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TOPOLOGICAL LEFT AMENABILITY OF SEMIDIRECT PRODUCTS

BY
H. D. JUNGHENN

ABSTRACT. Let S and T be locally compact topological semigroups and $S ⓣ T$ a semidirect product. Conditions are determined under which topological left amenability of S and T implies that of $S ⓣ T$, and conversely. The results are used to show that for a large class of semigroups which are neither compact nor groups, various notions of topological left amenability coincide.

1. **Introduction.** Let S and T be locally compact topological semigroups with identities (each denoted by 1) and $\tau: T\times S\to S$ a jointly continuous mapping such that $\tau(t, ss')=\tau(t, s)\tau(t, s')$, $\tau(tt', s)=\tau(t, \tau(t', s))$, $\tau(t, 1)=1$, and $\tau(1, s)=s(s, s'\in S; t, t'\in T)$. If multiplication on $S\times T$ is defined by

$$(s, t)(s', t')=(s\tau(t, s'), tt'),$$

then $S\times T$, with the usual product topology, becomes a locally compact topological semigroup with identity $(1, 1)$, called the *semidirect product* of S and T and denoted by $S ⓣ T$. The purpose of this paper is to determine when topological left amenability of S and T implies that of $S ⓣ T$. Positive results are obtained if, for example, T is a group and S is either compact or a group. More general results can be gotten by using a stronger amenability condition. The converse problem of determining topological left amenability of S and T from that of $S ⓣ T$ is also considered, and an application to topological wreath products is given in the final section.

2. **Preliminaries.** Let S be a locally compact topological semigroup (jointly continuous multiplication), $C(S)$ the Banach algebra of all bounded real-valued continuous functions on S (with the usual supremum norm), $C_0(S)$ the subalgebra of functions which vanish at infinity, and $M=M(S)$ the dual of $C_0(S)$. We shall, as usual, identify M with the space of bounded regular Borel measures on S (see, for example, [6]). M is a Banach algebra under convolution defined by

$$(\mu * \nu)(f)=\int_S\int_S f(st)\,d\mu(s)\,d\nu(t) \quad (f\in C_0(S); \mu, \nu\in M)$$

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The subset $P=P(S)$ of probability measures is a multiplicative subsemigroup of M, and the set P_c of probability measures with compact support is norm dense in P and spans a dense subspace of M.

A *mean* on M^*, the dual of M, is a positive linear functional Γ such that $\Gamma(1)=1$, where $1\in M^*$ is defined by $1(\mu)=\mu(S)$ $(\mu\in M)$. The set $Q(P_c)$, where $Q:M\to M^{**}$ denotes the canonical isometry, can be shown to be weak* dense in the set of all means on M. A *topological left invariant mean* (abbreviated TLIM) is a mean Γ on M^* such that $\Gamma(F*\mu)=\Gamma(F)$ for all $F\in M^*$ and $\mu\in P$, where $F*\mu\in M^*$ is defined by $(F*\mu)(\nu)=F(\mu*\nu)$. If a TLIM exists then S is said to be *topologically left amenable.*

Wong [13] noted that if S is a (locally compact topological) group with left Haar measure λ, then S is topologically left amenable if and only if $L^\infty(S,\lambda)$ has a TLIM (as defined, for example, in [5]). This is also a consequence of the following more general result (recalling that $L^1(S,\lambda)$ may be identified with the ideal in M of measures absolutely continuous with respect to λ):

Proposition 2.1. *Let S be a locally compact topological semigroup, M_1 any closed ideal of M which contains non-zero positive members. Then M^* has a* TLIM *if and only if M_1^* has a* TLIM. (*The notions of mean and* TLIM *on M_1^* are defined as for M^*.*)

Proof. Assume $\Gamma_1\in M_1^{**}$ is a TLIM. Let $R:M^*\to M_1^*$ denote the restriction operator and Γ the mean $\Gamma_1\circ R\in M^{**}$. Choose any $\nu\in M_1\cap P$. Then $R(F)*(\mu*\nu)=R(F*\mu)*\nu$ for all $\mu\in P$, $F\in M^*$, and therefore $\Gamma(F*\mu)=\Gamma_1(R(F*\mu))=\Gamma_1(R(F*\mu)*\nu)=\Gamma_1(R(F)*(\mu*\nu))=\Gamma_1(R(F))=\Gamma(F)$.

