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STABLE RANK AND THE $\bar{\partial}$-EQUATION

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ABSTRACT. Let G denote a plane domain bounded by finitely many closed, non-intersecting Jordan curves. We show the following refinement of the stable rank one property of $A(\overline{G})$: Suppose that for $f, g \in A(\overline{G})$, $g \neq 0$, there exists $\delta > 0$ such that $|f(z)| + |g(z)| \geq \delta$ $(z \in \overline{G})$. Then there exist $h, k \in A(\overline{G})$ such that

$$f + hg = \exp(k).$$

Also we obtain a partial result for the algebra $H^\infty(G)$.

Introduction. Let A be the disk algebra, consisting of all functions analytic on the unit disk and continuous on its closure. In [5] the authors proved the following result: Suppose $f_1, f_2 \in A$ and $|f_1(z)| + |f_2(z)| > 0$ for all z with $|z| \leq 1$. Then there are $g_1, g_2 \in A$ with $g_1^{-1} \in A$ and $g_1 f_1 + g_2 f_2 = 1$.

By a topological argument it is known that a branch of $\log g_1$ exists which is continuous on the closed unit disk (see [8, Corollary 1.21, p. 58]). So the function g_1 is an exponential, i.e., there exists $u \in A$ such that $g_1 = \exp(u)$.

By a modification of the method used in [5], we will show that this result holds for more general domains. Note that the example of an annulus and the functions $f(z) := z$, $g(z) := 0$ shows that not every invertible function is an exponential and that therefore the case $g = 0$ must be excluded.

Notation. In the sequel G denotes a bounded domain in the complex plane $\mathbf{C}$, whose boundary consists of finitely many closed, non-intersecting Jordan curves. Let $C(\overline{G})$ denote the algebra of all continuous, complex-valued functions in $\overline{G}$. We will consider the following algebras:

$$A(\overline{G}) := \{f \in C(\overline{G}) : f \text{ analytic in } G\},$$

$$A^n(\overline{G}) := \{f \in A(\overline{G}) : f^{(j)} \in A(\overline{G}),\ j = 1, \ldots, n\},$$

$$A^\infty(\overline{G}) := \bigcap_{n=1}^{\infty} A^n(\overline{G})$$

and

$$H^\infty(G) := \{f : f \text{ analytic in } G \text{ and bounded}\}.$$

The *zero set* of a function $g \in A(\overline{G})$ is denoted by Z_g, i.e.

$$Z_g := \{z \in \overline{G} : g(z) = 0\}.$$

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Especially we are interested in the *boundary zero set*, i.e. in the set $Z_g \cap \partial G$.

1. In [5] the authors used simply connected level sets to enclose the zeros of the functions f_1 and f_2. Since in the case of our domain G these level sets need not be simply connected, we replace them by "nicer" sets. Moreover, we only deal with the zero set of one function, which enables us to partly extend the results to $H^\infty(G)$.

LEMMA 1.1. *Suppose that $g \in A(\overline{G})$ has at least one zero in $\overline{G}$, but does not vanish identically. Then, given $\delta > 0$, there exist finitely many Jordan domains $H_1, \ldots, H_n$ with the following properties:*

(i) $Z_g \subset \bigcup_{j=1}^{n} H_j$.

(ii) $\overline{H}_j \cap \overline{H}_k = \emptyset \ (j \neq k)$.

(iii) $|g(z)| \leq \delta / 2 \ (z \in \bigcup_{j=1}^{n} [H_j \cap \overline{G}])$.

(iv) The function g does not vanish on the boundary set $\bigcup_{j=1}^{n} [\partial H_j \cap \overline{G}]$.

(v) Each set $\overline{H}_j \cap \overline{G}$ resp. $H_j \cap G$ is homeomorphic with a (resp. open) closed rectangle.

