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MINKOWSKI'S FUNDAMENTAL INEQUALITY FOR REDUCED POSITIVE QUADRATIC FORMS (II)

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Abstract

A convex polytope $\mathscr{D}(\boldsymbol{\alpha})$ was defined in Barnes (1978) as the set of Minkowski-reduced forms with prescribed diagonal coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$. A local minimum of the determinant $D(f)$ over $\mathscr{D}(\boldsymbol{\alpha})$ must occur at a vertex of $\mathscr{D}(\boldsymbol{\alpha})$. Here a criterion is obtained for a given vertex to provide a local minimum, completely analogous to Voronoï's criterion for a perfect form to be extreme.

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1. Introduction

We use the definitions introduced in Part I (Barnes, 1978): $\mathscr{M}$ is the polyhedral cone of Minkowski-reduced forms in n variables; $\mathscr{D}(\boldsymbol{\alpha})$ is the subset of $\mathscr{M}$ consisting of those positive forms $f(\mathbf{x}) = \sum_1^n a_{ij} x_i x_j$ for which

$$a_{ii} = \alpha_i \quad (i = 1, \ldots, n), \tag{1.1}$$

where necessarily

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n. \tag{1.2}$$

$\mathscr{D}(\boldsymbol{\alpha})$ is in fact a convex polytope.

The convexity of the determinantal surface $D(f) = \text{constant}$ implies that the minimum value of D over $\mathscr{D}(\boldsymbol{\alpha})$, or indeed any local minimum, is attained only at a vertex of $\mathscr{D}(\boldsymbol{\alpha})$. However, it is not necessarily true that a vertex v of $\mathscr{D}(\boldsymbol{\alpha})$ provides a local minimum of $D(f)$ for $f \in \mathscr{D}(\boldsymbol{\alpha})$; a vertex for which this is true we call *extreme with respect to* $\mathscr{D}(\boldsymbol{\alpha})$, *or*, for brevity, $\mathscr{D}$-*extreme*.

The main purpose of this article is to establish an analogue of Voronoï's (1907) well-known criterion for a form to be extreme in the classical sense, namely that it be perfect and eutactic. The analogue of a perfect form is clearly a *vertex* of $\mathscr{D}(\boldsymbol{\alpha})$.

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We construct the analogue of a eutactic form as follows. Recall first that f is Minkowski-reduced if and only if, for all $i = 1, \dots, n$ and all integral $\mathbf{x}$,

$$f(\mathbf{x}) \geqslant a_{ii} \quad \text{if g.c.d.}\,(x_i, x_{i+1}, \dots, x_n) = 1. \tag{1.3}$$

If $f \in \mathscr{D}(\boldsymbol{\alpha})$, denote by $\mathbf{m}_k$ $(k = 1, \dots, t)$ those $\mathbf{x}$, other than unit vectors, for which equality holds in (1.3) (as usual, we identify $\mathbf{x}$ and $-\mathbf{x}$ in such statements). Then we say that f is *$\mathscr{D}$-eutactic* if its adjoint F is expressible in the form

$$F(\mathbf{x}) = \sum_1^n A_{ij} x_i x_j = \sum_1^t \rho_k (\mathbf{m}_k' \mathbf{x})^2 + \sum_1^n \sigma_i x_i^2, \tag{1.4}$$

where the ρ_k, σ_i are real numbers and

$$\rho_k > 0 \quad (k = 1, \dots, t). \tag{1.5}$$

THEOREM 1. *A form $f \in \mathscr{D}(\boldsymbol{\alpha})$ is $\mathscr{D}$-extreme if and only if it is a vertex of $\mathscr{D}(\boldsymbol{\alpha})$ and is $\mathscr{D}$-eutactic.*

The proof of this theorem will be based on the ideas used in the proof of Voronoï's Theorem given in Barnes (1957); as there, we need

THEOREM 2 (Stiemke, 1915). *The system of linear inequalities*

$$\sum_{j=1}^p t_{ij} u_j = 0, \quad u_j > 0 \quad (i = 1, \dots, m;\ j = 1, \dots, p) \tag{1.6}$$

has a solution $\mathbf{u}$ if and only if every solution $\mathbf{v}$ of the dual system

$$\sum_{i=1}^m t_{ij} v_i \geqslant 0 \quad (j = 1, \dots, p) \tag{1.7}$$

satisfies

$$\sum_{i=1}^m t_{ij} v_i = 0 \quad (j = 1, \dots, p).$$

We shall also need the following simple geometrical result:

LEMMA 1. *Let K be an N-dimensional convex set and $\mathbf{p}$ a point of bd K, in a neighbourhood of which K is strictly convex and bd K is smooth. Let h be the tangent plane to K at $\mathbf{p}$ and h^+ the open half-space determined by h and containing int K. Let P be a convex polytope with a vertex at $\mathbf{p}$, and suppose that the whole of P in some punctured neighbourhood of $\mathbf{p}$ lies in h^+. Then there exists a punctured neighbourhood B of $\mathbf{p}$ such that if $\mathbf{q} \in B \cap P$ then $\mathbf{q} \in$ int K.*

2. Proof of Theorem 1

We begin with the appropriate analogue of the Lemma of Barnes (1957):

LEMMA 2. *If $f \in \mathscr{D}(\boldsymbol{\alpha})$, then f is $\mathscr{D}$-extreme if and only if there exists no non-trivial quadratic form $g(\mathbf{x}) = \sum_1^n b_{ij} x_i x_j$ satisfying*

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$$g(\mathbf{e}_i) = 0 \quad (i = 1, \ldots, n), \tag{2.1}$$

$$g(\mathbf{m}_k) \geqslant 0 \quad (k = 1, \ldots, t), \tag{2.2}$$

$$(F, g) = \sum A_{ij} b_{ij} \leqslant 0. \tag{2.3}$$

PROOF. (i) Suppose that (2.1), (2.2), (2.3) have a non-trivial solution g. Choose one with, say $\max |b_{ij}| = 1$, and set $f' = f + \varepsilon g$ where $\varepsilon > 0$ is small.

Then $f' \in \mathscr{D}(\boldsymbol{\alpha})$. For firstly

$$f'(\mathbf{e}_i) = f(\mathbf{e}_i) = \alpha_i \quad (i = 1, \ldots, n);$$

and, for any $k = 1, \ldots, t$ for which $\mathbf{m}_k$ satisfies (1.3) with equality for some i,

$$f'(\mathbf{m}_k) = f(\mathbf{m}_k) + \varepsilon g(\mathbf{m}_k) = \alpha_i + \varepsilon g(\mathbf{m}_k) \geqslant \alpha_i.$$

Next, there exist only finitely many other $\mathbf{x}$ in (1.3) which are necessary to specify $\mathscr{M}$ and, for all of these, $f(\mathbf{x}) > a_{ii} = \alpha_i$, whence also $f'(\mathbf{x}) > \alpha_i$ if ε is sufficiently small.

Now the tangent plane h to the determinantal surface (using φ as current co-ordinates in the coefficient space) at f is

$$(F, \varphi) = (F, f) = n \det f.$$

Since

$$(F, f') = (F, f) + \varepsilon(F, g) \leqslant (F, f),$$

it follows that f' lies in the closed half-space opposite to that containing the surface $\det \varphi = \det f$; hence, since this surface is strictly convex and $f' \neq f$, $\det f' < \det f$.

It follows from these results that f is not $\mathscr{D}$-extreme.

(ii) Suppose that (2.1), (2.2), (2.3) have only the trivial solution; let $f' = f + g$ $(g \neq 0)$ be any form in $\mathscr{D}(\boldsymbol{\alpha})$ close to f. Then

$$\begin{aligned} f'(\mathbf{e}_i) &= \alpha_i = f(\mathbf{e}_i), \quad \text{so that } g(\mathbf{e}_i) = 0 \quad (i = 1, \ldots, n); \\ f'(\mathbf{m}_k) &\geqslant \alpha_i = f(\mathbf{m}_k), \quad \text{so that } g(\mathbf{m}_k) \geqslant 0 \quad (k = 1, \ldots, t). \end{aligned}$$

Since g is non-trivial, our hypothesis implies that (2.3) is false, so that $(F, g) > 0$ and

$$(F, f') = (F, f) + (F, g) > (F, f). \tag{2.4}$$

We now apply Lemma 1, taking K to be the determinantal body $\det \varphi \geqslant \det f$, h the tangent plane $(F, \varphi) = (F, f)$, and P the polytope $\mathscr{D}(\boldsymbol{\alpha})$. Using in particular (2.4), we see that the hypotheses of the lemma hold and it follows that, if f' is sufficiently close to f in $\mathscr{D}(\boldsymbol{\alpha})$, but distinct from f, then $\det f' > \det f$. Thus f is $\mathscr{D}$-extreme.

