TRIVIAL ACTION ON THE TENSOR PRODUCT OF FINITE GROUPS

by R. J. HIGGS

(Received 31 March, 1987; revised 22 July, 1987)

Let G, H and K be finite groups such that K acts on both G and H. The action of K on G and H induces an action of K on their tensor product $G \otimes H$, and we shall denote the K-stable subgroup of $G \otimes H$ by $(G \otimes H)^K$. In section 1 of this note we shall obtain necessary and sufficient conditions for $(G \otimes H)^K = G \otimes H$. The importance of this result is that the direct product of G and H has Schur multiplier $M(G \times H)$ isomorphic to $M(G) \times M(H) \times (G \otimes H)$; moreover K acts on $M(G \times H)$, and $M(G \times H)^K$ is one of the terms contained in a fundamental exact sequence concerning the Schur multiplier of the semidirect product of K and $G \times H$ (see [**3**, (2.2.10) and (2.2.5)] for details). Indeed in section 2 we shall assume that G is abelian and use the fact that $M(G) \cong G \wedge G$ to find necessary and sufficient conditions for $M(G)^K = M(G)$.

1. To save repetition we shall continue to use the notation in the introduction. It will be convenient for any finite group L to let $\bar{L}$ denote $(L/L')/\Phi(L/L')$, where $\Phi(L)$ is the Frattini subgroup of L. Also since we shall be using tensor products it is most natural to adopt additive notation for abelian groups.

We begin by recalling that $G \otimes H$ is generated by pure tensors $g \otimes h$ for $g \in G$, $h \in H$, $(g \otimes h)^x = g^x \otimes h^x$ for $x \in K$, and $(G \otimes H)^K = \{z \in G \otimes H : z^x = z \text{ for all } x \in K\}$. Clearly if K acts trivially on G and H then $(G \otimes H)^K = G \otimes H$; our main result can, under suitably restricted circumstances, be regarded as a partial converse to this.

THEOREM 1. *Let Π denote the set of prime numbers which divide both the order of $\bar{G}$ and the order of $\bar{H}$. Then $(G \otimes H)^K = G \otimes H$ if and only if for each $p \in \Pi$ and each $x \in K$,*

$$\bar{g}^x = s(x)\bar{g} \quad \textit{and} \quad \bar{h}^x = t(x)\bar{h}$$

for all elements $\bar{g}$ and $\bar{h}$ of the Sylow p-subgroup of $\bar{G}$ and $\bar{H}$ respectively, where $s(x)$ and $t(x)$ are integers such that $s(x)t(x) \equiv 1 (\text{mod}\, p)$.

Proof. We have that $G \otimes H \cong G/G' \otimes H/H'$ under the isomorphism defined on pure tensors by $g \otimes h \mapsto gG' \otimes hH'$ *by* [**2**, (V.25.9)], so for notational convenience we shall assume henceforward that G and H are abelian. Next we note that if $\theta : G \to A$ and $\phi : H \to B$ are epimorphisms, then the homomorphism $\theta \otimes \phi : G \otimes H \to A \otimes B$ defined on pure tensors by $(\theta \otimes \phi)(g \otimes h) = \theta(g) \otimes \phi(h)$ is an epimorphism by [**4**, (V.5.2)], and so $(G \otimes H)/\ker(\theta \otimes \phi) \cong A \otimes B$. All further isomorphisms considered in this proof are constructed in this natural manner. Now we may express G and H as the direct sum of their respective Sylow subgroups,

$$G \cong \bigoplus_{i \in N} S_{p_i} \quad \text{and} \quad H \cong \bigoplus_{i \in N} T_{p_i}$$

Glasgow Math. J. **30** (1988) 271–274.

where p_i is the ith prime number. We then obtain from [**2**, (V.25.9)] that $G \otimes H \cong \bigoplus_{i \in N} (S_{p_i} \otimes T_{p_i})$, and consequently $(G \otimes H)^K \cong \bigoplus_{i \in N} (S_{p_i} \otimes T_{p_i})^K$. Thus we can now assume that both G and H are p-groups. However we now have that $G \otimes H$ is trivial if and only if G or H is trivial; to progress then we must assume that p divides the order of G and the order of H. Now $\overline{G \otimes H} \cong \bar{G} \otimes \bar{H}$ and $(\overline{G \otimes H})^K = \overline{G \otimes H}$ if and only if $(G \otimes H)^K = G \otimes H$, so that we may finally assume that G and H are elementary abelian p-groups.