Moreover, two more sets of Jordan domains $\{H_{1j}\}$, $\{H_{2j}\}$ $(j = 1, \ldots, n)$ exist, enjoying all properties (i)–(v) and satisfying

(vi) $\overline{H}_j \cap \overline{G} \subset H_{1j} \cap \overline{G} \subset \overline{H}_{1j} \cap \overline{G} \subset H_{2j} \cap \overline{G}$.

PROOF. First of all we "chase" the boundary zeros of g. Since by construction (iv) holds, it is then evident that only finitely many zeros can ly in the complement of the union of these "boundary" domains H_j. These we enclose by small rectangles. We shall work in the upper halfplane U: Let Γ denote a closed Jordan curve of ∂G and let Γ^* denote the unbounded component of $\hat{\mathbf{C}} \setminus \Gamma$. Remember that by the two-constant theorem [4, p. 299, Satz 2] there exists $\zeta \in \Gamma$ such that $g(\zeta) \neq 0$. (By assumption g does not vanish identically.) With the aid of the Riemann mapping theorem, we transform the simply connected domain Γ^* onto the upper halfplane U, sending the point ζ to infinity. It is known that this transform extends to a homeomorphism Φ of $\hat{\mathbf{C}}$ onto $\hat{\mathbf{C}}$ (this is the theorem of Schoenflies [6, Corollary 9.4, p. 282]).

Consider the function $\tilde{g} := g \circ \Phi^{-1}$. It is analytic in a domain $\tilde{G}$, one of whose boundary curves is the real axis. By the construction of Φ the boundary zero set K on the real axis is bounded and therefore compact. By uniform continuity we can find open disks D_{z_0} around each real boundary zero $z_0 \in K$ of $\tilde{g}$ such that

$$|\tilde{g}(z)| \leq \frac{\delta}{4} \quad (z \in D_{z_0} \cap \tilde{G}).$$

By the reflexion principle of Schwarz, the set K of real boundary zeros is totally disconnected, i.e., it cannot contain an interval.

So it is possible to choose the disks D_{z_0} such that $\tilde{g}$ does not vanish on the two intersecting points with the real axis. Since K is compact, finitely many of such disks D_{z_0} are sufficient to cover the real boundary zero set of $\tilde{g}$. Let $\tilde{H}_j$, $j = 1, \ldots, N$, denote the connected components of this union. Since in all intersecting points the function $\tilde{g}$ does not vanish, it is easy to construct rectangles $R_j \subset \tilde{H}_j$ which fulfill properties (i) – (v). To obtain the additional Jordan domains H_{1j}, H_{2j} just take slightly larger rectangles.

Now we transform back with the homeomorphism Φ to get the Jordan domains H_j, H_{1j} and H_{2j}. ∎

THEOREM 1.2. *Suppose $f, g \in A(\overline{G})$, $g \not\equiv 0$ and $|f(z)| + |g(z)| \geq \delta > 0$ for all $z \in \overline{G}$. Then there exists $h, k \in A(\overline{G})$ such that*

$$f + hg = \exp(k).$$

PROOF. If the function g has no zero at all in $\overline{G}$, just define $k := 0$ and $h := (1-f)/g$. So we may assume that g has at least one zero in $\overline{G}$.

STEP 1. There exist continuous functions h, k in $\overline{G}$ continuously differentiable in G such that $\partial h / \partial \bar{z}$ is bounded in $\overline{G}$ and

$$f + hg = \exp(k).$$

Take the simply connected domains H_j, H_{1j}, H_{2j} ($j = 1, \ldots, n$) from Lemma 1.1 and let χ_j denote a smooth function satisfying

$$\chi_j(z) = \begin{cases} 1 & z \in H_j \\ 0 & z \in \mathbf{C} \setminus H_{1j}. \end{cases}$$