Conversely, let Γ be a TLIM on M^*, and let (μ_n) be a net in P such that $\Gamma=\text{weak}^*-\lim_n Q(\mu_n)$. We may assume that $(\mu_n)\subset M_1$ (otherwise choose any $\nu\in M_1\cap P$ and replace μ_n by $\nu*\mu_n$, noting that $\Gamma(F)=\Gamma(F*\nu)=\lim_n Q(\mu_n)$ $F*\nu)=\lim_n Q(\nu*\mu_n)(F)(F\in M^*)$). Let $Q_1:M_1\to M_1^{**}$ denote the canonical injection, and let (μ_m) be a subnet such that $Q_1(\mu_m)$ weak* converges to some mean Γ_1 on M_1^*. Given $F_1\in M_1^*$, choose $F\in M^*$ such that $R(F)=F_1$. Then for any $\nu\in M_1\cap P$, $\Gamma_1(F_1*\nu)=\lim_m F_1(\nu*\mu_m)=\Gamma(F*\nu)=\Gamma(F)=\lim_m F(\mu_m)=\Gamma_1(F_1)$, so Γ_1 is a TLIM on M_1^*.

Using standard results from the theory of topological vector spaces it can be shown that S is topologically left amenable if and only if the following condition holds: (A) There exists a net (μ_n) in P_c (or, equivalently, in P) such that $\|\nu*\mu_n-\mu_n\|\to 0$ for each $\nu\in P_c$. (See [3] or [5], where the proof is given for the case S a group.) A related condition is the following: (B) There exists a net (μ_n) in P_c such that $\|\delta(s)*\mu_n-\mu_n\|\to 0$ uniformly in s on each compact subset of S. Here $\delta(s)\in P_c$ denotes Dirac measure at s. Clearly (B) implies (A), and if S is a group then the two conditions are equivalent [3]. Furthermore, if S is compact then (A) and (B) are each equivalent to the existence of a right zero in the semigroup $P(S)$. It is not known to the author if properties (A) and (B) are equivalent in general.

The space LUC(S) of *left uniformly continuous functions on* S is defined by $\text{LUC(S)} = \{f \in C(S) : s \to L(s)f$ is norm continuous$\}$, where $L(s)$ denotes the left translation operator on $C(S)$. LUC(S) is easily seen to be a closed translation invariant subalgebra of $C(S)$ which contains the constant function 1 and which coincides with $C(S)$ if S is compact or discrete. (See [1, 9] for other properties of LUC(S).) A *mean* on LUC(S) is a positive linear functional μ on LUC(S) such that $\mu(1) = 1$. If for each $f \in \text{LUC(S)}$ and $s \in S$, $\mu(L(s)f) = \mu(f)$, then μ is a *left invariant mean* (LIM) and LUC(S) is said to be *left amenable.* If S is a group then LUC(S) is the space $UC_r(S)$ defined in [5]. In this case S is topologically left amenable if and only if LUC(S) is left amenable [5; Theorem 2.3.2]. The same is true if S is compact.

3. **Main results.** Throughout this section S and T denote locally compact topological semigroups with identities and $X = S \circledT T$ a semi-direct product of S and T, as defined in section 1.

THEOREM 3.1. *If S and T have property* (B) *and T is a group, then X has property* (B).

Proof. Let $(\lambda_i) \subset P_c(S)$ and $(\nu_j) \subset P_c(T)$ be nets such that $\|\delta(s) * \lambda_i - \lambda_i\|$ and $\|\delta(t) * \nu_j - \nu_j\|$ tend to zero uniformly on compact subsets of S and T respectively. For each i and j define $\mu_{ij} \in \text{P}(X)$ by

$$\mu_{ij}(f) = \iint\limits_{ST} f((1, t)(s, 1))\, d\nu_j(t)\, d\lambda_i(s), \; (f \in C_0(X)).$$

