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CUP PRODUCTS AND GROUP EXTENSIONS

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Abstract

Let G be a finitely generated group and let R be a commutative ring, regarded as a G-module with G acting trivially. We shall determine when the cup product of two elements of $H^1(G, R)$ is zero. Our method will use the interpretation of $H^2(G, R)$ as extensions of G by R. This will give an alternative demonstration of results of Hillman and Würfel.

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1. Introduction

Throughout this paper G is a finitely generated group, p is 0 or a prime, and $k = \mathbb{Z}/p\mathbb{Z}$ regarded as a $\mathbb{Z}G$-module with G acting trivially. The kernel of the cup product $\cup: H^1(G, k) \otimes H^1(G, k) \to H^2(G, k)$ was studied in [3], [4] and [7], these papers depending in part on cochain calculations. We shall offer a different approach, using the interpretation of $H^2(G, k)$ as extensions of G by k. With the exception of Theorem 2(ii), our results are essentially those of [4] and [7].

For any group H we set $H^* = H'H^p$ when $p \neq 0$. As usual, we shall identify $H^1(H, k)$ with $\operatorname{Hom}(H, k)$. If H is nilpotent, then $\tau(H)$ will indicate the torsion subgroup of H (in other words the elements of finite order in H), and when $p = 0$, we define G^* by $G^*/G' = \tau(G/G')$.

Let $H^1(G, k) \wedge H^1(G, k)$ denote the alternating product, and $H^1(G, k) \odot H^1(G, k)$ the symmetric product. Since the cup product is anticommutative, it induces homomorphisms

$$\gamma: H^1(G, k) \wedge H^1(G, k) \to H^2(G, k) \qquad \text{if } p \neq 2,$$

and

$$\theta: H^1(G, k) \odot H^1(G, k) \to H^2(G, k) \qquad \text{if } p = 2,$$

as described in [3]. We shall prove

THEOREM 1. *Suppose $p \neq 0$, and $f, g \in \mathrm{Hom}(G, k)$ are linearly independent over k with kernels H, K respectively.*

(i) *$f \cup g = 0$ if and only if $H^* K^* \neq G^*$.*
(ii) *If $p \neq 2$, then $\ker \gamma \cong G^*/[G^*, G]G^p$.*
(iii) *If $p = 2$, then $\ker \theta \cong G^*/[G^*, G]G^{**}$.*

THEOREM 2. *Let $p = 0$, let $f, g \in \mathrm{Hom}(G, \mathbb{Z})$ be linearly independent over $\mathbb{Z}$, and let $H = \ker f \cap \ker g$.*

(i) *If $K/[G^*, G] = \tau(G/[G^*, G])$, then $\ker \gamma \cong G^*/K$.*
(ii) *Suppose r is the index of $\langle \overline{f}, \overline{g} \rangle$ in $H^1(G/H, \mathbb{Z})$, where $\overline{f}$ and $\overline{g}: G/H \to \mathbb{Z}$ are the homomorphisms induced by f and g respectively. If $T/[H, G] = \tau(G/[H, G])$, then $f \cup g$ has finite additive order in $H^2(G, \mathbb{Z})$ if and only if $G'/[H, G]$ is infinite, and in this case the order is $\frac{1}{r}\,\mathrm{l.c.m.}(r, |G^*/G'T|)$.*

We use the following method: as in [4] we consider the five term exact sequence associated with the group extension $1 \to G^* \to G \to G/G^* \to 1$:

$$0 \to H^1(G/G^*, k) \to H^1(G, k) \to H^1(G^*, k)^G \xrightarrow{\delta} H^2(G/G^*, k) \to H^2(G, k).$$

It will be important to describe the map δ accurately. This will be done by using group extensions (Lemma 3) and the well known structure of $H^2(G/G^*, k)$ (Lemmas 4 and 5). The motivation for this paper was to show that the approach of [4] could be modified so as to avoid complicated cochain calculations.