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PROOF OF THEOREM 1. Using the coefficients b_{ij} of the form g, write (2.1)–(2.3) as

$$(2.5)\qquad \begin{cases} \begin{array}{r} b_{ii} \geqslant 0 \\ -b_{ii} \geqslant 0 \end{array} & (i = 1, \dots, n), \\ \sum_1^n b_{ij} m_{ik} m_{jk} \geqslant 0 & (k = 1, \dots, t), \\ -\sum A_{ij} b_{ij} \geqslant 0. \end{cases}$$

We identify this with the system (1.7), with $b_{11}, b_{12}, \dots, b_{nn}$ playing the part of the variables $v_1, v_2, \dots, v_m$. Then (1.6) becomes, using variables $\lambda_i, \mu_i\ (i = 1, \dots, n)$, $\rho_k\ (k = 1, \dots, t), \nu$,

$$(2.6)\qquad \begin{cases} \lambda_i \delta_{ij} - \mu_i \delta_{ij} + \sum_1^t \rho_k m_{ik} m_{jk} - \nu A_{ij} = 0 \quad (i, j = 1, \dots, n), \\ \lambda_i > 0, \quad \mu_i > 0, \quad \rho_k > 0, \quad \nu > 0. \end{cases}$$

(a) Suppose that f is $\mathscr{D}$-extreme. By Lemma 2, every solution of (2.5) is trivial and so certainly has equality throughout; it follows at once that f is a vertex of $\mathscr{D}(\boldsymbol{\alpha})$. Also, by Stiemke's Theorem, (2.6) has a solution; dividing through by ν, we may suppose that the solution has $\nu = 1$; multiplying by $x_i x_j$ and summing, we obtain

$$F(\mathbf{x}) = \sum_1^n A_{ij} x_i x_j = \sum_1^t \rho_k (\mathbf{m}_k' \mathbf{x})^2 + \sum_1^n (\lambda_i - \mu_i) x_i^2$$

which gives (1.4), with (1.5), noting that $\sigma_i = \lambda_i - \mu_i$ is unrestricted in sign.

(b) Suppose next that f is a vertex of $\mathscr{D}(\boldsymbol{\alpha})$ and is $\mathscr{D}$-eutactic. Then (2.6) has a solution and so, by Stiemke's Theorem, any solution g of (2.5) satisfies (2.5) with equality throughout; since f is a vertex of $\mathscr{D}(\boldsymbol{\alpha})$ it then follows that $g \equiv 0$. It now follows from Lemma 2 that f is $\mathscr{D}$-extreme.

3. An example

As noted in Part I, the quaternary form

$$(3.1)\quad f(\mathbf{x}) = ax_1^2 + ax_1x_2 - ax_1x_3 - ax_1x_4 + bx_2^2 - bx_2x_4 + cx_3^2 + cx_3x_4 + dx_4^2$$

is a vertex of $\mathscr{D}(\boldsymbol{\alpha}) = \mathscr{D}(a, b, c, d)$ which is however not $\mathscr{D}$-extreme for some values of a, b, c, d (where we still of course assume (1.2), that is

$$(3.2)\qquad 0 < a \leqslant b \leqslant c \leqslant d).$$

THEOREM 2. *Suppose that* $a < b$. *Then the form* (3.1), *subject to* (3.2), *is $\mathscr{D}$-extreme if and only if*

$$(3.3)\qquad ad < bc.$$

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PROOF. It is easily verified that for the form (3.1), neglecting unit vectors, there are just seven relations (1.3) that hold with equality, namely

$$f(-1,1,0,0)=b, \quad f(1,0,1,0)=c,$$
$$f(1,0,0,1)=f(0,1,0,1)=f(0,0,-1,1)=f(0,1,-1,1)=f(-1,1,-1,1)=d.$$

These suffice to establish that f is a vertex of $\mathscr{D}(\boldsymbol{\alpha})$.

[*Note*: If $a=b$, there is one further relation $f(1,-1,1,0)=c$ and the following analysis therefore does not apply.]