Let $R(x)$ and $L(x)$ denote, respectively, the right and left translation operators (by $x \in X$) on $C(X)$, and define $W: C(X) \to C(T)$ by $(Wf)(t) = f(1, t)$. For any $s' \in S$, $t' \in T$, and $f \in C_0(X)$ we have

$$\begin{aligned}\delta(s', t') * \mu_{ij}(f) &= \iint\limits_{ST} f((s', t')(1, t)(s, 1))\, d\nu_j(t)\, d\lambda_i(s) \\ &= \iint\limits_{ST} [WL(s', 1)R(s, 1)f](t', t)\, d\nu_j(t)\, d\lambda_i(s) \\ &= \iint\limits_{ST} [WL(s', 1)R(s, 1)f](t)\, d\nu_j(t)\, d\lambda_i(s) + \alpha(i, j, s', t', f) \\ &= \iint\limits_{TS} f(s'\tau(t, s), t)\, d\lambda_i(s)\, d\nu_j(t) + \alpha(i, j, s', t', f) \qquad (1)\end{aligned}$$

where $|\alpha(i, j, s', t', f)| \le \|\delta(t') * \nu_j - \nu_j\|\, \|f\|$. Let K_j denote the support of ν_j, and for each $t \in T$ define $g_t \in C(S)$ by $g_t(s) = f(\tau(t, s), t)$. The double integral in (1)

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may then be written

$$\iint_{TS} g_t(\tau(t^{-1}, s')s)\, d\lambda_i(s)\, d\nu_j(t) = \int_{K_j} [\delta(\tau(t^{-1}, s')) * \lambda_i](g_t)\, d\nu_j(t)$$

$$= \int_{K_j} [\delta(\tau(t^{-1}, s')) * \lambda_i - \lambda_i](g_t)\, d\nu_j(t) + \int_T \lambda_i(g_t)\, d\nu_j(t). \quad (2)$$

Note that the second integral on the right in (2) is $\mu_{ij}(f)$.

Now let C and K be compact subsets of S and T respectively. Given $\varepsilon > 0$, choose j such that $\|\delta(t') * \nu_j - \nu_j\| < \varepsilon$ for all $t' \in K$. Since $\tau(K_j^{-1}xC)$ is compact in S we may choose i such that $\|\delta(\tau(t^{-1}, s')) * \lambda_i - \lambda_i\| < \varepsilon$ for all $t \in K_j$ and $s' \in C$. It follows from (1) and (2) that $|\delta(s', t') * \mu_{ij}(f) - \mu_{ij}(f)| \le 2\varepsilon \|f\|$ for all $s' \in C$, $t' \in K$ and $f \in C_0(X)$.

Let the linear space $E = M(X)^{S \times T}$ have the topology of uniform convergence on compact subsets of $S \times T$, where $M(X)$ carries the norm topology. For each i and j define $V_{ij} \in E$ by $V_{ij}(s, t) = \delta(s, t) * \mu_{ij} - \mu_{ij}$. The above argument shows that 0 is in the closure in E of the set $A = \{V_{ij} : i, j\}$, hence there exists a net (V_n) in A which converges to 0. The corresponding net of measures (μ_n) then has the required properties.

REMARKS. Theorem 3.1 holds for the direct product case even if T is not a group, as an examination of the proof (which simplifies) reveals. In general, however, the theorem fails if T is not a group. As an example, let S and G be compact topological groups and let $T = G \cup \{0\}$, where 0 is an isolated zero of T. Define $\tau : T \times S \to S$ as follows: $\tau(G, s) = \{s\}$, $\tau(0, s) = 1$. Then if S is non-trivial, $S \, ⓣ \, T$ has at least two left zeros and therefore cannot be topologically left amenable.

It is not known to the author if the property (A)-analog of Theorem 3.1 holds (except, of course, in the trivial cases S compact or S a group). However, one can show the following: If S and T have property (A) and T is a group, then there exists a net $(\mu_n) \subset P_c(X)$ such that $\|(\lambda \otimes \nu) * \mu_n - \mu_n\| \to 0$ for every $\lambda \in P(S)$ and $\nu \in P(T)$ (where $\lambda \otimes \nu$ denotes the product measure).

The converse of Theorem 3.1 holds even if T is not a group. In fact, we have the following result:

PROPOSITION 3.2. *Let X satisfy condition* (A) (*respectively,* (B)). *Then S and T satisfy condition* (A) (*respectively,* (B)).