2. Notation

Mappings will mostly be written on the left, and modules will be left modules. Let $A, B \leq H$ be groups, let $X \subseteq H$, and let M be a $\mathbb{Z}H$-module. Then we use the notation H' for the commutator subgroup of

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H, $\langle X \rangle$ for the subgroup generated by X, $|X|$ for the order X, $[A, B]$ for $\langle a^{-1}b^{-1}ab | a \in A, b \in B \rangle$, and M^H for $\{m \in M | hm = m$ for all $h \in H\}$. The restriction map from $H^2(H, M)$ to $H^2(A, M)$ will be denoted by $\mathrm{res}_{H,A}$, and the lowest common multiple of two positive integers by l.c.m. If θ is a map, then $\operatorname{im}\theta$ will indicate the image of θ, and $\ker\theta$ the kernel of θ. Suppose $A, B \triangleleft H$, A is abelian and B acts trivially on M. Then we can also view M as a $\mathbb{Z}[H/B]$-module, and we can make A into a $\mathbb{Z}H$-module by letting H act via conjugation so that $h \cdot a = hah^{-1}$ for $a \in A$ and $h \in H$; we shall use these well known observations without further comment in the future.

3. Preliminary results

Most of the lemmas in this section are well known. For the purposes of this paper, the theory on page 294 of [2] instead of Lemma 3 would be sufficient.

LEMMA 3. *Let A be an abelian normal subgroup of the group H, let $K = H/A$, let M be a $\mathbb{Z}K$-module, and view A as a $\mathbb{Z}K$-module with K acting on A by conjugation. Let $f \in \mathrm{Hom}_{\mathbb{Z}K}(A, M)$ and let*

$$\delta: H^1(A, M)^H = \mathrm{Hom}_{\mathbb{Z}K}(A, M) \to H^2(K, M)$$

be the transgression map associated with the group extension $1 \to A \to H \to K \to 1$. Suppose $\chi: K \times K \to A$ is a factor set representing the element in $H^2(K, A)$ corresponding to the above extension. Then (after choosing the notation correctly) $-\delta(f)$ is an element of $H^2(K, M)$ which is represented by the factor set $f\chi: K \times K \to M$. In particular if f is surjective, then $\delta(f)$ is represented by a group extension of the form

$$1 \to A/\ker f \to H/\ker f \to K \to 1.$$

PROOF. To ensure that $-\delta(f)$ and $f\chi$ represent the same element in $H^2(K, A)$, we need to choose the notation correctly, and the notation of [6, IV.4 and XI.9] will suffice. Let T be a set of coset representatives for A in H, let $^-: H \to K$ denote the natural epimorphism, and let $B(\mathbb{Z}H)$ denote the (normalized) bar resolution [6, page 114]. Thus $B_n(\mathbb{Z}H)$ is the free $\mathbb{Z}H$-module with free generators $\{[x_1|\dots|x_n] | x_i \in H\backslash 1\}$, and f is represented by any $\hat{f} \in \mathrm{Hom}_{\mathbb{Z}H}(B_1(\mathbb{Z}H), M)$ such that $\hat{f}([a]) = f(a)$ for all $a \in A\backslash 1$; we shall define $\hat{f}$ by $\hat{f}([at]) = f(a)$ for all $\alpha \in A$ and $t \in T$ $(at \neq 1)$, and assume $1 \in T$. Let $\partial: B_2(\mathbb{Z}H) \to B_1(\mathbb{Z}H)$ be the boundary map defined by

$$\partial([x|y]) = x[y] - [xy] + [x]$$

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for $x, y \in H\backslash 1$. Then $\delta(f)$ is represented by the factor set $\psi: K \times K \to M$ satisfying $\psi(\overline{x}, \overline{y}) = \hat{f}\partial([x|y])$ for $x, y \in H$ (cf. the "connection" of [6, page 349]). In fact if we write $x = ar$, $y = bs$, $xy = ct$ $(a, b, c \in A;\ r, s, t \in T)$, then $\psi(\overline{x}, \overline{y}) = f(ts^{-1}r^{-1})$. But the factor set χ can be defined by $\chi(\overline{x}, \overline{y}) = rst^{-1}$ (see [6, page 111]), and we deduce that $-\delta(f)$ and $f\chi$ represent the same element of $H^2(K, M)$.