It remains then to prove the theorem when G, H, and hence also $G \otimes H$, are elementary abelian p-groups; as such we may regard all three as vector spaces over Z_p. Let $\{g_1, \ldots, g_m\}$ and $\{h_1, \ldots, h_n\}$ be bases for G and H respectively over Z_p. Then $\{g_i \otimes h_j : 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $G \otimes H$ over Z_p by [**4**, p. 142]. Let $x \in K$, $g_u^x = \sum_{i=1}^{m} a_{ui} g_i$ for $1 \le u \le m$, and $h_v^x = \sum_{j=1}^{n} b_{vj} h_j$ for $1 \le v \le n$, where a_{ui}, b_{vj} are integers. Then

$$(g_u \otimes g_v)^x = \sum_{i,j} a_{ui} b_{vj} (g_i \otimes g_j).$$

Thus $g_u \otimes g_v$ is fixed by x if and only if

$$a_{ui} b_{vj} \equiv \begin{cases} 1 (\operatorname{mod} p) & \text{for} \quad (i, j) = (u, v), \\ 0 (\operatorname{mod} p) & \text{otherwise} \end{cases}$$

which yields that $a_{uu} b_{vv} \equiv 1 (\operatorname{mod} p)$ and $a_{ui} \equiv b_{vj} \equiv 0 (\operatorname{mod} p)$ for all $i \ne u$ and all $j \ne v$. Since these conditions are required to be true for all values of u and v, we conclude that $a_{uu} \equiv a_{11} (\operatorname{mod} p)$ for $1 \le u \le m$, $b_{vv} \equiv b_{11} (\operatorname{mod} p)$ for $1 \le v \le n$, and $a_{11} b_{11} \equiv 1 (\operatorname{mod} p)$. We have thus obtained the desired result.

The most interesting consequences of the theorem occur when we set $H = G$ and consider a single action of K on G.

COROLLARY 1. *Suppose the orders of G/G' and K are relatively prime, and let Π denote the set of prime numbers which divide the order of G/G'. Then $(G \otimes G)^K = G \otimes G$ if and only if every $x \in K$ either inverts or acts trivially on the Sylow p-subgroup of G/G' for each $p \in \Pi$.*

Proof. By Theorem 1 we have that $(G \otimes G)^K = G \otimes G$ if and only if for each $p \in \Pi$ and each $x \in K$, $\bar{g}^x = s(x)\bar{g}$ for all elements $\bar{g}$ of the Sylow p-subgroup of $\bar{G}$, where $s(x)$ is an integer with $s(x)^2 \equiv 1 (\operatorname{mod} p)$. However the only element/elements in Z_p whose square is $[1]$, are $[1]$ for $p = 2$ and $[\pm 1]$ for $p \ne 2$.

It remains, because of [**1**, (5.1.4)], to show that if G is an abelian p-group and $x \in K$ inverts $\bar{G}$, then x inverts G. In this situation x^2 acts trivially on $\bar{G}$ and hence on G, so that x is an automorphism of G of order 2. Let $T = \langle x \rangle$ and $C_T(G) = \{g \in G : g^x = g\}$. Then $G = C_T(G) \times [T, G]$ by [**1**, (5.2.3)], and $C_T(G) \le \Phi(G)$. Thus $C_T(G)$ must be trivial, and the desired conclusion is yielded by [**1**, (10.1.4)].

