(\psi) = 0$, the theorem follows from equation (17).

The theorem provides (non-conjugate) solutions of $\Lambda_p^n(\hat{\phi}) = 0$ and $\Lambda_{-p}^n(\hat{\psi}) = 0$ in terms of known (conjugate) solutions of $L_p^n(\phi) = 0$ and $L_{-p}^n(\psi) = 0$ respectively.

7. Construction of a new p^n-analytic function from a given p^n-analytic function

The last stage is to show that the transformations used in sections 5 and 6, when applied in succession, lead to a new pair of n-conjugate functions of ξ, η and so to a new p^n-analytic function of ζ, $\bar{\zeta}$. It is shown further that the relation between this new p^n-analytic function and the original p^n-analytic function is symmetric.

THEOREM 7.1 *If* $\phi(x, y)$ *and* $\psi(x, y)$ *are n-conjugate functions so that* $w(z, \bar{z}) = \phi(x, y)+iy^{-p}\psi(x, y)$ *is* p^n*-analytic, and functions* $\hat{\phi}_n(\xi, \eta)$, $\hat{\psi}_n(\xi, \eta)$ *and* $\hat{w}_n(\zeta, \bar{\zeta})$ *are defined so that*

$$\hat{\phi}_n+i\eta^{-p}\hat{\psi}_n \equiv \hat{w}_n = (r/a)^{p+2-2n}\bar{w}_n = (r/a)^{p+2-2n}(\bar{z}/a)^n\bar{w},$$

then $\Lambda_p^n(\hat{\phi}_n) = 0$, $\Lambda_{-p}^n(\hat{\psi}_n) = 0$ *and* $\hat{\phi}_n$, $\hat{\psi}_n$ *are n-conjugate functions so that* $\hat{w}_n$ *is* p^n*-analytic.* (The relation between w_n and w has been amended slightly by the introduction of a factor a^{-n}.)

The first stage in the proof of this theorem is to prove the special case $n = 1$. Parsons has proved this theorem for the particular case $p = 1$; a proof for general p can be modelled directly on Parsons' proof but an alternative method is given here.

For any function $w = \phi+iy^{-p}\psi$, define the two functions g and h by the relations

$$g = \partial\phi/\partial x-y^{-p}\,\partial\psi/\partial y, \quad h = \partial\psi/\partial x+y^{-p}\,\partial\phi/\partial y.$$

Then it can be proved that

$$y^p g = \frac{\partial}{\partial \bar{z}}(y^p w) + \frac{\partial}{\partial z}(y^p \bar{w}), \quad iy^{-p} h = \frac{\partial w}{\partial \bar{z}} - \frac{\partial \bar{w}}{\partial z}.$$

The condition that ϕ and ψ be conjugate functions and w a p-analytic function is $g = h = 0$ and in this case

$$\frac{\partial}{\partial \bar{z}}(y^p w) = -\frac{\partial}{\partial z}(y^p \bar{w}) = \frac{ip}{4y} y^p(\bar{w}+w), \tag{18}$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = \frac{ip}{4y}(\bar{w}-w). \tag{19}$$

The function $\hat{w}_1(\zeta, \bar{\zeta})$ is shown to be p-analytic by verifying that $\hat{g}_1 = \hat{h}_1 = 0$, where

$$\eta^p \hat{g}_1 = \frac{\partial}{\partial \bar{\zeta}}(\eta^p \hat{w}_1) + \frac{\partial}{\partial \zeta}(\eta^p \bar{\hat{w}}_1),$$

$$i\eta^{-p} \hat{h}_1 = \frac{\partial \hat{w}_1}{\partial \bar{\zeta}} - \frac{\partial \bar{\hat{w}}_1}{\partial \zeta}.$$