Now suppose f is surjective. If $f\chi$ is represented by an extension of the form $1 \to M \to E \to K \to 1$, then there exists a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \searrow & \\ & & \big\downarrow{\scriptstyle -f} & & \big\downarrow{\scriptstyle \theta} & & K \to 1 \\ 1 & \longrightarrow & M & \longrightarrow & E & \nearrow & \end{array}$$

for some group homomorphism $\theta: H \to E$, necessarily surjective, and the result follows.

LEMMA 4. *Let p be a prime, let G be a finite elementary abelian p-group, and let $(f_1, \dots, f_n)$ be a k-basis for $H^1(G, k)$.*

(i) *If $p = 2$, then the set $\{f_i \cup f_j | 1 \leq i, j \leq n\}$ is a k-basis for $H^2(G, k)$.*

(ii) *If p is odd, then the set $\{f_i \cup f_j,\ \beta f_l | 1 \leq i < j \leq n,\ 1 \leq l \leq n\}$ is a k-basis for $H^2(G, k)$, where $\beta: H^1(G, k) \to H^2(G, k)$ is the Bockstein map. In particular if $\chi \in H^2(G, k)$, then $\operatorname{res}_{G,A}\chi = 0$ for all cyclic subgroups A of G if and only if $\chi = \sum_{i<j} \lambda_{ij} f_i \cup f_j$ for some $\lambda_{ij} \in k$.*

LEMMA 5. *Let G be a free abelian group, and let $(f_1, \dots, f_n)$ be a $\mathbb{Z}$-basis for $H^1(G, \mathbb{Z})$. Then the set $\{f_i \cup f_j | 1 \leq i < j \leq n\}$ is a $\mathbb{Z}$-basis for $H^2(G, \mathbb{Z})$.*

PROOF. Lemmas 4 and 5 follows from the Künneth theorem (see [1, page 101] and [5, VI.15]). For information of the Bockstein map, see [1, 2.23].

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LEMMA 6. *Let p be a prime, let G be an elementary abelian p-group, and let $(f_1, f_2, \dots, f_n)$ be a k-basis for $H^1(G, k)$. Write $K_1 = \ker f_1$ and $K_2 = \ker f_2$. Suppose $\chi \in H^2(G, k)$ is represented by the group extension*

$$0 \to k \to E \xrightarrow{\theta} G \to 1.$$

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(i) *If* $\theta^{-1}(K_1)$ *and* $\theta^{-1}(K_2)$ *are elementary abelian, then* $\chi = \lambda f_1 \cup f_2$ *for some* $\lambda \in k$.

(ii) *If* p *is odd, then* E *has exponent* p *if and only if* $\chi = \sum_{i<j} \lambda_{ij} f_i \cup f_j$ *for some* $\lambda_{ij} \in k$.

PROOF. Suppose $\theta^{-1}(K_1)$ and $\theta^{-1}(K_2)$ are elementary abelian. By Lemma 4 we may write

$$\chi = \sum_{i \le j} \lambda_{ij} f_i \cup f_j \quad \text{if } p = 2,$$

and

$$\chi = \sum_{i<j} \lambda_{ij} f_i \cup f_j + \sum_i \lambda_i \beta f_i \quad \text{if } p \text{ is odd},$$

where $\lambda_{ij}, \lambda_i \in k$. Since $\theta^{-1}(K_1)$ is elementary abelian, $\operatorname{res}_{G,K_1} \chi = 0$ and we see that $\lambda_i = \lambda_{ij} = 0$ if $i \ne 1$. Also $\theta^{-1}(K_2)$ is elementary abelian, hence $\operatorname{res}_{G,K_2} \chi = 0$ and we deduce that $\lambda_1 = \lambda_{1j} = 0$ if $j \ne 2$. This proves (i).