Proof. We prove only that if X satisfies (A) then so does S. Let (μ_n) be a net in $P_c(X)$ such that $\|\mu * \mu_n - \mu_n\| \to 0$ for all $\mu \in P(X)$. Define a net $\lambda_n)$ in $P_c(S)$ by $\lambda_n(g) = \int_{S \times T} g(s)\, d\mu_n(s, t)$, $(g \in C_0(S))$, or, equivalently, $\lambda_n(A) = \mu_n(A \times T)$ (A a Borel subset of S). Given $\lambda \in P_c(S)$ define $\mu \in P_c(X)$ by $\mu(f) = \int_S f(s, 1)\, d\lambda(s)$, $(f \in C_0(X))$. Let $g \in C_0(S)$ and define $f \in C(X)$ by $f(s, t) = g(s)$.

Then,

$$\int_S g(ss')\,d\lambda(s) = \int_S [R(s',1)f](s,1)\,d\lambda(s)$$
$$= \int_{S\times T} [R(s',1)f](s,t)\,d\mu(s,t)$$
$$= \int_{S\times T} f(s\tau(t,s'),t)\,d\mu(s,t),$$

so

$$(\lambda * \lambda_n)(g) = \int_S \int_{S\times T} f(s\tau(t,s'),t)\,d\mu(s,t)\,d\lambda_n(s')$$
$$= \int_{S\times T}\int_{S\times T} f(s\tau(t,s'),t)\,d\mu(s,t)\,d\mu_n(s',t')$$
$$= \int_{S\times T}\int_{S\times T} f(s\tau(t,s'),tt')\,d\mu(s,t)\,d\mu_n(s',t')$$
$$= \int_{S\times T} f\,d\mu * \mu_n.$$

Since $\lambda_n(g) = \int_{S\times T} f\,d\mu_n$ it follows that $\|\lambda * \lambda_n - \lambda_n\| \leq \|\mu * \mu_n - \mu_n\|$ and hence $\|\lambda * \lambda_n - \lambda_n\| \to 0$. The proofs of the remaining statements are similar.

If S and T are *both* groups then Theorem 3.1 follows from the remark at the end of section 2, and the next result, which is of some independent interest.

THEOREM 3.3. (a) *If* LUC(S) *and* LUC(T) *are left amenable and if the set* $D = \{t \in T : \tau(t,S)$ *is dense in* $S\}$ *is dense in* T (*which is trivially the case if* T *is a group*), *then* LUC(X) *is left amenable.* (b) *If* LUC(X) *is left amenable and* S *is compact then* LUC(S) *and* LUC(T) *are left amenable.*

Proof. Let λ and ν be LIM's on LUC(S) and LUC(T), respectively, and define bounded linear operators V: LUC(X) $\to$ LUC(S) and W: LUC(X) $\to$ LUC(T) by $(Vf)(s) = f(s,1)$; $(Wf)(t) = \lambda(VL(1,t)f$, $(s \in S, t \in T, f \in \text{LUC}(X))$. We shall show that the mean $\mu = \nu \circ W$ is a LIM on LUC(X).

Note first the following identities: $VL(s,1) = L(s)V$, $WL(1,t) = L(t)W$, and $WL(s,1) = W$. The first two are easily established. To verify the third, let $t \in D$, $s \in S$ and $f \in \text{LUC}(X)$. Then $\lambda(VL(\tau(t,s),t)f) = \lambda(VL(s,1)L(1,t)f) = \lambda(L(s)VL(1,t)f) = \lambda(VL(1,t)f)$. From the definition of D and the fact that D is dense in T it follows that $\lambda(VL(s,t)f) = \lambda(VL(1,t)f)$ for all $s \in S$, $t \in T$, and this establishes the identity.

The proof that μ is a LIM follows easily from the above identities: For each $s \in S$, $t \in T$ and $f \in \text{LUC}(X)$ we have $\mu(L(s,t)f) = \nu(WL(1,t)L(s,1)f) =$

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$\nu(L(t)WL(s,1)f)=\nu(Wf)=\mu(f)$.