The identity (1.4) is now

$$\begin{aligned}\sum_{1}^{4} A_{ij}x_ix_j &= \rho_1(-x_1+x_2)^2+\rho_2(x_1+x_3)^2+\rho_3(x_1+x_4)^2\\ &\quad+\rho_4(x_2+x_4)^2+\rho_5(-x_3+x_4)^2+\rho_6(x_2-x_3+x_4)^2\\ &\quad+\rho_7(-x_1+x_2-x_3+x_4)^2+\sum_1^4\sigma_ix_i^2,\end{aligned} \tag{3.4}$$

and this yields

$$\begin{aligned}\rho_4 = A_{23}+A_{24} &= \tfrac{1}{8}a(ab+ac-2ad-2bc)+\tfrac{1}{8}a(-ab-ac+4bc)\\ &= \tfrac{1}{4}a(bc-ad).\end{aligned}$$

If now $ad \geqslant bc$, then $\rho_4 \leqslant 0$ and so, by Theorem 2, f is not $\mathscr{D}$-extreme. If however $ad<bc$, it is easy to verify that (3.4) is soluble with all $\rho_k>0$; one may take ρ_7 sufficiently small and positive and then determine $\rho_1,\dots,\rho_6$ from the relations

$$\rho_1=-A_{12}-\rho_7, \quad \rho_2=A_{13}-\rho_7, \quad \rho_3=A_{14}+\rho_7,$$
$$\rho_4=\rho_5=\tfrac{1}{4}a(bc-ad), \quad \rho_6=-A_{23}-\rho_7.$$

4. A refinement of Theorem 1

Among the inequalities (1.3) there is a finite set which implies all remaining inequalities; that is, there is a finite set S of vectors $\mathbf{x}$ for which the corresponding equations $f(\mathbf{x})=a_{ii}$ define the facets of $\mathscr{M}$. In determining $\mathscr{D}(\boldsymbol{\alpha})$ and its vertices, it suffices of course to consider only this minimal set of inequalities. However, the application of Theorem 1 is complicated by the need to know all the examples of equality in (1.3) for a given form f. Fortunately it turns out to be necessary, in testing whether a vertex f of $\mathscr{D}(\boldsymbol{\alpha})$ is $\mathscr{D}$-extreme, to consider only vectors in S:

THEOREM 3. *Let f be a vertex of $\mathscr{D}(\boldsymbol{\alpha})$ and $\mathbf{m}_k$ $(k=1,\dots,r)$ $(r\leqslant t)$ be the vectors of S, other than unit vectors, for which equality holds in* (1.3) *Then f is $\mathscr{D}$-extreme if and only if f is eutactic with respect to this set of vectors, that is, if*

$$F(\mathbf{x})=\sum_1^n A_{ij}x_ix_j=\sum_1^r \rho_k'(\mathbf{m}_k'\mathbf{x})^2+\sum_1^n\sigma_i'x_i^2 \tag{4.1}$$

for some real ρ_k', σ_i' satisfying $\rho_k'>0$ $(k=1,\dots,r)$.

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For the proof of Theorem 4 it clearly suffices to establish the corresponding strengthening of Lemma 2, namely

LEMMA 3. *Let* $f \in \mathscr{D}(\boldsymbol{\alpha})$ *and let* $\mathbf{m}_k$ $(k = 1, \ldots, r)$ *be those* $\mathbf{x} \in S$, *other than unit vectors, for which equality holds in* (1.3). *Then* f *is* $\mathscr{D}$*-extreme if and only if there exists no non-trivial quadratic form* g *satisfying*

$$g(\mathbf{e}_i) = 0 \quad (i = 1, \ldots, n), \quad g(\mathbf{m}_k) \geqslant 0 \quad (k = 1, \ldots, r), \quad (F, g) \leqslant 0. \tag{4.2}$$

PROOF. Only one modification is needed in the proof given for Lemma 2, namely for the assertion in part (i) that $f' \in \mathscr{D}(\boldsymbol{\alpha})$. We now have

$$f'(\mathbf{e}_i) = f(\mathbf{e}_i) = \alpha_i \quad (i = 1, \ldots, n)$$

and, for each $k = 1, \ldots, r$ and the corresponding i,

$$f'(\mathbf{m}_k) = f(\mathbf{m}_k) + \varepsilon g(\mathbf{m}_k) \geqslant f(\mathbf{m}_k) = \alpha_i.$$

Next, if $\mathbf{x} \in S$ but is not a unit vector or one of $\mathbf{m}_1, \ldots, \mathbf{m}_r$, our hypothesis implies that $f(\mathbf{x}) > a_{ii}$, whence also $f'(\mathbf{x}) > a_{ii} = \alpha_i$ if ε is sufficiently small. Since S is finite, it follows that $f'(\mathbf{x}) \geqslant \alpha_i$ for all $\mathbf{x} \in S$ (and corresponding i), and hence, by definition of S, for all integral $\mathbf{x}$.

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