It is easy to show (and is well known) that E has exponent p if and only if $\operatorname{res}_{G,A} \chi = 0$ for all cyclic subgroups A of G. Thus we obtain (ii) from Lemma 4(ii).

LEMMA 7. *Let* G *be a free abelian group, let* $(f_1, \dots, f_n)$ *be a* $\mathbb{Z}$*-basis for* $H^1(G, \mathbb{Z})$, *and let* $K = \ker f_1 \cap \ker f_2$. *Suppose* $\chi \in H^2(G, \mathbb{Z})$ *is represented by the group extension*

$$0 \to \mathbb{Z} \to E \xrightarrow{\theta} G \to 1.$$

(i) *If* $[E, \theta^{-1}(K)] = 1$, *then* $\chi = r f_1 \cup f_2$ *where either* $r = 0$ *or* $r = \pm|\ker \theta / E'|$.

(ii) *If* $\chi = r f_1 \cup f_2$ *where* $r \in \mathbb{Z}$, *then* $[E, \theta^{-1}(K)] = 1$.

PROOF. By Lemma 5, write $\chi = \sum_{i<j} \lambda_{ij} f_i \cup f_j$ where $\lambda_{ij} \in \mathbb{Z}$, and set $L = \ker \theta$ and $K_i = \ker f_i$ $(1 \le i \le n)$.

(i) Since $\theta^{-1}(K_1)$ is abelian, $\operatorname{res}_{G,K_1} \chi = 0$ and we see that $\lambda_{ij} = 0$ if $i \ne 1$, and then $\theta^{-1}(K_2)$ abelian imples that $\operatorname{res}_{G,K_2} \chi = 0$, and hence $\lambda_{1j} = 0$ if $j \ne 2$. Thus $\chi = r f_1 \cup f_2$ for some $r \in \mathbb{Z}$. Suppose $r = \pm 1$. If $E' \ne L$, then there exists a prime p such that $E' \subseteq L^p$. Let $\pi: \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ denote the natural surjection. Since $(\pi f_1, \dots, \pi f_n)$ is a $\mathbb{Z}/p\mathbb{Z}$-basis for $H^1(G, \mathbb{Z}/p\mathbb{Z})$, it follows from Lemma 4 that $\pi f_1 \cup \pi f_2 \ne 0$, and hence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to E/L^p \to G \to 1$$

is nonsplit. This contradicts $E' \subseteq L^p$. Therefore $E' = L$.

In general if $0 \neq r \in \mathbb{Z}$, let $\mu_r : \mathbb{Z} \to \mathbb{Z}$ denote multiplication by r. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & E_1 & \searrow & \\ & & \Big\downarrow \mu_r & & \Big\downarrow \varphi & & G \to 1 \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & E & \nearrow & \end{array}$$

for some map φ, where the top sequence represents $f_1 \cup f_2$ and the bottom sequence $rf_1 \cup f_2$. This shows that $|L/E'| = |r|$, and (i) follows.

(ii) Since $\mathrm{res}_{K_1} f_1 \cup f_2 = \mathrm{res}_{K_2} f_1 \cup f_2 = 0$, we see that $\theta^{-1}(K_1)$ and $\theta^{-1}(K_2)$ are abelian, and hence $[\theta^{-1}(K_1)\theta^{-1}(K_2),\ \theta^{-1}(K)] = 1$. But $[a, b]^s = [a^s, b]$ for $a, b \in E$, $s \in \mathbb{Z}$, and $\theta^{-1}(K_1)\theta^{-1}(K_2)$ has finite index in E, and the proof of (ii) is easily completed.