For the converse, assume S is compact and μ is a LIM on LUC(X). Given $f\in$ LUC(S) = $C(S)$ define $f'\in C(X)$ by $f'(s,t)=f(s)$. The compactness of S and the joint continuity of τ imply that $f'\in$ LUC(X). Define $\lambda(f)=\mu(f')$. Then the identity $(L(s)f)'=L(s,1)f'$ implies that λ is a lim. A similar argument shows that LUC(T) is left amenable.

Remarks. Theorem 3.3(a) is valid if S and T are merely topological semigroups, not necessarily locally compact, and LUC(X) is replaced by any translation invariant left introverted (see [2, p. 540] for definition) subspace F of $C(X)$ containing the constant functions, provided LUC(S) and LUC(T) are replaced by the spaces $\{f(\cdot,1):f\in F\}$ and $\{f(1,\cdot):f\in F\}$, respectively.

If S and T are discrete then part (a) of Theorem 3.3 reduces to a result of M. Klawe [7, Prop. 3.4], whose proof, quite different from ours, is based on Day's fixed point theorem.

The example given after Theorem 3.1 shows that Theorem 3.3(a) fails in general if D is not dense in T. Note that if T is a group then $D=T$.

4. **Wreath Products.** The wreath product construction may be used to produce non-trivial examples of locally compact topological semigroups which are neither compact nor groups but for which conditions (A) and (B) of section 2 are equivalent.

Let T be a discrete group which acts on the right on the phase space Y. For example, we could take $Y=T$, and right multiplication as the action. Let U be a compact topological semigroup with identity and S the product space U^Y with the product topology and coordinate multiplication. Define $\tau:T\times S\to S$ by $\tau(t,s)(y)=s(yt)$, where yt denotes the action of t on y. The semidirect product $S \textcircled{\tau} T$ is called the *(abstract) wreath product* of U and T and is denoted by $UwrT$. (A survey of the algebraic theory of wreath products of semigroups and their applications may be found in [12]. See also [4, 8], where topological questions are considered.) Note that if U is not a group and T is not finite, then $UwrT$ is neither compact nor a group.

Theorem 4.1. *The following are equivalent:*

(a) *$UwrT$ has property (A).*

(b) *$UwrT$ has property (B).*

(c) LUC($UwrT$) *is left amenable.*

(d) LUC(U) $(=C(U))$ *and* LUC(T) $(=C(T))$ *are left amenable.*

Proof. Since S is compact and T is a group, properties (A) and (B) are equivalent for each of these semigroups (see section 2). It follows from Theorem 3.1 and Proposition 3.2 that (a) and (b) are equivalent. A similar application of Theorem 3.3 shows that (a) and (c) are equivalent.

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To show that (c) and (d) are equivalent it suffices, by Theorem 3.3, to prove that $C(U)$ is left amenable if and only if $C(S)$ is left amenable. Since U is a continuous homomorphic image of S, the sufficiency follows from a result of Day [2, p. 540]. An interesting proof of the necessity uses the structure theory of compact topological semigroups, according to which each such semigroup R has minimal right ideals and a smallest two-ideal $K(R)$, and the minimal right ideals are precisely of the form eR, where $e^2 = e \in K(R)$ [10]. A result of Rosen relates this structure theory to the existence of a LIM on $C(R)$: $C(R)$ has a LIM if and only if R has exactly one minimal right ideal [11]. Applying this to the present setting it thus suffices to show that if U has exactly one minimal right ideal then the same is true for S. For each $y \in Y$ let $P_y : S \to U$ denote the projection mapping $s \to s(y)$. Since $P_y(S) = U$ it follows easily that $P_y(K(S)) = K(U)$. In particular, if $e^2 = e \in K(S)$ then $e(y)$ is an idempotent in $K(U)$ and hence $e(y)U$ is a minimal right ideal of U. Therefore, for any pair of idempotents $e_1, e_2 \in K(S)$, $e_1(y)U = e_2(y)U$ for all $y \in Y$, and so $e_1S = e_2S$.

REMARK. Using similar techniques one can show that properties (A) and (B) are equivalent to a third amenability property: There exists a net (μ_n) in $P_c(UwrT)$ such that $\|\delta(x) * \mu_n - \mu_n\| \to 0$ for each $x \in UwrT$.

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THE GEORGE WASHINGTON UNIVERSTIY
WASHINGTON, D.C. 20052
U.S.A.