By Lemma 1.1 (iii), (v) the sets $\overline{G}_j := \overline{G} \cap \overline{H}_{2j}$ are homeomorphic with closed rectangles and $|g(z)| \leq \delta / 2$ for all $z \in \bigcup_{j=1}^{n} \overline{G}_j$. This implies

$$|f(z)| \geq \frac{\delta}{2} \quad (z \in \bigcup_{j=1}^{n} \overline{G}_j).$$

By [8, Corollary 1.21, p. 58] there exists a continuous branch $\log_j f$ of $\log f$ in $\overline{G}_j$, which we extend continuously to $\mathbf{C}$. Then we define the continuous functions

$$k := \sum_{j=1}^{n} \chi_j \log_j f, \quad h := \frac{\exp k - f}{g}.$$

Using the properties (i)–(iv) of Lemma 1.1, it is evident that h, k are continuously differentiable and that $\partial h / \partial \bar{z}$ is bounded in $\overline{G}$. (Note that h vanishes identically in $\bigcup_{\nu=1}^{n} \overline{G}_\nu$, especially "near" Z_g.)

STEP 2. There exist $h, k \in A(\overline{G})$ such that

$$f + hg = \exp(k).$$

With the functions from Step 1, we define the continuous function

$$F := \frac{f}{f+hg} = f \exp(-k).$$

Clearly,

$$\frac{F}{f} \cdot f + \frac{1-F}{g} \cdot g = 1.$$

Of course, F/f and $(1-F)/g$ are differentiable in G, but not necessarily analytic. Therefore we seek for a $u \in C(\overline{G})$, which is continuously differentiable in G such that

$$\frac{\partial}{\partial \bar{z}} \left[\frac{\exp(ug)}{f+hg} \right] = 0,$$

which implies the analyticity of $\frac{F}{f} \exp(ug)$ and $F \exp(ug)$. This yields the inhomogeneous $\bar{\partial}$-equation

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{f+hg} \cdot \frac{\partial h}{\partial \bar{z}} =: \mu(z).$$

We show that one solution u is given by

$$u(z) := \frac{1}{2\pi i} \int_G \frac{\mu(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}, \quad (z \in G).$$

Since μ is bounded on $\overline{G}$ the convolution u is continuous. From [3, Satz 2.1, Steps 3 and 4, p. 208 f.] it follows that u satisfies the desired $\bar{\partial}$-equation.

We now replace the function F by $F \exp(ug)$. Then the function

$$\alpha := \frac{F}{f} \exp(ug) = \exp[ug - k]$$

is a member of $A(\overline{G})$. Since the function $ug - k$ is continuous in $\overline{G}$, we have

$$ug - k \in A(\overline{G}).$$

So the function α is an exponential. Also the function

$$\beta := \frac{1 - F\exp(ug)}{g} = \left[h + f \frac{1 - \exp(ug)}{g} \right] \exp(-k)$$

is continuous up to all the boundary of G and analytic in G. Since the identity

$$\alpha f + \beta g = 1$$

holds, we are done. ∎

COROLLARY. *Let A denote one of the algebras $A^n(\overline{G})$, $n \in \mathbf{N}$, or $A^\infty(\overline{G})$. Suppose $f, g \in A$, $g \not\equiv 0$ and $|f(z)| + |g(z)| \geq \delta > 0$ for all $z \in \overline{G}$. Then there are $h, k \in A$ such that*

$$f + hg = \exp(k).$$

PROOF. Theorem 1.2 implies the existence of functions $\tilde{h}, \tilde{k} \in A(\overline{G})$ with the desired property. There exists $\delta_0 > 0$ such that

$$|\exp(\tilde{k}(z))| \geq 2\delta_0$$

for all $z \in \overline{G}$. Now, by a version of the theorem of Mergelyan [9, Theorem 13.3], there exist rational functions r with poles off $\overline{G}$, approximating $\tilde{h}$. Especially, there exists r such that

$$|\tilde{h}(z) - r(z)|\,|g(z)| \leq \delta_0 \quad (z \in \overline{G}).$$

Since the connected component $\exp(A(\overline{G}))$ is open in the group of invertible elements in $A(\overline{G})$ [7, Theorem 10.44], there exists $k \in A(\overline{G})$ such that

$$f + rg = \exp(k).$$

Now the left side is a member of A and an easy calculation shows that therefore $k \in A$. ∎

We now present a partial result concerning the so-called stable rank of the algebra $H^\infty(G)$, see [2].