PROOF OF THEOREM 1. Consider the five term exact sequence associated with the group extension $1 \to G^* \to G \to G/G^* \to 1$:

$$0 \to H^1(G/G^*, k) \xrightarrow{\theta} H^1(G, k) \to H^1(G^*, k)^G \xrightarrow{\delta} H^2(G/G^*, k) \xrightarrow{\varphi} H^2(G, k)$$

where θ and φ are the inflation maps, and δ is the transgression map. Choose $\overline{f}, \overline{g} \in H^1(G/G^*, k)$ such that $\theta(\overline{f}) = f$, $\theta(\overline{g}) = g$. If $f \cup g = 0$, then $\overline{f} \cup \overline{g} = \delta(u)$ for some $u \in H^1(G^*, k)^G$; note that $u \neq 0$, so u is onto and $G^*/L \cong k$ where $L = \ker u$. Using Lemma 3, we see that $\delta(u)$ is represented by a group extension of the form

$$0 \to k \to G/L \xrightarrow{\pi} G/G^* \to 1$$

for some homomorphism π. Since $\mathrm{res}_{G/G^*, H/G^*} \overline{f} \cup \overline{g} = 0$, it follows that $\pi^{-1}(H/G^*)$ is elementary abelian, and hence $H^* \subseteq L$. Similarly $K^* \subseteq L$ and we deduce that $H^*K^* \neq G^*$.

Conversely suppose $H^*K^* \neq G^*$. Choose a subgroup M such that $H^*K^* \subseteq M < G^*$ and M is maximal under these conditions. Then $M \lhd G$ and $G^*/M \cong k$ because $[G, G^*] = [HK, G^*] = [H, G^*][K, G^*] \subseteq H^*K^*$. Since H/M and K/M are elementary abelian, application of Lemma 6(i) shows that

$$0 \to k \to G/M \to G/G^* \to 1$$

is represented by $\lambda\overline{f} \cup \overline{g}$ for some $\lambda \in k$. Now G/M is *not* elementary abelian, hence $\lambda \neq 0$ and it follows from Lemma 3 that $\overline{f} \cup \overline{g} \in \mathrm{im}\,\delta$. Therefore $f \cup g = 0$ which proves (i).

Now suppose p is odd and let

$$\overline{\gamma}: H^1(G/G^*, k) \wedge H^1(G/G^*, k) \to H^2(G/G^*, k)$$

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be the map induced by the cup product. Then $\overline{\gamma}$ is a monomorphism by Lemma 4(ii) and $\varphi\overline{\gamma} = \gamma(\theta \wedge \theta)$, and hence

$$\ker \gamma \cong \ker \gamma(\theta \wedge \theta) = \ker \varphi\overline{\gamma} \cong \ker \varphi \cap \operatorname{im} \overline{\gamma} = \operatorname{im} \delta \cap \operatorname{im} \overline{\gamma}$$

because θ is an isomorphism and $\ker \varphi = \operatorname{im} \delta$. Since δ is a monomorphism, we deduce that

$$\ker \gamma \cong \left\{ v \in H^1(G^*, k)^G \mid \delta(v) = \sum_{i<j} \lambda_{ij} f_i \cup f_j \text{ for some } \lambda_{ij} \in k \right\}.$$

If $v \neq 0$ and $N = \ker v$, then $\delta(v)$ is represented by an extension of the form

$$0 \to k \to G/N \to G/G^* \to 1$$

by Lemma 3. It now follows from Lemma 6(ii) that G/N has exponent p if and only if $\delta(v) = \sum_{i<j} \lambda_{ij} f_i \cup f_j$ for some $\lambda_{ij} \in k$. Therefore $\ker \gamma \cong \operatorname{Hom}(G^*/[G^*, G]G^p, k)$. But G is finitely generated, hence G^* is finitely generated and we conclude that $\ker \gamma \cong G^*/[G^*, G]G^p$ as required.