2. In this section we shall assume with the notation of section 1 that G is abelian. Now K acts on $M(G)$ via $[\alpha]^x = [\alpha^x]$ for all $[\alpha] \in M(G)$ and $x \in K$, where α denotes a complex-valued cocycle of G and $\alpha^x(g, h) = \alpha(g^x, h^x)$ for all $g, h \in G$. Also the action of K on $G \otimes G$ induces an action of K on $G \wedge G = G \otimes G / \langle g \otimes g : g \in G \rangle$ in the obvious way. However these two actions are related for there exists an isomorphism $\theta : M(G) \to G \wedge G$ such that $\theta([\alpha]^x) = \theta([\alpha])^x$ for all $[\alpha] \in M(G)$ and $x \in K$ (see [**3**, (2.6.6) and (2.6.7)]). Thus imitating Theorem 1 we obtain:

THEOREM 2. *Let Π denote the set of prime numbers such that G has p-rank at least two for each $p \in \Pi$. Then $M(G)^K = M(G)$ if and only if for each $p \in \Pi$ and each $x \in K$ either* (i) *x acts as an element of $SL(2, p)$ on the Sylow p-subgroup of $\bar{G}$ if G has p-rank* 2, *or* (ii) *x inverts or acts trivially on the Sylow p-subgroup of $\bar{G}$ if G has p-rank greater than* 2.

Proof. The reduction to the case when G is an elementary abelian p-group is almost exactly as in the proof of Theorem 1 with the symbol '$\otimes$' replaced by '$\wedge$'. The only minor difference is that if G is an abelian p-group then $G \wedge G$ is trivial if and only if G is cyclic; this accounts for the need to consider only those primes p for which G has p-rank at least two.

It thus remains to prove the theorem for G an elementary abelian p-group. Let $\{g_1, \ldots, g_m\}$ be a basis for G over Z_p. Then $\{g_i \wedge g_j : 1 \le i < j \le m\}$ is a basis for $G \wedge G$ over Z_p, where $g_i \wedge g_j$ is the image of $g_i \otimes g_j$ in $G \wedge G$. Let $x \in K$ and $g_u^x = \sum_{i=1}^{m} a_{ui} g_i$ for $1 \le u \le m$, where the a_{ui} are integers. Then

$$(g_u \otimes g_v)^x = \sum_i a_{ui} a_{vi} (g_i \otimes g_i) + \sum_{i<j} (a_{ui} a_{vj} - a_{uj} a_{vi})(g_i \otimes g_j) \\ + \sum_{i<j} a_{uj} a_{vi} ((g_i \otimes g_j) + (g_j \otimes g_i)).$$

Thus $(g_u \wedge g_v)^x = g_u \wedge g_v$ for $u < v$ if and only if

$$a_{ui} a_{vj} - a_{uj} a_{vi} \equiv \begin{cases} 1 (\operatorname{mod} p) & \text{for} \quad (i, j) = (u, v) \\ 0 (\operatorname{mod} p) & \text{for all} \quad (i, j) \neq (u, v) \quad \text{with} \quad 1 \le i < j \le m. \end{cases}$$

Now assuming henceforward that G does have p-rank at least 3, these conditions yield firstly that $a_{ui} \equiv a_{vj} \equiv 0 (\operatorname{mod} p)$ for all $(i, j) \neq (u, v)$ with $1 \le i < j \le m$, and hence secondly that $a_{uu} a_{vv} \equiv 1 (\operatorname{mod} p)$. Since these conditions are required to be true for all values of u and v with $u < v$, we conclude that $a_{uu} \equiv a_{11} (\operatorname{mod} p)$ for $1 \le u \le m$ and $a_{11} \equiv \pm 1 (\operatorname{mod} p)$. We have thus obtained the desired result in all cases.

We conclude this note with the obvious remark that if p does not divide the order of K in case (ii) of Theorem 2, then the condition stated there can be strengthened to: x either inverts or acts trivially on the Sylow p-subgroups of G if G has p-rank greater than 2.

REFERENCES

1. D. Gorenstein, *Finite groups,* (Chelsea, New York, 1980).
2. B. Huppert, *Endliche Gruppen I,* (Springer, 1967).
3. G. Karpilovsky, *The Schur multiplier,* (Oxford University Press, 1987).
4. S. Maclane, *Homology,* (Springer, 1963).

DEPARTMENT OF MATHEMATICS,
UNIVERSITY COLLEGE.
DUBLIN 4,
IRELAND.