THEOREM 1.3. Suppose $f \in H^\infty(G)$, $g \in A(\overline{G})$, $g \not\equiv 0$ and $|f(z)| + |g(z)| \geq \delta > 0$ for all $z \in G$. Moreover, assume that there exists a bounded analytic branch of $\log f$ in $\{z \in G : |g(z)| < \delta/2\}$. Then there exist $h, k \in H^\infty(G)$ such that

$$f + hg = \exp(k).$$

PROOF. We may assume that g has at least one zero in $\overline{G}$. Then we first of all construct smooth functions h, k in G such that h, k and $\partial h/\partial \bar{z}$ are bounded in G and $f + hg = \exp(k)$.

Take the sets of simply connected domains $H_j, H_{1j}, H_{2j}, j = 1, \ldots, n$ from Lemma 1.1 and let χ_j denote a smooth function satisfying

$$\chi_j(z) = \begin{cases} 1 & z \in H_j \\ 0 & z \notin \overline{H}_{1j}. \end{cases}$$

By properties (iii), (v) of Lemma 1.1 the set $H_{2j} \cap G$ is simply connected and $|g(z)| < \delta/2$ $(z \in H_{2j} \cap G)$. By hypothesis, there exists a branch $\log f$ such that $\|\log f\|_\infty \leq L$ in each domain $G \cap H_{2j}$. Define the following functions in G:

$$k := \sum_{j=1}^{n} \chi_j \log f, \quad h := \frac{\exp(k) - f}{g}.$$

Now this function k is bounded. Using Lemma 1.1, it is easy to see that h, k are smooth in G and that $h, \partial h / \partial \bar{z}$ is bounded in G. Note that we have

$$\frac{\partial h}{\partial \bar{z}} \cdot g = \frac{\partial k}{\partial \bar{z}} \exp(k) = \sum_{j=1}^{n} \frac{\partial \chi_j}{\partial \bar{z}} \log f.$$

Moreover, the functions χ_j are constant near the zeros of g and $\log f$ is bounded.

Now we proceed as in Step 2 in the proof of Theorem 1.2. ∎

REMARKS. 1) For $g \not\equiv 0$ and sufficiently small δ, all components of $E_\delta := \{z \in G : |g(z)| < \delta\}$ are simply connected. The crucial point in the hypothesis of Theorem 1.3 is the boundedness. (Assuming the contrary, there exists for every $\delta > 0$ a bounded residual domain D_δ of a domain in E_δ. By the maximum principle, D_δ contains at least one of the finitely many residual domains of G. So there exists an increasing sequence $(D_\nu)_{\nu=1}^{\infty}$ of bounded, simply connected domains such that $|g(z)| \leq 1/\nu$ for all $z \in \partial D_\nu$. The union $D := \bigcup_{\nu=1}^{\infty} D_\nu$ is simply connected and bounded, and g vanishes identically on the continuum ∂D. Therefore, $\partial D \subset \partial G$, and the two-constant theorem [4, p. 299, Satz 2] implies the contradiction $g = 0$.)

2) Theorem 1.3 complements the result in [2, Theorem 2].

SUPPLEMENT. Dr. R. Mortini has kindly pointed out to me the thesis of L.A. Laroco, *Stable Rank and Approximation Theorems in H^∞*, University of California at Berkeley, 1988. He proved that, for the case of the open unit disk $\mathbf{D}$, the existence of a bounded logarithm of f is necessary and sufficient for the conclusion of Theorem 1.3.

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