The case $p = 2$ is similar but easier; one uses Lemma 4(i) instead of Lemmas 4(ii) and 6(ii). Since this argument is identical to that of [4, Section 3], we omit it.

Proof of Theorem 2. Consider the five term exact sequence associated with the group extension $1 \to G^* \to G \to G/G^* \to 1$:

$$0 \to H^1(G/G^*, \mathbb{Z}) \xrightarrow{\theta} H^1(G, \mathbb{Z}) \to H^1(G^*, \mathbb{Z})^G$$
$$\xrightarrow{\delta} H^2(G/G^*, \mathbb{Z}) \xrightarrow{\varphi} H^2(G, \mathbb{Z}),$$

where θ and φ are the inflation maps, and δ is the transgression map. If $\overline{\gamma}\colon H^1(G/G^*, \mathbb{Z}) \wedge H^1(G/G^*, \mathbb{Z}) \to H^2(G/G^*, \mathbb{Z})$ is the homomorphism induced by the cup product, then $\overline{\gamma}$ is an isomorphism by Lemma 5, and $\varphi\overline{\gamma} = \gamma(\theta \wedge \theta)$. Since θ is an isomorphism, δ is a monomorphism and we deduce that

$$\ker \gamma \cong \ker \gamma(\theta \wedge \theta) = \ker \varphi\overline{\gamma} \cong \ker \varphi = \operatorname{im} \delta$$
$$\cong \operatorname{Hom}(G^*/[G^*, G], \mathbb{Z}) \cong G^*/K$$

(because G is finitely generated implies $G^*/[G^*, G]$ is finitely generated) which proves (i). The argument of this section is identical to that of [4, Section 2].

Now let (e, h) be a $\mathbb{Z}$-basis for $H^1(G/H, \mathbb{Z})$. By anticommutativity of the cup product $e \cup e = h \cup h = 0$, hence $f \cup g = \pm r e \cup h$, so we may assume that $r = 1$ and that $(\overline{f}, \overline{g})$ is a $\mathbb{Z}$-basis for $H^1(G/H, \mathbb{Z})$. Choose a $\mathbb{Z}$-basis $(f_1, \dots, f_n)$ of $H^1(G/G^*, \mathbb{Z})$ such that $\theta(f_1) = f$ and $\theta(f_2) = g$. Suppose

$f \cup g$ has finite additive order s. Then there exists $u \in \mathrm{Hom}(G^*/[G^*, G], \mathbb{Z})$ such that $\delta(u) = sf_1 \cup f_2$. Note that $u \neq 0$ so if $L = \ker u$, then $G^*/L \cong \mathbb{Z}$. Let $v: G^* \to \mathbb{Z}$ be an epimorphism with kernel L, so $u = tv$ for some $t \in \mathbb{Z}$. Then $t\delta(v) = sf_1 \cup f_2$, thus $t|s$ by Lemma 5 and we deduce that $t = \pm 1$. Also application of Lemma 3 shows that $\delta(v)$ is represented by an extension of the form

$$0 \to \mathbb{Z} \to G/L \to G/G^* \to 1.$$

Therefore $[H, G] \subseteq L$ and $|G^*/G'L| = s$ by Lemma 7. But it is easy to show that $G'/[H, G]$ is cyclic, hence $L = T$ and we conclude that $|G^*/G'T| = s$.

Conversely suppose $G'/[H, G]$ is infinite. Since $G'/[H, G]$ is cyclic, it follows that $G^*/T \cong \mathbb{Z}$. If $w: G^* \to \mathbb{Z}$ is an epimorphism with kernel T, then $\delta(w)$ is represented by an extension of the form

$$0 \to \mathbb{Z} \to G/T \to G/G^* \to 1$$

by Lemma 3. From Lemma 7(i), this extension also represents $\pm l f_1 \cup f_2$ where $l = |G^*/G'T|$. It follows that $lf \cup g = 0$ and hence $f \cup g$ has finite order. This complete the proof of Theorem 2.

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