This is conveniently done in two stages corresponding to the successive transformations used in obtaining $\hat{w}_1$ from w. First $\hat{w}_1$ is replaced by $(r/a)^p \bar{w}_1$ and the relation $\zeta = a^2/\bar{z}$ is used to obtain

$$\eta^p \hat{g}_1 = -\frac{a^{p-2}}{r^p}\left\{z^2 \frac{\partial}{\partial z}(y^p \bar{w}_1) + \bar{z}^2 \frac{\partial}{\partial \bar{z}}(y^p w) - \tfrac{1}{2} p y^p (z\bar{w}_1 + \bar{z} w_1)\right\},$$

$$i\eta^{-p} \hat{h}_1 = -\frac{r^p}{a^{p+2}}\left\{z^2 \frac{\partial \bar{w}_1}{\partial z} - \bar{z}^2 \frac{\partial w_1}{\partial \bar{z}} + \tfrac{1}{2} p (z\bar{w}_1 - \bar{z} w_1)\right\}.$$

Now w_1 is replaced by wz/a and because w is p-analytic the conditions given in (18) and (19) can be used to show that $\hat{g}_1 = \hat{h}_1 = 0$ and so $\hat{w}_1$ is p-analytic as required.

This theorem leads to a second preliminary result which is needed before the main theorem can be proved.

LEMMA. *If $\phi(x, y)$ and $\psi(x, y)$ are conjugate functions so that $w(z, \bar{z}) = \phi(x, y) + iy^{-p}\psi(x, y)$ is p-analytic, and the function $W(\zeta, \bar{\zeta})$ is defined so that*

$$(\bar{\zeta}/a)^{n-1} W = (r/a)^{p+2-2n}(z/a)^n \bar{w},$$

then $W(\zeta, \bar{\zeta})$ is p-analytic.

Using the relation $\zeta = a^2/\bar{z}$, it is easily proved that $W = (r/a)^p(\bar{z}/a)\bar{w}$ and so by the case $n = 1$ of theorem 7.1 which has just been proved, W is p-analytic.

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The main theorem can now be proved. Since w is p^n-analytic, $w = \sum_{s=0}^{n-1} (\bar{z}/a)^s w^{(s)}$ where the functions $w^{(s)}$ are p-analytic. Hence

$$\begin{aligned}\hat{w}_n &= (r/a)^{p+2-2n} \sum_{s=0}^{n-1} (z/a)^s (\bar{z}/a)^n \bar{w}^{(s)} \\ &= \sum_{s=0}^{n-1} (r/a)^{p+2-2(n-s)} (\bar{z}/a)^{n-s} \bar{w}^{(s)}.\end{aligned}$$

Since $w^{(s)}$ is p-analytic, the lemma shows that

$$(r/a)^{p+2-2(n-s)} (\bar{z}/a)^{n-s} \bar{w}^{(s)} = (\bar{\zeta}/a)^{n-s-1} W^{(n-s)}$$

where $W^{(n-s)}$ is p-analytic. Hence $\hat{w}_n = \sum_{s=0}^{n-1} (\bar{\zeta}/a)^{n-s-1} W^{(n-s)}$ and so $\hat{w}_n$ is p^n-analytic as required.

Finally, an elementary calculation shows that the relation between $w(z, \bar{z})$ and $\hat{w}_n(\zeta, \bar{\zeta})$ is symmetric:

THEOREM 7.2 *If* $\hat{\hat{w}}_{nn}(z, \bar{z}) = (\rho/a)^{p+2-2n} (\bar{\zeta}/a)^n \bar{\hat{w}}_n(\zeta, \bar{\zeta})$, *then* $\hat{\hat{w}}_{nn} = w$.

Theorems 7.1, 7.2 are generalizations of results given by Parsons [4] for the case $n = p = 1$.

References

[1] J. C. Burns, 'The iterated equation of generalized axially symmetric potential theory. I Particular solutions', *Journ. Austr. Math. Soc.*, 7 (1967), 263—276.

[2] E. Beltrami, *Opere Matematiche*, vol. 3 (1911), 349—382.

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol 2 (Interscience, 1962): supplement to chapter 4, especially p. 389.

[4] D. H. Parsons, 'Irrotational flow of a liquid with axial symmetry', *Proc. Edin. Math. Soc.* (2) 13 (1963), 201—204.

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